KYBERNETIKA — VOLUME 15 (1979), NUMBER 1

Two-Level Feedback Control for Interconnected Distributed Parameter Systems of First Order

LESZEK TRYBUS

A decentralized optimal regulator problem is studied for a set of interconnected distributed subsystems described by linear symmetric hyperbolic equations of first order. Coupling is provided through boundary conditions. Long tubular reactors divided into sections with zonal or pointwise controls can be examples of such plants. Then subsystems are meant as sections of the reactor. The overall quadratic optimization problem is decomposed into two levels using the infeasible method. Matrix gain coefficients of first level regulators are characterized by Riccati integro-differential equations. Coordination variables can be adjusted at the second level using the gradient technique.

I. INTRODUCTION

A few hierarchical feedback control design methods have been developed for large scale (lumped parameter) systems, both deterministic and stochastic. Short surveys of existing results can be found in the papers by Schwarz and Wend [1], and Singh and Titli [2]. Multilevel optimization techniques are also of increasing interest for interconnected distributed systems [3], [4], [5], where reduction of computational complexity seems particularly attractive. Good introduction to this subject provides the paper by Pradin and Titli [4]. However, they have made some slightly restrictive mathematical assumptions as a consequence of infinite dimensional couplings considered.

Here more realistic situation is studied, i.e. the case when the couplings are finite dimensional and introduced through boundary conditions. Subsystems are described by linear symmetric hyperbolic equations of first order. Such systems represent counter-current tubular reactors of sectional structure, so for instance absorption and distillation columns, pushed furnaces, tunnel kilns, etc. The subsystems can be meant as sections of the reactor (Fig. 1). Quadratic cost functional is considered to derive a decentralized feedback control for the overall system.

The optimization problem is formulated in Section II and decomposed into two levels in the next section using the infeasible method. Open-loop control for i-th subsystem is developed in Section IV. Then, first level feedback regulators are derived and characterized by Riccati integro-differential equations. Typical gradient algorithm is proposed to coordinate the regulators (Section VI). Some remarks on more general forms of boundary couplings are given in the concluding section.

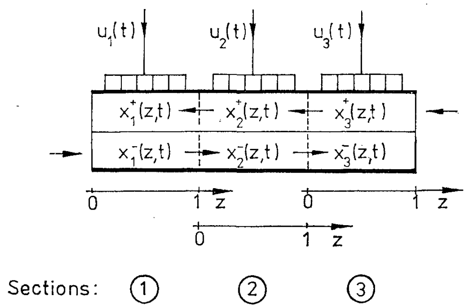

Fig. 1. Scheme of a tubular reactor composed of three cascade sections; x_i^-, x_i^+ — state variables of counter-current flows in i-th section, u_i — control, $i = 1, 2, 3$; z — normalized spatial variable; t — time.

The two-level regulator developed here is actually being implemented to feedback control of a multisectional tunnel kiln, where the state is reconstructed using a Pearson-type decentralized filter [6].

II. PROBLEM FORMULATION

Suppose the overall distributed system consists of r subsystems that are described by the following first-order linear symmetric hyperbolic equations

$$\text{(1)} \qquad \frac{\partial x_i(z, t)}{\partial t} = \Lambda_i(z) \frac{\partial x_i(z, t)}{\partial z} + A_i(z)\, x_i(z, t) + B_i(z)\, u_i(t)\,,$$

$$x_i(z, t_0) = x_{i0}(z) \quad i = 1, \dots, r\,,$$

in the region $\{(z, t) : z \in [0, 1],\ t \in [t_0, t_1],\ t_1 < \infty\}$, where x_i is an n_i-state and u_i — m_i-control. It is assumed that $n_i \times n_i$ function matrices Λ_i, A_i, and B_i (of dimension $n_i \times m_i$) have the properties

(i) $\Lambda_i(z)$ is continuously differentiable and $A_i(z)$, $B_i(z)$ are continuous in $[0, 1]$;

(ii) $\Lambda_i(z)$ is diagonal (normal form of first order partial differential equations [7]) and its entries satisfy the inequalities
$\lambda_1(z) \leqq \lambda_2(z) \leqq \ldots \leqq \lambda_{n'_i}(z) < 0 < \lambda_{n'_i+1}(z) \leqq \lambda_{n'_i+2}(z) \leqq \ldots \leqq \lambda_{n_i}(z)$
in $[0, 1]$, n'_i – a nonnegative integer;

(iii) if $\lambda_k(z^*) = \lambda_{k+1}(z^*)$ for some $z^* \in [0, 1]$ then $\lambda_k(z) = \lambda_{k+1}(z)$ for all $z \in [0, 1]$.

