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Influence of Resolution Power of an Instrument on Estimation of Basic Material Spatial Structure Parameters

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Two basic material spatial structure parameters (the mean number of particles per unit test volume and the mean value of the size distribution of particles) are expressed as functions not only of planar structure parameters but also of resolution power of used instrument, for example optical, electron or X-ray microscope. Various situations arising by using areal and lineal metallographic analysis are considered. An exact solution is given for Rayleigh's distribution of diameters of spherical particles randomly spaced in the studied material specimen.

1. INTRODUCTION

A very important property of each instrument for establishing the morphology of evaluated object is its resolution power. According to Abbe's theory, it is defined as the smallest distance of two points of investigated object which may be clearly recognized one from another using the given instrument. For example in optical microscope we can observe particles the size of which is equal to at least 0·35 times wawelength of light used for lighting studied object. The resolution power of X-ray microscopes is approximately the same. Electron microscopes for usual practice reach a resolution power about 3 nm and the most perfect types of these microscopes approach the theoretical value 0·3 nm [9]. With regard to the resolution power of the human eye (about 0·3 mm), it follows an effective magnification for the optic microscope 1500, for usual electron microscope 100 000 and for the most perfect electron microscope 1 000 000.

Comprehensibly these facts should be taken into account in processing of measurements obtained by metallographic analysis of the polished plane using optic and electronic microscopy and in comparing results gained at different conditions. Otherwise in the first case the calculated material planar structure parameters are biased and a consequence of it is the bias of material spatial structure parameters calculated from them and in the second case we compare the populations of measurements that are in a different way deprived of the smallest particles.

The aim of this contribution is to show the deformation of the distribution of an investigated characteristic of spherical particles (e.g. the diameter of planar section of a particle; the chord length intercepted on circular particle section by a line applied randomly to the microstructure in the polished plane) called just by the inability of the instrument and the human eye to record the particles of small sizes and to derive the expressions for estimating two basic material spatial structure parameters (the mean number of particles per unit test volume and the mean value of the size distribution of particles). Hitherto published solutions of these problems (see e.g. [1] to [5]) are always based on the assumption of absolute resolution power of the instrument.

The contribution is divided into five chapters. After mathematical formulation of problems in Chap. 2, the influence of resolution power in areal and lineal metallographic analysis is studied in Chap. 3 and 4 respectively. The derived results are applied on Rayleigh's distribution of spherical particle diameter in Chap. 5.

2. FORMULATION OF PROBLEMS

In model constructing we shall consider the three-dimensional euclidean space E_3, we shall locate randomly and mutually indenpendently the points $S_1, S_2, \ldots$ in it and round each point S_j $(j = 1, 2, \ldots)$ we shall circumscribe a sphere r_j with radius $0{\cdot}5\xi_j$. We shall assume that the positive random variables ξ_j are mutually independent and identically distributed with known continuous distribution function $F(x)$ and probability density $f(x)$, depeding on k_0 unknown parameters $\theta_1, \ldots, \theta_{k_0}$. The ν-th moment of random variable ξ_j we shall denote α_ν e.g.

$$\alpha_\nu = \int_0^\infty x^\nu f(x)\,\mathrm{d}x \tag{2.1}$$

and we shall assume that α_3 is finite [2]. These moments together with the parameters $\theta_1, \ldots, \theta_{k_0}$ and the mean number N_V of sphere centres per unit test volume we shall call spatial structure parameters.

Let a, b, c denote the coordinate axes in E_3. Now let us consider two planes V_a and V_b defined by equations $a = \text{const.}$ and $b = \text{const.}$ respectively.

