KYBERNETIKA — VOLUME 14 (1978), NUMBER 5

Stackelberg Solution Concept for General Multistage Games

JAROSLAV DOLEŽAL

The so-called Stackelberg solution concept for two-player nonzero-sum games is applied to a general class of multistage games and a set of necessary optimality conditions is obtained using certain results of discrete optimal control theory and theory of mathematical programming. These necessary conditions have the form of a discrete non-linear two-point boundary-value problem in general setting. To solve such problem certain iterative procedures must be employed to compute the Stackelberg solution in practical cases.

For a more special class of linear multistage games with both cost functionals being quadratic it is possible to derive an explicite computational scheme for the determination of the open-loop Stackelberg solution, which can be easily implemented. As an illustration, a simple example of this type is solved in detail and comparison with equilibrium and noninferior solution concepts is performed. The presented theory can found interesting applications in economical problems, management, etc.

1. INTRODUCTION

In this contribution a general class of two-player non-zero-sum multistage games is investigated from the point of view of the so-called Stackelberg solution concept rediscovered in [1]. This concept was applied earlier to differential games in [2–4] and also some aspects concerning the application of this concept to multistage games using the dynamic programming approach were pointed out. The possibility to obtain analogous results for multistage games and discrete optimal control problems with two objectives was studied by the author in [5–7] applying the mathematical programming techniques.

It is well-known that in the case of nonzero-sum games the notion of optimality is generally nonunique and it can be defined in several different ways assuming always certain mode of behaviour of each participating player. The mostly studied solution concepts are equilibrium, minimax and non-inferior ones, which were applied to multistage games in [8]. However, the all mentioned solution concepts require that the formulation of a game in question is in certain sense symmetric with respect to each player. As this is not always true, the player having more favourable position can improve his cost functional taking advantage of this fact. As a typical example of such situation, the two-player nonzero-sum game can be mentioned, the case in which one player does not know the cost functional of the other player.

For the further studied class of two-player nonzero-sum multistage games a set of necessary optimality conditions is derived for the so-called open-loop Stackelberg solution using the mathematical programming approach, e.g., see [9–10]. These conditions are then applied to the case of linear multistage games with quadratic cost functionals and the more explicit form of the open-loop Stackelberg solution is computed. In this way a discrete analogy of the above mentioned results [2–4] is obtained.

2. PROBLEM FORMULATION

For the purpose of this study the following definition of the Stackelberg solution concept for two-player nonzero-sum games will be satisfactory, see [1–4].

Definition 2.1. Given a two-player nonzero-sum game, where Player 1, resp. Player 2, minimizes a cost functional $J_1(u, v)$, resp. $J_2(u, v)$, selecting the appropriate strategies u and v from the admissible strategy sets $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively. The strategy pair $(\tilde{u}, \tilde{v})$ is called a *Stackelberg solution* with Player 2 as a *leader* and Player 1 as a *follower* if for any $v \in \boldsymbol{V}$ and any $u \in \boldsymbol{U}$

$$J_2(\tilde{u}, \tilde{v}) \leqq J_2(u^0(v), v)\,, \tag{1}$$

where the mapping $u^0 \colon \boldsymbol{V} \to \boldsymbol{U}$ is defined by the relation

$$J_1(u^0(v), v) = \min_{u \in \boldsymbol{U}} J_1(u, v) \tag{2}$$

and

$$\tilde{u} = u^0(\tilde{v})\,. \tag{3}$$

Further details and consequences of this definition can be found in [2–7]. At this place let us only remark that Stackelberg solution with Player 2 as the leader is optimal for Player 2 if he announces his move first and if the aim of Player 1 is to minimize J_1, while that of Player 2 is to minimize J_2. Such situation can for example arise if the information pattern of a game in question is biased in the sense that Player 1 knows only his cost functional, while Player 2 knows both cost functionals. Player 1 is thus forced to follow using Stackelberg strategy $\tilde{u}$ if Player 2 announces his Stackelberg strategy $\tilde{v}$.

This solution concept will be further applied to multistage games. General two-player nonzero-sum multistage game is supposed to be described in the following way, see [5–8].

The dynamic behaviour of the system is determined by a vector difference equation

$$x_{k+1} = f_k(x_k, u_k, v_k), \quad k = 0, 1, \dots, K-1, \quad x_0 = \text{given}, \tag{4}$$

where the positive integer K denotes the prescribed number of stages, $x_k \in E^n$ denotes state of the system at the stage k, $u_k \in E^m$ and $v_k \in E^p$ are controls (decisions) at the stage k, and finally f_k: $E^n \times E^m \times E^p \to E^n$. If not otherwise stated, all vectors are treated as column-vectors except of the gradients of various functions, which are always assumed to be row-vectors.