The state x_i and the matrix Λ_i can be partitioned according to (ii) as follows (compare [7])

(2) $$x_i = \begin{bmatrix} x_i^- \\ x_i^+ \end{bmatrix}, \quad \Lambda_i = \begin{bmatrix} \Lambda_i^- & 0 \\ 0 & \Lambda_i^+ \end{bmatrix}, \quad x_i^- \in R^{n'_i}, \quad x_i^+ \in R^{n_i - n'_i}$$

in order to represent better the counter-current character of the system. For instance, in a distillation column x_i^-, x_i^+ describe the state (composition and temperature) of vapor moving up and of liquid flowing down, respectively. Boundary conditions providing interconnection between subsystems are given by

(3) $$\begin{aligned} &x_1^-(0, t) = 0\,, \\ &x_i^-(0, t) = M_i^- x_{i-1}(1, t)\,, \quad i = 2, \ldots, r\,, \\ &x_r^+(1, t) = 0\,, \\ &x_i^+(1, t) = M_i^+ x_{i+1}^+(0, t)\,, \quad i = 1, \ldots, r-1\,, \end{aligned}$$

where M_i^-, M_i^+ are $n'_i \times n'_i$, $(n_i - n'_i) \times (n_i - n'_i)$ matrices, respectively.

From the conditions (3) we see that the state $x_i^-(0, t)$ of "minus flow" at the beginning of i-th subsystem depends on the final value $x_{i-1}^-(1, t)$ of $(i-1)$-st one (see Fig. 1). Similarly, the state $x_i^+(1, t)$ of "plus flow" at the end is deduced from $x_{i+1}^+(0, t)$ of $(i+1)$-st subsystem. In this way a cascade structure of the overall counter-current system is provided. This structure is in fact "pure" cascade and does not include parallel branches passing by certain subsystems.

Observe that if $n'_{i-1} = n'_i$ and M_i^- is the unit matrix then the state x^- is continuous between $(i-1)$-st and i-th subsystems.

One can show using the method of energy inequalities (see [7], [8] for details) that under the assumptions (i)–(iii) there exists a unique L^2-solution x_i, $i = 1, \ldots$ $\ldots, r$, to the equations (1), (3) if x_{i0} and u_i are square integrable. Such x_i is defined as a limit of C^1-functions.

Quadratic cost functional is taken as

(4) $$J = \frac{1}{2}\int_{t_0}^{t_1} \left\{ \sum_{i=1}^{r} \left[\int_0^1 \int_0^1 x_i(z, t)^T Q_i(z, z')\, x_i(z', t)\, \mathrm{d}z\, \mathrm{d}z' + u_i(t)^T R_i\, u_i(t) \right] + \right.$$
$$\left. + \sum_{i=1}^{r-1} x_i^-(0, t)^T S_i^- x_i^-(0, t) + \sum_{i=2}^{r} x_i^+(1, t)^T S_i^+\ x_i^+(1, t) \right\} \mathrm{d}t\,,$$

where Q_i, R_i, S_i^-, S_i^+ are $n_i \times n_i$, $m_i \times m_i$, $n_i' \times n_i'$, and $(n_i - n_i') \times (n_i - n_i')$ matrices, respectively. Moreover:

– $Q_i(z, z')$ is continuous and such that

$$Q_i(z, z') = Q_i(z', z)^T \,, \tag{5}$$

$$\int_0^1 \int_0^1 \varphi_i(z)^T \, Q_i(z, z') \, \varphi_i(z) \, \mathrm{d}z \, \mathrm{d}z' \geqq 0 \quad \text{for all} \quad \varphi_i \in L^2[0, 1] \quad \text{of dimension} \quad n_i \,;$$

– R_i, S_i^-, S_i^+ are positive definite and symmetric.

The problem considered is as follows: Find the optimal decentralized feedback control $\hat{u}_i$, $i = 1, \ldots, r$, for the series of subsystems (1), (3) which minimizes the functional (4).

III. DECOMPOSITION OF THE PROBLEM

We shall apply the infeasible method of decomposition [4] that is particularly convenient for systems with state variable couplings as in (3). Introduce a set of interconnection variables s_i^-, s_i^+, called sometimes pseudocontrols, defined as follows

$$\begin{aligned} s_i^-(t) &= M_i^- \, x_{i-1}^-(1, t) \,, \quad i = 2, \ldots, r \,, \\ s_i^+(t) &= M_i^+ \, x_{i+1}^+(0, t) \,, \quad i = 1, \ldots, r - 1 \,. \end{aligned} \tag{6}$$

So the i-th subsystem's equations become

$$\begin{aligned} \frac{\partial x_i}{\partial t} &= \Lambda_i(z) \frac{\partial x_i}{\partial z} + A_i(z) \, x_i + B_i(z) \, u_i \,, \\ x_i(z, t_0) &= x_{i0}(z) \,, \quad i = 1, \ldots, r \,, \\ x_i^-(0, t) &= s_i^-(t) \,, \quad i = 2, \ldots, r \,, \\ x_i^+(1, t) &= s_i^+(t) \,, \quad i = 1, \ldots, r - 1 \,, \\ x_1^-(0, t) &= 0 \,, \qquad x_r^+(1, t) = 0 \,. \end{aligned} \tag{7}$$

These represent a simple counter-current exchanger, as shown in Fig. 2, with the controls u_i, s_i^-, s_i^+.