Let $R_x \subset E_3$ be the union of all spheres r_j, that means

$$R_x = \bigcup r_j\,; \tag{2.2}$$

then

$$R_y = R_x \cap V_a \tag{2.3}$$

is the set of circular planar sections of spheres in the plane V_a. The diameters $\eta_1, \eta_2, \ldots$ of these sections are positive random variables which are mutually independent and identically distributed with distribution function $G(y)$ and probability density $g(y)$. It may be shown that for absolute resolution power ($d_* = 0$) the functions $g(y)$ and $G(y)$ are of the form (see e.g. [1] to [3])

$$g(y) = \frac{y}{\alpha_1} \int_y^\infty \frac{1}{\sqrt{(x^2 - y^2)}} f(x)\,\mathrm{d}x \tag{2.4}$$

and

$$G(y) = 1 - \frac{1}{\alpha_1} \int_y^\infty \left[\int_t^\infty \frac{t}{\sqrt{(x^2 - t^2)}} f(x)\,\mathrm{d}x \right] \mathrm{d}t = \tag{2.5}$$

$$= 1 - \frac{1}{\alpha_1} \int_y^\infty \sqrt{(x^2 - y^2)}\, f(x)\,\mathrm{d}x\,. \tag{2.5b}$$

Let $\bar{c} = V_a \cap V_b$. Then

$$R_p = R_y \cap V_b \tag{2.6}$$

is the set of chords that the line $\bar{c}$ intercepts on circular sections in the plane V_a. The chord lengths $\psi_1, \psi_2, \ldots$ are positive random variables mutually independent and identically distributed with distribution function $K(p)$ and probability density $k(p)$, depending on $g(y)$ and therefore on $f(x)$ too. It may be shown that for absolute resolution power the functions $k(p)$ and $K(p)$ are of the form (see e.g. [2] and [3])

$$k(p) = \frac{p}{\beta_1} \int_p^\infty \frac{1}{\sqrt{(y^2 - p^2)}}\, g(y)\,\mathrm{d}y \tag{2.7a}$$

$$= \frac{p}{\alpha_1 \beta_1} \iint\limits_{\substack{p<y<\infty \\ y<x<\infty}} \frac{y}{\sqrt{[(y^2 - p^2)(x^2 - y^2)]}}\, f(x)\,\mathrm{d}x\,\mathrm{d}y \tag{2.7b}$$

$$= \frac{2p}{\alpha_2} [1 - F(p)] \tag{2.7c}$$

and

$$K(p) = \frac{1}{\alpha_2} \left\{ \int_0^p x^2 f(x)\,\mathrm{d}x - p^2[1 - F(p)] \right\}, \tag{2.8}$$

where $F(p)$ is the distribution function of the random variable ξ_j

$$F(p) = P(\xi_j \leqq p) = \int_0^p f(x)\,\mathrm{d}x \tag{2.9}$$

and β_ν is the ν-th moment of the random variable η_j

$$\beta_\nu = \int_0^\infty y^\nu \, g(y) \, \mathrm{d}y \,. \tag{2.10}$$

Let γ_ν denote the ν-th moment of the random variable ψ_j

$$\gamma_\nu = \int_0^\infty p^\nu \, k(p) \, \mathrm{d}p \,. \tag{2.11}$$

The moments β_ν and γ_ν together with parameters N_A and N_L are called the planar structure parameters, N_A being the mean number of circular sections per unit test area of the investigated plane V_a and N_L the mean number of considered chords per unit length of test line $\bar{c}$.

The definition of moments α_ν, β_ν and γ_ν will be extended for $\nu < 0$ too.

Due to the fact that the spatial structure is not directly observable, the parameters of this structure must be expressed as functions of planar structure parameters, the size of which we can estimate using processing of measurement results obtained on polished plane. We shall give our attention to two basic spatial structure parameters: N_V and α_1. It is known (see e.g. [2] to [4]) that for absolute resolution power

$$N_V = \frac{N_A}{\alpha_1} \tag{2.12}$$

holds. The first moment α_1 is, of course, spatial structure parameter too; however, using the relation (see e.g. [2] and [5])

$$\alpha_1 = \frac{\pi}{2\beta_{-1}} \,, \tag{2.13}$$

we get N_V also as function of only planar structure parameters in the form

$$N_V = \frac{2N_A}{\pi} \beta_{-1} \,. \tag{2.14}$$

Further we know that

$$N_A = \frac{N_L}{\beta_1} \tag{2.15}$$

and analogically to (2.13)

$$\beta_1 = \frac{\pi}{2\gamma_{-1}} \tag{2.16}$$

holds, so that with regard to (2.12) we have

$$N_V = \frac{N_L}{\alpha_1 \beta_1} \,; \tag{2.17}$$

inserting (2.13) and (2.16) in (2.17), we obtain

$$N_V = \frac{2N_L}{\pi} \frac{\beta_{-1}}{\beta_1} \tag{2.18}$$

and

$$N_V = \frac{4N_L}{\pi^2} \beta_{-1} \cdot \gamma_{-1} \tag{2.19}$$

respectively.