Introduce further a set of coordination variables $\varrho_i^-(t)$, $i = 2, \ldots, r$, $\varrho_i^+(t)$, $i = 1, \ldots, r - 1$ (Lagrange multipliers) and consider the augmented functional

$$\bar{J} = J + \sum_{i=2}^{r} \int_{t_0}^{t_1} \varrho_i^-(t)^T \left[M_i^- \, x_{i-1}^-(1, t) - s_i^-(t) \right] \mathrm{d}t \, + \tag{8}$$

$$+\sum_{i=1}^{r-1}\int_{t_0}^{t_1}\varrho_i^+(t)^T\,[M_i^+\,x_{i+1}^+(0,t)-s_i^+(t)]\,\mathrm{d}t$$

that should be minimized with respect to u_i, s_i, and maximized with respect to ϱ_i.

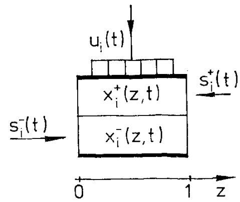

Fig. 2. Representation of i-th subsys.em using interconnection variables s_i^-, s_i^+.

This can be written in the additive form

$$\bar{J}=\sum_{i=1}^{r}J_i \tag{9}$$

using

$$\begin{aligned}(10)\quad J_i=\int_{t_0}^{t_1}\Big\{&\frac{1}{2}\int_0^1\int_0^1 x_i(z,t)^T\,Q_i(z,z')\,x_i(z',t)\,\mathrm{d}z\,\mathrm{d}z'+\tfrac{1}{2}\,u_i(t)^T\,R_i\,u_i(t)+\\&+\tfrac{1}{2}\,s_i^-(t)^T\,S_i^-\,s_i^-(t)+\tfrac{1}{2}\,s_i^+(t)^T\,S_i^+\,s_i^+(t)+\\&+\varrho_{i+1}^-(t)^T\,M_{i+1}^-\,x_i^-(1,t)-\varrho_i^-(t)^T\,s_i^-(t)+\\&+\varrho_{i-1}^+(t)^T\,M_{i-1}^+\,x_i^+(0,t)-\varrho_i^+(t)^T\,s_i^+(t)\Big\}\,\mathrm{d}t\,,\quad i=2,\ldots,r-1\,,\end{aligned}$$

where the identities

$$s_i^-(t)=x_i^-(0,t)\,,\quad s_i^+(t)=x_i^+(1,t) \tag{11}$$

have been applied. J_1 and J_r are like (10) but without $(\varrho_i^-)^T\,s_i^-$ and $(\varrho_i^+)^T\,s_i^+$, respectively.

Using the fact that (u_i, s_i) appear in J_i only, one can decompose the overall problem into two levels: first – minimization of J_i with respect to (u_i, s_i) for some specific ϱ, and second – maximization of $\bar{J}$ over ϱ (see [2], [9], the strong Lagrange duality) to provide coordination of first level subproblems.

So in i-th minimization subproblem we deal with the quadratic cost (10) for the first order symmetric hyperbolic system (7), subjected the boundary pseudocontrols s_i^-, s_i^+ (Fig. 2) and the control u_i acting through distributed gain coefficient $B_i(z)$. Similar problem has been extensively treated by Russell [8]. Assume that u_i, s_i are of class L^2 and ϱ_i is bounded. Define a few matrices

$$
\begin{aligned}
(12) \qquad T_{i1}^- &= (S_i^-)^{-1}\,, \quad T_{i2}^- = T_{i1}^- A_i^-(0)\,,\\
N_{i+1}^- &= (A_i^-(1))^{-1}\,(M_{i+1}^-)^T\,,\\
T_{i1}^+ &= (S_i^+)^{-1}\,, \quad T_{i2}^+ = T_{i1}^+ A_i^+(1)\,,\\
N_{i-1}^+ &= (A_i^+(0))^{-1}\,(M_{i-1}^+)^T\,.
\end{aligned}
$$

We have the following result:

Proposition 1. The optimal control $\hat{u}_i(t)$, interconnection variables $\hat{s}_i^-(t)$, $\hat{s}_i^+(t)$, and the resulting solution $\hat{x}_i(z, t)$ of (7), minimize the cost (10) if and only if

$$
\begin{aligned}
(13) \qquad \hat{u}_i(t) &= -R_i^{-1} \int_0^1 B_i(z)^T\, p_i(z, t)\,\mathrm{d}z\,, \quad i = 1, \ldots, r\,,\\
\hat{s}_i^-(t) &= T_{i1}^-\, \varrho_i^-(t) + T_{i2}^-\, p_i^-(0, t)\,, \qquad i = 2, \ldots, r\,,\\
\hat{s}_i^+(t) &= T_{i1}^+\, \varrho_i^+(t) - T_{i2}^+\, p_i^+(1, t)\,, \qquad i = 1, \ldots, r-1\,,
\end{aligned}
$$

where $p_i = p_i(z, t)$ is the solution of the adjoint system

$$
\begin{aligned}
(14) \qquad \frac{\partial p_i}{\partial t} &= \frac{\partial}{\partial z}(A_i(z)\, p_i) - A_i(z)^T\, p_i - \int_0^1 Q_i(z, z')\, \hat{x}_i(z', t)\,\mathrm{d}z'\,,\\
p_i(z, t_1) &= 0\,, \qquad i = 1, \ldots, r\,,\\
p_i^+(0, t) &= N_{i-1}^+\, \varrho_{i-1}^+(t)\,, \quad i = 2, \ldots, r\,,\\
p_i^-(1, t) &= -N_{i+1}^-\, \varrho_{i+1}^-(t)\,, \quad i = 1, \ldots, r-1\,,\\
p_1^+(0, t) &= 0\,, \quad p_r^-(1, t) = 0\,.
\end{aligned}
$$

Proof. Consider an arbitrary i with the exception of 1 and r. While proving we shall omit the subscript i for simplification. The minimum condition is

$$
I(u - \hat{u},\, s - \hat{s}) = \left\langle \frac{\partial J}{\partial u}\bigg|_{\hat{u}},\, u - \hat{u} \right\rangle + \left\langle \frac{\partial J}{\partial s}\bigg|_{\hat{s}},\, s - \hat{s} \right\rangle = 0
$$

for all square integrable u, s ($\langle .,. \rangle$ denotes the inner product in $L^2[t_0, t_1]$). This can be replaced by

$$(15) \qquad I(w, v) = 0 \quad \text{for all} \quad w, v \in L^2[t_0, t_1] ,$$

where

$$w = u - \hat{u} , \quad v = s - \hat{s} = \operatorname{col}[v^-, v^+] .$$

Denoting $x(z, t) = x(z, t; w, v)$ we have for our problem

$$(16) \quad I(w, v) = \int_{t_0}^{t_1} \left\{ \int_0^1 \int_0^1 \hat{x}(z, t)^T Q(z, z') x(z', t) \, dz \, dz' + \hat{u}(t)^T R w(t) + \right.$$
$$+ [\hat{s}^-(t)^T S^- - \varrho^-(t)^T] v^-(t) + [\hat{s}^+(t)^T S^+ - \varrho^+(t)^T] v^+(t) +$$
$$\left. + \varrho_{+1}^-(t)^T M_{+1}^- x^-(1, t) + \varrho_{-1}^+(t)^T M_{-1}^+ x^+(0, t) \right\} dt ,$$

where for instance ϱ_{+1}^- denotes ϱ_{i+1}^-, M_{+1}^- denotes M_{i+1}^-, etc. Observe that

$$(17) \qquad x(z, t_0; w, v) = x(z, t_0; u, s) - x(z, t_0; \hat{u}, \hat{s}) = 0 .$$

We shall show that if (13), (14), (17) are satisfied then (15) holds. Employing the adjoint equation, the first term of (16) can be transformed as follows

$$\int_{t_0}^{t_1} \int_0^1 \int_0^1 \hat{x}(z, t)^T Q(z, z') x(z', t) \, dt \, dz \, dz' =$$
$$= \int_{t_0}^{t_1} \int_0^1 \left[- \frac{\partial p}{\partial t} + \frac{\partial}{\partial z} (Ap) - A^T p \right]^T x \, dt \, dz =$$
$$= \int_0^1 [p(z, t_0)^T x(z, t_0) - p(z, t_1)^T x(z, t_1)] \, dz + \int_{t_0}^{t_1} [p(1, t)^T A(1) x(1, t) -$$
$$- p(0, t)^T A(0) x(0, t)] \, dt + \int_{t_0}^{t_1} \int_0^1 p^T \left(\frac{\partial x}{\partial t} - A \frac{\partial x}{\partial z} - Ax \right) dt \, dz ,$$

where integration by parts over t and z has been applied. The component $p(z, t_0)^T$ $x(z, t_0)$ vanishes by (17) and the last term in brackets is equal to $B(z) w(t)$ by (7). Partitioning as in (2), one can write the product $p(1, t)^T A(1) x(1, t)$ as

$$p^-(1, t)^T A^-(1) x^-(1, t) + p^+(1, t)^T A^+(1) v^+(t) ,$$

where the second boundary condition of (7) has been used. Similarly

$$p(0, t)^T A(0) x(0, t) = p^-(0, t)^T A^-(0) v^-(t) + p^+(0, t)^T A^+(0) x^+(0, t) .$$