From relations (2.12) to (2.19) we can see that we are able the basic spatial structure parameters express as functions of N_A and β_v, the size of which we can estimate from areal analysis, but we are unable to express them as functions only of parameters N_L and γ_v, the size of which we can estimate from lineal analysis using the readings from automatic image analyzer e.g. Quantimet of older type or Scandig.

All hitherto introduced relations are valid for absolute resolution power. Now we shall investigate the changes arising in case of fixed resolution point that we denote $d_*(d_* > 0)$. Solving further problems, we can always assume, without detriment to generality, that

$$d_* < \gamma_1 \tag{2.20}$$

and consequently also

$$d_* < \beta_1 \,. \tag{2.21}$$

The influence of resolution power of an instrument in application of areal analysis is given in Chap. 3 and of lineal analysis in Chap. 4.

3. INFLUENCE OF RESOLUTION POWER IN AREAL ANALYSIS

In application of areal analysis only those diameters of particle sections are registered, whose diameter $\eta_j > d_*$. Therefore, the corresponding probability density $g^*(y)$ has the form

$$g^*(y) = \frac{g(y)}{\int_{d_*}^{\infty} g(v)\,\mathrm{d}v} = \frac{g(y)}{1 - G(d_*)} \quad \text{for} \quad y > d_*\,, \qquad = 0 \quad \text{for} \quad 0 < y \leqq d_*\,, \tag{3.1}$$

where $g(y)$ and $G(y)$ are given in (2.4) and (2.5) respectively; analogically to (2.10), the v-th moment β_v^* of the distribution $g^*(y)$ will be

$$\beta_v^* = \int_{d_*}^{\infty} y^v\, g^*(y)\,\mathrm{d}y = \frac{1}{1 - G(d_*)} \int_{d_*}^{\infty} y^v\, g(y)\,\mathrm{d}y\,. \tag{3.2}$$

 Now we shall prove the validity of these two inequalities

(3.3a) $$\beta_\nu < \beta_\nu^* \quad \text{for} \quad \nu > 0\,,$$

(3.3b) $$\beta_\nu \geqq \beta_\nu^* \quad \text{for} \quad \nu < 0$$

(for $\nu = 0$ is, of course, $\beta_0 = \beta_0^* = 1$).

Proving (3.3a), we shall go out from the inequality

(3.4) $$\int_0^{d_*} y^\nu g(y)\,\mathrm{d}y < d_*^\nu \int_0^{d_*} g(y)\,\mathrm{d}y\,.$$

With the aid of (15.4.5) in [6]

(3.5) $$\beta_\nu^{2\nu} \leqq \beta_{\nu-1}^\nu \beta_{\nu+1}^\nu\,, \quad \nu > 0\,,$$

we can prove, with regard to (2.1), that

(3.6) $$d_*^\nu < \beta_1^\nu \leqq \beta_\nu\,,$$

so that we may rewrite (3.4) as follows

$$\int_0^{d_*} y^\nu g(y)\,\mathrm{d}y - \int_0^\infty y^\nu g(y)\,\mathrm{d}y \int_0^{d^*} g(y)\,\mathrm{d}y < 0$$

and that, after adding the integral $\int_0^\infty y^\nu g(y)\,\mathrm{d}y$ and arrangement, can be transformed to the inequality

(3.7) $$[1 - G(d_*)] \int_0^\infty y^\nu g(y)\,\mathrm{d}y < \int_{d_*}^\infty y^\nu g(y)\,\mathrm{d}y$$

which, however, is only another form of inequality (3.3a).

Proving the inequality (3.3b), we shall use gradual inequalities

(3.8) $$\int_{d_*}^\infty g(y)\,\mathrm{d}y \left[\int_0^{d_*} y^{-\nu} g(y)\,\mathrm{d}y\right] \geqq \int_{d_*}^\infty g(y)\,\mathrm{d}y \left[d_*^{-\nu} \int_0^{d_*} g(y)\,\mathrm{d}y\right] \geqq$$
$$\geqq \int_0^{d_*} g(y)\,\mathrm{d}y \int_{d_*}^\infty y^{-\nu} g(y)\,\mathrm{d}y$$

or

(3.9) $$\int_{d_*}^\infty g(y)\,\mathrm{d}y \left[\int_0^{d_*} y^{-\nu} g(y)\,\mathrm{d}y\right] - \left[1 - \int_{d_*}^\infty g(y)\,\mathrm{d}y\right] \int_{d_*}^\infty y^{-\nu} g(y)\,\mathrm{d}y \geqq 0$$

and hence

(3.10) $$\int_{d_*}^\infty g(y)\,\mathrm{d}y \int_0^\infty y^{-\nu} g(y)\,\mathrm{d}y - \int_{d_*}^\infty y^{-\nu} g(y)\,\mathrm{d}y \geqq 0$$

which, however, is only another form of inequality (3.3b). To be able to take up the influence of resolution point d_* ($d_* > 0$) on the change of spatial structure parameters N_V and α_1, we need, except for the proof of inequality (3.3b), to find the limits of moments β_{-1} and β^*_{-1} using the variable d_*. Firstly it holds

$$\beta^*_{-1} = \frac{\int_{d_*}^{\infty} y^{-1}\, g(y)\, \mathrm{d}y}{\int_{d_*}^{\infty} g(y)\, \mathrm{d}y} \leqq \frac{\frac{1}{d_*}\int_{d_*}^{\infty} g(y)\, \mathrm{d}y}{\int_{d_*}^{\infty} g(y)\, \mathrm{d}y} = \frac{1}{d_*}. \tag{3.11}$$

Secondly we have

$$\beta_{-1}^{-1} = \left[\int_0^{\infty} y^{-1}\, g(y)\, \mathrm{d}y\right]^{-1} = \left[\int_0^{d_*} y^{-1}\, g(y)\, \mathrm{d}y + \int_{d_*}^{\infty} y^{-1}\, g(y)\, \mathrm{d}y\right]^{-1} \leqq \tag{3.12}$$

$$\leqq d_*\left[d_*\int_0^{d_*} y^{-1}\, g(y)\, \mathrm{d}y + \int_{d_*}^{\infty} g(y)\, \mathrm{d}y\right]^{-1} \leqq d_*,$$

being

$$d_*\int_0^{d_*} y^{-1}\, g(y)\, \mathrm{d}y + \int_{d_*}^{\infty} g(y)\, \mathrm{d}y \geqq 1, \tag{3.13}$$

while

$$\int_0^{d_*} y^{-1}\, g(y)\, \mathrm{d}y \geqq \frac{1}{d_*}\int_0^{d_*} g(y)\, \mathrm{d}y = \frac{1}{d_*}\left[1 - \int_{d_*}^{\infty} g(y)\, \mathrm{d}y\right]. \tag{3.14}$$

From (3.3b), (3.11) and (3.14) it follows that

$$\beta^*_{-1} \leqq \beta_{-1}. \tag{3.15}$$

In areal analysis we are able to register only those particles, whose circular sections have the diameter $\eta_j > d_*$ in the polished plane. Therefore, between the observable mean number N_A^* of such sections and real mean number N_A of these sections per unit test area, the inequality

$$N_A^* = N_A \int_{d_*}^{\infty} g(y)\, \mathrm{d}y < N_A \tag{3.16}$$

must always hold. Therefore, in practice the equations (2.13) and (2.14) have in fact the form

$$\alpha_1^* = \frac{\pi}{2\beta^*_{-1}} \tag{3.17}$$

and

$$N_V^* = \frac{2N_A^*}{\pi} \beta_{-1}^* \tag{3.18}$$

respectively. In view of (3.15) and (3.16), we must state that

$$\alpha_1^* \geqq \frac{\pi}{2} d_* \geqq \frac{\pi}{2\beta_{-1}} = \alpha_1 \tag{3.19}$$

and

$$N_V^* \leqq \frac{2}{\pi} \frac{N_A^*}{d_*} \leqq \frac{N_A}{\beta_{-1}} = N_V \,. \tag{3.20}$$

For real shapes of probability densities $f(x)$ we can make this conclusion: the application of the relations (2.12) and (2.13), derived for absolute resolution power, in the case of reduced resolution power leads in areal analysis to:

a) overevaluating real mean value α_1 of the distribution of sphere diameter,

b) underevaluating real mean number N_V of sphere centres per unit test volume.