Now

$$I(w, v) = - \int_0^1 p(z, t_1)^T x(z, t_1)\, \mathrm{d}z + \int_{t_0}^{t_1} \left\{ \left[\int_0^1 p(z, t)^T B(z)\, \mathrm{d}z + \hat{u}(t)^T R \right] w(t) + \right.$$
$$+ [\hat{s}^-(t)^T S^- - \varrho^-(t)^T - p^-(0, t)^T A^-(0)]\, v^-(t) +$$
$$+ [\hat{s}^+(t)^T S^+ - \varrho^+(t)^T + p^+(1, t)^T A^+(1)]\, v^+(t) +$$
$$+ [\varrho^-_{+1}(t)^T M^-_{+1} + p^-(1, t)^T A^-(1)]\, x^-(1, t) +$$
$$\left. + [\varrho^+_{-1}(t)^T M^+_{-1} - p^+(0, t)^T A^+(0)]\, x^+(0, t) \right\} \mathrm{d}t\,.$$

Hence the condition (15) holds by (13), and the terminal and boundary conditions of (14). Modification of the derivation for $i = 1$, $i = r$ is obvious. This completes the proof.

V. FEEDBACK REGULATORS FOR SUBSYSTEMS

To find feedback forms of $\hat{u}_i$, $\hat{s}_i^-$, $\hat{s}_i^+$ we shall seek a representation (see [8], [10])

$$p_i(z, t) = \int_0^1 P_i(z, z', t)\, \hat{x}_i(z', t)\, \mathrm{d}z' + \xi_i(z, t)\,, \tag{18}$$

where $n_i \times n_i$ function matrix P_i has the property

$$P_i(z, z', t) = P_i(z', z, t)^T\,. \tag{19}$$

According to (2), P_i can be partitioned as follows

$$P_i = \begin{bmatrix} P_i^- \\ P_i^+ \end{bmatrix} = [P_{i_-}, P_{i_+}]\,, \tag{20}$$

with P_i^- of dimension $n_i' \times n_i$, P_{i-} of dimension $n_i \times n_i'$, etc. We claim that

Proposition 2. The matrix $P_i = P_i(z, z', t)$ and vector $\xi_i = \xi_i(z, t)$ satisfy the following equations

$$\frac{\partial P_i}{\partial t} - \frac{\partial}{\partial z}(A_i(z) P_i) - \frac{\partial}{\partial z'}(P_i A_i(z')) + A_i(z)^T P_i + P_i A_i(z') - \tag{21}$$
$$- \int_0^1 P_i(z, z', t) B_i(z')\, \mathrm{d}z'\, R_i^{-1} \int_0^1 B_i(z)^T P_i(z, z', t)\, \mathrm{d}z -$$
$$- P_{i-}(z, 0, t)\, A_i^-(0)\, (S_i^-)^{-1} A_i^-(0)\, P_i^-(0, z', t) -$$
$$- P_{i+}(z, 1, t)\, A_i^+(1)\, (S_i^+)^{-1} A_i^+(1)\, P_i^+(1, z', t) = - Q_i(z, z')\,,$$

$$P_i(z, z', t_1) = 0\,, \quad i = 1, \dots, r\,,$$
$$P_{i+}(z, 0, t) = 0\,, \quad P_i^+(0, z', t) = 0\,,$$
$$P_{i-}(z, 1, t) = 0\,, \quad P_i^-(1, z', t) = 0$$

and

$$(22) \qquad \frac{\partial \xi_i}{\partial t} - \frac{\partial}{\partial z}(A_i(z)\,\xi_i) + A_i(z)^T \xi_i - P_{i-}(z, 0, t)\,A_i^-(0)\,(S_i^-)^{-1}\,[\varrho_i^-(t) + A_i^-(0)\,\xi_i^-(0, t)] + P_{i+}(z, 1, t)\,A_i^+(1)\,(S_i^+)^{-1}\,[\varrho_i^+(t) - A_i^+(1)\,\xi_i^+(1, t)] - \int_0^1 P_i(z, z', t)\,B_i(z')\,\mathrm{d}z'\,R_i^{-1}\int_0^1 B_i(z')^T\,\xi_i(z', t)\,\mathrm{d}z' = 0\,,$$

$$\xi_i(z, t_1) = 0\,, \quad i = 1, \dots, r\,,$$
$$\xi_i^+(0, t) = N_{i-1}^+\,\varrho_{i-1}^+(t)\,, \quad i = 2, \dots, r\,,$$
$$\xi_i^-(1, t) = -N_{i+1}^-\,\varrho_{i+1}^-(t)\,, \quad i = 1, \dots, r-1\,,$$
$$\xi_1^+(0, t) = 0\,, \quad \xi_r^-(1, t) = 0\,.$$

Remark. The last terms in the equations (21) (on the left) and (22) disappear at $i = r$ and the last but ones – at $i = 1$.