4. INFLUENCE OF RESOLUTION POWER IN LINEAL ANALYSIS

In application of lineal analysis we shall consider two cases, which may occur owing to the existence of resolution point d_*:

a) there are recorded the chords only on particles whose circular section in the polished plane has diameter $\eta_j \geqq d_*$;

b) there are recorded only those chords whose length $\psi_j > d_*$.

The first case occurs owing to the inability of used instrument to recognize the particles with diameter smaller than d_*. The second case is typical for automatic image analysers based on scanning; the distance between two neighbouring lines gives just the resolution power.

From the relations (2.12) to (2.19) we can see that the lineal analysis enables us to get only the estimate of β_1. Lineal analysis together with areal analysis make possible to obtain the estimates of α_1 and N_V. All these relations are functions of the moment γ_{-1}. Let us investigate the influence of resolution point on the size of this moment in both cases under consideration.

In case a) we locate the line $\bar{c}$ on the polished plane V_a, covered by circular sections the diameter of which has probability density $g^*(y)$, given in (3.1). Let us suggest that for the probability density $g(y)$, having the form (2.4), the corresponding ν-th moment γ_ν of the chord length distribution $k(p)$, given in (2.7a), has the form

$$\gamma_\nu = \frac{\Gamma(\frac{1}{2})\,\Gamma\left(\frac{\nu+2}{2}\right)}{2\Gamma\left(\frac{\nu+3}{2}\right)}\,\frac{\beta_{\nu+1}}{\beta_1} \quad \text{for} \quad \nu = -1, 0, 1, \ldots, \tag{4.1}$$

where $\Gamma(n)$ is gamma function and β_ν is given by (2.10). In the case a) the probability density $k(p)$ passes on the form

$$k^*(p) = \frac{p}{\beta_1^*}\int_p^\infty \frac{g^*(y)}{\sqrt{(y^2 - p^2)}}\,\mathrm{d}y \tag{4.2}$$

and the relation (4.1) on the form

$$\gamma_\nu^* = \frac{\Gamma(\frac{1}{2})\,\Gamma\left(\frac{\nu+2}{2}\right)}{2\Gamma\left(\frac{\nu+3}{2}\right)}\,\frac{\beta_{\nu+1}^*}{\beta_1^*} \quad \text{for} \quad \nu = -1, 0, 1, \ldots, \tag{4.3}$$

where β_ν^* is given in (3.2). For $\nu = -1$, we get

$$\gamma_{-1} = \frac{\pi}{2\beta_1} \tag{4.4}$$

and

$$\gamma_{-1}^* = \frac{\pi}{2\beta_1^*}\,. \tag{4.5}$$

In view of (3.3a), we have

$$\gamma_{-1}^* < \gamma_{-1}\,, \tag{4.6}$$

therefore, the moment γ_{-1} is again underevaluated and, due to this fact, the same property has the estimate of the spatial parameter N_V, defined by (2.19). From (3.3b) it namely follows that

$$\beta_{-1} \geqq \beta_{-1}^* \tag{4.7}$$

and besides that, in accordance with (2.15), we have

$$N_L^* = \beta_1^* N_A^* < \beta_1^* N_A \int_{d_*}^\infty g(y)\,\mathrm{d}y = N_A \int_{d_*}^\infty y\,g(y)\,\mathrm{d}y < N_A\beta_1 = N_L\,. \tag{4.8}$$

Therefore, we get

$$N_V^* = \frac{4N_L^*}{\pi^2}\,\beta_{-1}^*\gamma_{-1}^* < \frac{4N_L}{\pi^2}\,\beta_{-1}\gamma_{-1} = N_V\,. \tag{4.9}$$

Now let N^*_{VL} denote the mean number of sphere centres per unit test volume gained using lineal analysis (the case a)) and N^*_{VA} that gained using areal analysis and expressed by (3.18). We see that

$$N^*_{VL} = N^*_{VA} \tag{4.10}$$

holds.

In the case b) we locate the line $\bar{c}$ on the polished plane V_a, covered by circular sections whose diameters have probability density $g^*(y)$, given in (3.1); but, in comparison with the case a), we are able to record from the originated chords only those, whose length $\psi_j > d_*$. The corresponding probability density $k^{**}(p)$ of detectable chord length has the form

$$k^{**}(p) = \frac{k^*(p)}{\int_{d_*}^{\infty} k^*(p)\,\mathrm{d}p}, \tag{4.11}$$

where $k^*(p)$ is given in (4.2) and the ν-th moment γ^{**}_ν of this distribution is given by

$$\gamma^{**}_\nu = \int_{d_*}^{\infty} p^\nu\, k^{**}(p)\,\mathrm{d}p = \frac{1}{\int_{d_*}^{\infty} k^*(p)\,\mathrm{d}p} \int_{d_*}^{\infty} p^\nu\, k^*(p)\,\mathrm{d}p\,. \tag{4.12}$$

By similar way as in the case of moments β_ν and β^*_ν for $\nu < 0$, we can prove that

$$\gamma^*_{-1} \geqq \gamma^{**}_{-1}\,. \tag{4.13}$$

With respect to (4.6), this inequality can be extended in this way

$$\gamma_{-1} > \gamma^*_{-1} \geqq \gamma^{**}_{-1}\,. \tag{4.14}$$

The application of the procedure b) in (2.19) leads to an expression that we shall denote N^{**}_{VL} and for which

$$N^{**}_{VL} = \frac{4N^*_L}{\pi^2}\,\beta^*_{-1}\gamma^{**}_{-1} < \frac{4N^*_L}{\pi^2}\,\beta^*_{-1}\gamma^*_{-1} = N^*_{VL} \tag{4.15}$$

holds. From (3.20), (4.10) and (4.15) it follows that

$$N_V \geqq N^*_{VA} = N^*_{VL} > N^{**}_{VL}\,. \tag{4.16}$$

Now we can conclude that the procedure b) gives a more expressive underevaluating spatial parameter N_V than the procedure a); with regard to (3.15) and (4.13), we gain

$$N^{**}_{VL} < \frac{4N^*_L}{\pi^2}\,\frac{1}{d^2_*}\,. \tag{4.17}$$

5. DETERMINING CORRECTION FACTOR SIZE UNDER THE ASSUMPTION OF RAYLEIGH'S DISTRIBUTION OF SPHERICAL PARTICLE DIAMETERS

In two foregoing chapters our attention was concentrated only on demonstration of the resolution power influence on the size of spatial structure parameters in the case when these parameters are calculated accordng to the expressions, derived for ab olute resolution power. In this chapter, we shall introduce the expressions for relevant correction factors under the assumption of Rayleigh's distribution of spherical particle diameter

(5.1) $$f(x) = \frac{x}{\mu}\, e^{-x^2/2\mu}\,, \quad x > 0\,, \quad \mu > 0\,,$$

where μ is a parameter of the distribution. It may be proved [1] that the corresponding distribution of circular section diameters is also of the same type, that means, it holds

(5.2) $$g(y) = \frac{y}{\alpha_1} \int_y^\infty \frac{f(x)}{\sqrt{(x^2 - y^2)}}\, dx = \frac{y}{\mu}\, e^{-y^2/2\mu}\,,$$

$(y > 0;\ \mu > 0)$, where

(5.3) $$\alpha_1 = \int_0^\infty x\, f(x)\, dx = \sqrt{\frac{\pi\mu}{2}}\,.$$

A consequence of this property is the equality

(5.4) $$\beta_1 = \alpha_1\,.$$

In addition to this, we have

(5.5) $$\beta_{-1} = \int_0^\infty \frac{1}{\mu}\, e^{-y^2/2\mu}\, dy = \sqrt{\frac{\pi}{2\mu}}\,.$$