Proof. The subscript i will be omitted as before. Using the symbolic functions $\delta(z)$, $\delta(z-1)$ one can transform the equation (14) to the form

$$(23) \qquad \frac{\partial p}{\partial t} - \frac{\partial}{\partial z}(A(z)\,p) = -A(z)^T\,p - \int_0^1 Q(z, z')\,\hat{x}(z', t)\,\mathrm{d}z' - \delta(z)\begin{bmatrix} 0 \\ A^+(0) \end{bmatrix} N_{-1}^+\varrho_{-1}^+ - \delta(z-1)\begin{bmatrix} A^-(1) \\ 0 \end{bmatrix} N_{+1}^-\varrho_{+1}^-\,,$$

$$p(z, t_1) = 0\,,$$
$$p^+(0, t) = 0\,, \quad p^-(1, t) = 0\,.$$

To derive (21) and (22) we shall transform the difference

$$\Delta(p) = \frac{\partial p}{\partial t} - \frac{\partial}{\partial z}(Ap)$$

in two ways and then make one equation using those two equivalent representations. First

$$(24) \qquad \Delta(p) = \int_0^1 [-A(z)^T\,P(z, z', t) - Q(z, z')]\,\hat{x}(z', t)\,\mathrm{d}z' - A(z)^T\,\xi(z, t) - \delta(z)\begin{bmatrix} 0 \\ A^+(0) \end{bmatrix} N_{-1}^+\varrho_{-1}^+ - \delta(z-1)\begin{bmatrix} A^-(1) \\ 0 \end{bmatrix} N_{+1}^-\varrho_{+1}^-$$

by (18) and (23). On the other hand, applying (18) directly in $\Delta(p)$, one obtains

$$\Delta(p) = \int_0^1 \left\{ \frac{\partial P(z, z', t)}{\partial t} \hat{x}(z', t) + P(z, z', t) \frac{\partial \hat{x}(z', t)}{\partial t} - \right. \tag{25}$$

$$\left. - \frac{\partial}{\partial z} (A(z) P(z, z', t)) \hat{x}(z', t) \right\} dz' + \frac{\partial \xi(z, t)}{\partial t} - \frac{\partial}{\partial z} (A(z) \xi(z, t)) .$$

Employing (7) in the second term above and using integration by parts over z' yields

$$\int_0^1 P(z, z', t) \frac{\partial \hat{x}(z', t)}{\partial t} dz' = P(z, 1, t) A(1) \hat{x}(1, t) - P(z, 0, t) A(0) \hat{x}(0, t) + \tag{26}$$

$$+ \int_0^1 \left\{ - \frac{\partial}{\partial z'} (P(z, z', t) A(z')) + P(z, z', t) A(z') \right\} \hat{x}(z', t) dz' -$$

$$- \int_0^1 P(z, z', t) B(z') R^{-1} \int_0^1 B(z'')^T \left[\int_0^1 P(z'', z''', t) \hat{x}(z''', t) dz''' + \xi(z'', t) \right] dz' dz'' ,$$

where

$$P(z, 1, t) A(1) \hat{x}(1, t) = P_-(z, 1, t) A^-(1) \hat{x}^-(1, t) + P_+(z, 1, t) . \tag{27a}$$

$$. A^+(1) \left[T_1^+ \varrho^+(t) - T_2^+ \int_0^1 P^+(1, z', t) \hat{x}(z', t) dz' - T_2^+ \xi^+(1, t) \right] ,$$

$$P(z, 0, t) A(0) \hat{x}(0, t) = P_-(z, 0, t) A^-(0) . \tag{27b}$$

$$. \left[T_1^- \varrho^-(t) + T_2^- \xi^-(0, t) + T_2^- \int_0^1 P^-(0, z', t) \hat{x}(z', t) dz' \right] +$$

$$+ P_+(z, 0, t) A^+(0) \hat{x}^+(0, t) .$$

The first part of the last term in (26) can be written as

$$\int_0^1 \left[\int_0^1 P(z, z', t) B(z') dz' R^{-1} \int_0^1 B(z)^T P(z, z', t) dz \right] \hat{x}(z', t) dz'$$

after changing variables. Notice that $P_-(z, 1, t)$ in (27a) and $P_+(z, 0, t)$ in (27b) are equal to zero by the boundary conditions of (21). Now we can construct one equation basing on (24) and transformed (25). Collecting in that equation the terms involving integrals over z' (and $\hat{x}(z', t)$) yields the Riccati equation (21) that is clearly symmetric in the sense (19). This what remains is

$$\frac{\partial \xi}{\partial t} - \frac{\partial}{\partial z} (A(z) \xi) + A(z)^T \xi + P_+(z, 1, t) A^+(1) [T_1^+ \varrho^+(t) - T_2^+ \xi^+(1, t)] - \tag{28}$$

$$- P_-(z, 0, t)\, \Lambda^-(0)\, [T_1^- \varrho^-(t) + T_2^- \xi^-(0, t)] - \int_0^1 P(z, z', t)\, B(z')\, \mathrm{d}z'\, R^{-1}\,.$$

$$. \int_0^1 B(z')^T \xi(z', t)\, \mathrm{d}z' +$$

$$+ \delta(z) \begin{bmatrix} 0 \\ \Lambda^+(0) \end{bmatrix} N_{-1}^{+} \varrho_{-1}^{+}(t) + \delta(z-1) \begin{bmatrix} \Lambda^-(1) \\ 0 \end{bmatrix} N_{+1}^{-} \varrho_{+1}^{-}(t) = 0\,.$$

Assume that

$$\xi^+(0, t) = 0 \quad \text{and} \quad \xi^-(1, t) = 0\,.$$

Moving the last two terms from (28) into the above conditions and replacing T_1 by S^{-1} as in (12) one obtains (22). The other boundary conditions of (21) follow $P_-(z, 1, t)$, $P_+(z, 0, t)$ by symmetry. From $p(z, t_1) = 0$ one obtains $P(z, z', t_1) = 0$ and $\xi(z, t_1) = 0$. The proof is complete.

Using (18), the optimal control $\hat{u}_i(t)$ (see (13)) can be written in the feedback form

$$\hat{u}_i(t) = - \int_0^1 K_i(z, t)\, \hat{x}_i(z, t)\, \mathrm{d}z + \zeta_i(t)\,, \tag{29}$$

where

$$K_i(z, t) = R_i^{-1} \int_0^1 B_i(z')^T P_i(z', z, t)\, \mathrm{d}z'\,, \tag{30}$$

$$\zeta_i(t) = -R_i^{-1} \int_0^1 B_i(z)^T \xi_i(z, t)\, \mathrm{d}z\,. \tag{31}$$

Observe that the Riccati equations (21) can be solved off-line independently for each subsystem. So we have obtained a series of optimal regulators for subsystems, however uncoordinated yet.

VI. COORDINATION OF THE CONTRCLLERS

As shown by Pearson [9], the task of the coordinator is to perform maximization of the overall Lagrange functional $\bar{J}$, given by (8), with respect to the coordination variables ϱ_i, $i = 1, \ldots, r$. Gradient of $\bar{J}$ results from the difference between the optimal interconnection variables $\hat{s}_i$ and the actually occurring couplings, i.e.

$$\operatorname*{grad}_{\varrho_i^-} \bar{J} = M_i^- \hat{x}_{i-1}^-(1, t) - \hat{s}_i^-(t)\,, \quad i = 2, \ldots, r\,,$$

$$\operatorname*{grad}_{\varrho_i^+} \bar{J} = M_i^+ \hat{x}_{i+1}^+(0, t) - \hat{s}_i^+(t)\,, \quad i = 1, \ldots, r-1\,,$$

where

$$\hat{s}_i^-(t) = T_{i1}^- \varrho_i^-(t) + T_{i2}^- \left[\int_0^1 P_i^-(0, z', t)\, \hat{x}_i(z', t)\, \mathrm{d}z' + \xi_i^-(0, t)\right], \tag{32}$$

$$\hat{s}_i^+(t) = T_{i1}^+ \varrho_i^+(t) - T_{i2}^+ \left[\int_0^1 P_i^+(1, z', t)\, \hat{x}_i(z', t)\, \mathrm{d}z' + \xi_i^+(1, t)\right]$$

by (13) and (18). Therefore, the steepest descent algorithm may be used as a strategy for the coordinator

$$\varrho_i^-(t)^{(k+1)} = \varrho_i^-(t)^{(k)} + k_c[M_i^- \hat{x}_{i-1}^-(1, t) - \hat{s}_i^-(t)]^{(k)}\,, \quad i = 2, \ldots, r\,, \tag{33}$$

$$\varrho_i^+(t)^{(k+1)} = \varrho_i^+(t)^{(k)} + k_c[M_i^+ \hat{x}_{i+1}^+(0, t) - \hat{s}_i^+(t)]^{(k)}\,, \quad i = 1, \ldots, r - 1\,,$$

where k denotes the iteration index at the coordination level and k_c is the iteration constant.