In areal analysis with point resolution d_* the equation (5.2) becomes

(5.6) $$g^*(y) = \frac{g(y)}{\int_{d_*}^\infty g(v)\, dv} = \frac{y}{\mu}\, e^{(d_*^2 - y^2)/2\mu} \quad \text{for} \quad y > d_*\,,$$

$$= 0 \quad \text{for} \quad 0 < y \leqq d_*\,.$$

For the ν-th moment β_ν^* we get

(5.7) $$\beta_\nu^* = \int_{d_*}^\infty y^\nu\, g^*(y)\, dy = \frac{1}{\mu}\, e^{d_*^2/2\mu} \int_{d_*}^\infty y^{\nu+1} e^{-y^2/2\mu}\, dy\,;$$

 the substitution

$$\frac{y^2}{2\mu} = z \tag{5.8}$$

gives

$$\beta_\nu^* = e^{d_*^2/2\mu}(2\mu)^{\nu/2} \int_{d_*^2/2\mu}^{\infty} z^{\nu/2} e^{-z}\, dz = e^{d_*^2/2\mu}(2\mu)^{\nu/2}\, \Gamma\left(1 + \frac{\nu}{2};\ \frac{d_*^2}{2\mu}\right), \tag{5.9}$$

where

$$\Gamma(n, x) = \int_x^{\infty} t^{n-1} e^{-t}\, dt \tag{5.10}$$

is the incomplete gamma function. For numerical calculations, it is useful to express the incomplete gamma function as a function of a complement $Q(\chi^2/k)$ of the χ^2-distribution function which is tabulated. It holds, namely,

$$\Gamma(n, x) = \Gamma(n)\, Q(\chi^2 = 2x \mid k = 2n)\,, \tag{5.11}$$

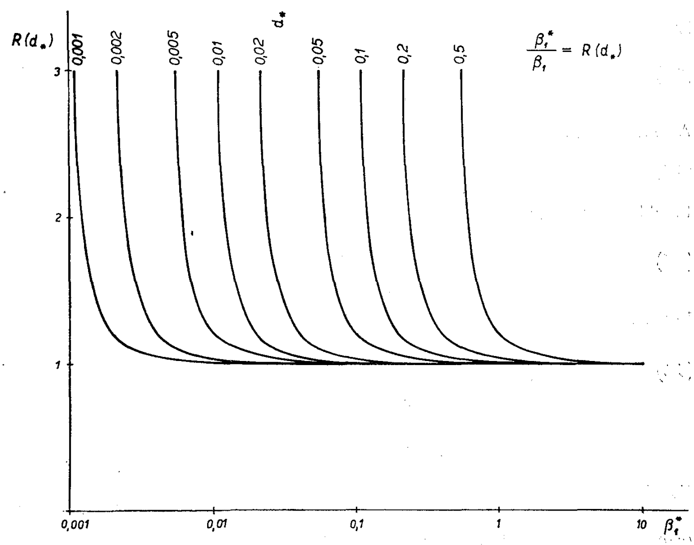

Fig. 1.

where

$$Q(\chi^2 \mid k) = 1 - P(\chi^2 \mid k) = \left[2^{k/2}\,\Gamma\left(\frac{k}{2}\right)\right]^{-1} \int_{\chi^2}^{\infty} t^{k/2-1} e^{t/2}\, dt \tag{5.12}$$

($0 \leqq \chi^2 < \infty$), $P(\chi^2 \mid k)$ being the χ^2-distribution function and k the number of degrees of freedom.

Now, inserting (5.11) in (5.9) and expressing the parameter μ as a function of the moment β_1 and β_{-1}, using (5.3) and (5.4) and (5.5) respectively, we obtain the moments β_1^* and β_{-1}^* in the forms

$$\beta_1^* = \beta_1 e^{\chi^2/2} Q(\chi^2 \mid 3)\,, \tag{5.13}$$

where

$$\chi^2 = \frac{\pi}{2}\frac{d_*^2}{\beta_1^2}\,, \tag{5.14}$$

and

$$\beta_{-1}^* = \beta_{-1} e^{\chi^2/2} Q(\chi^2 \mid 1)\,, \tag{5.15}$$

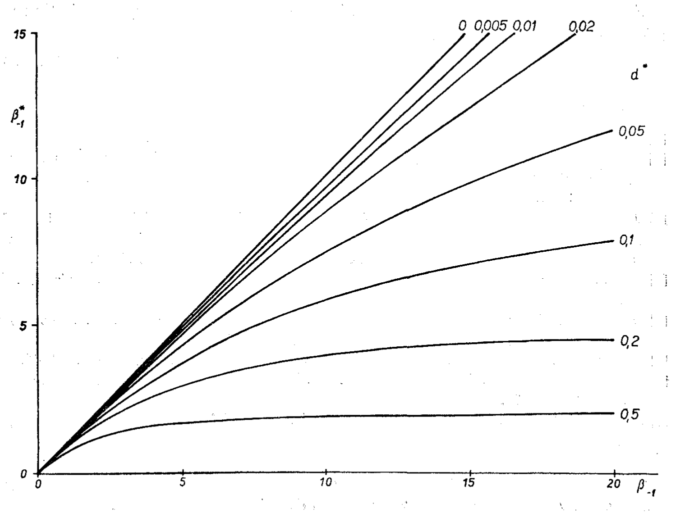

Fig. 2.

where

$$\chi^2 = \frac{2}{\pi} d_*^2 \beta_{-1}^2 \,. \tag{5.16}$$

From (5.13) and (5.15) it is clear that the correction factor for converting β_ν in β_ν^* ($\nu = \pm 1$) has not simple form, in spite of equality of moments α_1 and β_1.

Fig. 1 represents the relation (5.15) for various levels of d_* ($0 \leqq d_* \leqq 0{\cdot}5$). From the courses of curves d_* we can verify the validity of proved inequality (3.3b) for $d_* > 0$; in addition to this, we can see the expressive influence of increasing d_* on the underevaluating β_{-1}. For $d_* = 0$, we have $\beta_{-1} = \beta_{-1}^*$.

In Fig. 2 the ratio

$$\frac{\beta^*_1}{\beta_1} = R(d_*)$$

is plotted as a function of the resolution point d_* ($0{\cdot}001 \leqq d_* \leqq 0{\cdot}5$). The choice of this graphical form was called by the fact that the greatest differences between β_1^* and β_1 arise in the domain of small values. From the courses of curves d_* it can be again verified the validity of the inequality (3.3a). For calculated moment β_1^* and given d_*, we can read off the value of the ratio $R(d_*)$ and by means of it to determine the unknown moment β_1.

For calculating curves in Fig. 1 and 2, we used the tables of χ^2-distribution and relevant expansions introduced in [7].

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REFERENCES

[1] S. D. Wicksell: The corpuscle problem. Biometrika *17* (1925), 84—89.

[2] V. Horálek: Statistické modely pro zkoušení a kontrolu výrobků, materiálů a surovin. (Statistical models for testing and inspection of products, materials and raw materials.) Charles University, Prague 1961 (dissertation).

[3] J. Likeš: Stanovení počtu a velikosti částic dispersní fáze metodami kvantitativní metalografie. (Determining number and size of particles of a dispersion phase by methods of quantitative metallography.) Rozpravy ČSAV, Series of Techn. Sci. *74* (1964), No. 1.

[4] R. L. Fulmann: Measurement of particle sizes in opague bodies. Journal of Metals *5* (1953), 447—452.

[5] V. Horálek: Příspěvek k otázce hodnocení struktury materiálu. (A contribution to study of material structure.) Aplikace matematiky *3* (1958), 376—383.

[6] H. Cramér: Mathematical Methods of Statistics. Princeton Univ. Press, Princeton 1946.

[7] M. Abramowitz, I. A. Stegun: Handbook of Mathematical Functions. Dover Publication, Inc., N. York 1970.

[8] J. Ježek, J. Vobořil: Elektronová metalografie jemnozrnných fází. (Electron Metallography of Finegrained Phases.) NČSAV, Praha 1972.

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