So we have obtained

Conclusion. A decentralized optimal feedback control of the overall linear first order distributed system (1), (3), composed of r subsystems coupled together in a cascade, can be performed applying the local feedback controls $\hat{u}_i$ given by (29), $i = 1, \ldots, r$, that should be coordinated using the variables ϱ_i adjusted according to the gradient algorithm (33) at the second level.

To solve the Riccati equations (21) one recommends transformation to a Chandrasekhar-type representation as in [11] and the method of characteristics to solve resulting equations, or polynomial approximation applying the orthogonal collocation technique [12].

VII. FINAL REMARKS

The overall distributed system has been of the assumed "pure" cascade structure. However, more complicated tubular reactors quite often include parallel branches passing by certain sections. An example of such structure is shown in Fig. 3 (controls have been omitted). In such cases one of boundary conditions in (3), suppose the last one, should be replaced by

$$x_i^+(1, t) = \sum_{j=i}^{r-1} M_{i,j+1}^+\, x_{j+1}^+(0, t)\,, \quad i = 1, \ldots, r - 1\,.$$

Taking the interconnection variable s_i^+

$$s_i^+(t) = \sum_{j=i}^{r-1} M_{i,j+1}^+\, x_{j+1}^+(0, t)$$

and a coordination one ϱ_i^+, we can construct a Lagrange functional $\bar{J}$. Its last term is (compare (8))

$$\sum_{i=1}^{r-1} \int_{t_0}^{t_1} \varrho_i^+(t)^T \left[\sum_{j=i}^{r-1} M_{i,j+1}^+ x_{j+1}^+(0, t) - s_i^+(t) \right] \mathrm{d}t$$

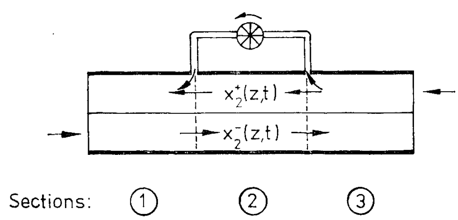

Fig. 3. Tubular reactor with a parallel branch.

Hence i-th component J_i of $\bar{J}$ has the term

$$\int_{t_0}^{t_1} \{ [\sum_{j=2}^{i} \varrho_{j-1}^+(t)^T M_{j-1,i}^+] x_i^+(0, t) - \varrho_i^+(t)^T s_i^+(t) \} \mathrm{d}t$$

that depends on x_i^+ and s_i^+ only. So again J_i can be minimized independently for each subsystem and the theory presented can be applied.

(Received February 11, 1978.)

REFERENCES

[1] H. Schwarz, H. D. Wend: New results on the two-level decomposition and coordination of the linear regulator problem. Proc. of VI-th IFAC Congr., Boston 1975, p. 25.3.

[2] M. G. Singh, A. Titli: Hierarchical feedback control for large dynamical systems. Int. J. Systems Sci. *8* (1977), 1, 31—49.

[3] D. A. Wismer: Distributed multilevel systems, In: Optimization Methods for Large Scale Systems. McGraw-Hill, New York 1971.

[4] B. Pradin, A. Titli: Methods of decomposition coordination for the optimization of interconnected distributed parameter systems. Proc. of VI-th IFAC Congr., Boston 1975, p. 15.3.

[5] J. C. Lord, K. S. P. Kumar: A multilevel distributed filter concept for computerized traffic control applications. Proc. of IEEE Conf. Dec. Contr., Clearwater, Florida, November 1976.

[6] L. Trybus: Pearson-type decentralized filter for a first-order distributed systems of sectional structure. Systems Science (submitted).

[7] R. Courant, D. Hilbert: Methods of Mathematical Physics, Part II: Partial Differential Equations. Interscience, New York 1962.

[8] D. L. Russell: Quadratic performance criteria in boundary control of linear symmetric hyperbolic systems. SIAM J. Contr. *11* (1973), 3, 475—509.

[9] J. D. Pearson: Dynamic decomposition techniques. In: Optimization Methods for Large Scale Systems. McGraw-Hill, New York 1971.

[10] J. L. Lions: Contrôle optimal de systèmes gouvernés par des équations aux derivées partielles. Russian Edition: Mir, Moscow 1972.

[11] J. Casti, L. Ljung: Some new analytic and computational results for operator Riccati equations. SIAM J. Contr. *13* (1975), 4, 817—826.

[12] B. A. Finlayson: The Method of Weighted Residuals and Variational Principles. Academic Press, New York 1972.

Dr Leszek Trybus, Ignacy Łukasiewicz Technical University, 35—959 Rzeszów, ul. W. Pola 2. Poland.