The cost functionals J_1 and J_2 are given by the relations

$$J_i = g^i(x_K) + \sum_{k=0}^{K-1} h^i_k(x_k, u_k, v_k), \quad i = 1, 2, \tag{5}$$

where g^i: $E^n \to E^1$ and h^i_k: $E^n \times E^m \times E^p \to E^1$. Here and henceforth it is assumed that Player 2 is the leader.

Definition 2.2. Any sequence

$$u = \{u_0, u_1, \dots, u_{K-1} \mid u_k \in E^m,\ k = 0, 1, \dots, K-1\}$$

is denoted as an admissible *open-loop strategy* of Player 1. An open-loop strategy of Player 2 is defined in a quite analogical way.

Now it is possible to apply the Stackelberg solution concept to the class of just defined multistage games. The aim is to obtain necessary optimality conditions for this problem to be able to determine open-loop Stackelberg solutions in a general case, at least in principal.

No further constraints concerning x, u, and v will be imposed. Such constraints, if present, would cause considerable troubles, because the complexity of the further presented relations would be increased. Then it is also not excluded the possible failure of the described approach.

3. NECESSARY OPTIMALITY CONDITIONS

It is easy to see that the application of the Stackelberg solution concept results in a two-level optimization. This concept can be also applied to resolve optimizations problems with two objective functions [7]. The multistage case was treated there using the general results of [9]. In this way the two-level discrete maximum principle was obtained, however, certain convexity requirements were necessary.

As we shall see such convexity assumptions will not be necessary in the studied case. Namely, in our approach the maximum condition is not used in an explicite way. Then the necessary conditions for discrete optimal control problems [9, Theorem 5] can be applied not taking into the account the pertinent convexity assumptions. Such conditions can be derived quite easily [9, Proposition 3]. Similar technique was presented earlier in [10].

Turning back to the studied problem we see that if Player 2 is the leader, the corresponding mapping u^0 in (2) is determined by the solution of a discrete optimal control problem given by system equations (4) and cost functional J_1 with v being now a sequence of the parameters. Assume for a moment that to each v there exists a unique corresponding u determined by (2). Recall that $\boldsymbol{U} = E^m$ and $\boldsymbol{V} = E^p$ in the studied unconstrained case. The desired mapping u^0 can be described implicitely by a set of necessary optimality conditions for the just mentioned discrete optimal control problem. In general, let the following assumption be valid.

Assumption 3.1. All functions appearing in (4) and (5) are twice continuously differentiable in corresponding domains of definition.

From [9–10] it follows that then there exist row-vectors $\lambda_k \in E^n$, $k = 0, 1, \ldots, K$ satisfying

$$\lambda_k = \frac{\partial}{\partial x} H^1_{k+1}(x_k, u_k, v_k)\,, \quad k = 0, 1, \ldots, K - 1\,, \tag{6}$$

$$\lambda_K = - \frac{\partial}{\partial x} g^1(x_K)\,,$$

where the Hamiltonian H^1_{k+1} is written as

$$H^1_{k+1}(x, u, v) = -h^1_k(x, u, v) + \lambda_{k+1} f_k(x, u, v)\,, \quad k = 0, 1, \ldots, K - 1\,, \tag{7}$$

and

$$\frac{\partial}{\partial u} H^1_{k+1}(x_k, u_k, v_k) = 0\,, \quad k = 0, 1, \ldots, K - 1\,. \tag{8}$$

Here $x_0, x_1, \ldots, x_K$ denote the state trajectory corresponding to the pair of sequences (u, v). The relations (4), (6) and (8) are the multistage analogy of condition (2) in the definition of Stackelberg solution. Hence, a Stackelberg solution $(\tilde{u}, \tilde{v})$ must necessarily satisfy these relations, and in the same time the cost functional J_2 should be minimized. This is a nonstandard form of the discrete optimal control problem and, therefore, several preliminary steps are necessary if we want to apply the results of [9–10].

Assumption 3.2. The $n \times n$-dimensional matrices

$$\frac{\partial}{\partial x} f_k(x, u, v), \quad k = 0, 1, \ldots, K - 1$$

are regular for any $x \in E^n$, $u \in E^m$ and $v \in E^p$.

From (6) and (8) we then obtain (arguments are omitted for the sake of simplicity)

$$\lambda_{k+1} = \left(\lambda_k + \frac{\partial h_k^1}{\partial x}\right)\left(\frac{\partial f_k}{\partial x}\right)^{-1} = F_k^T(x_k, \lambda_k, u_k, v_k), \tag{9}$$

$$\lambda_K = -\frac{\partial g^1}{\partial x},$$

$$\frac{\partial H_{k+1}^1}{\partial u} = -\frac{\partial h_k^1}{\partial u} + \left(\lambda_k + \frac{\partial h_k}{\partial x}\right)\left(\frac{\partial f_k}{\partial x}\right)^{-1}\frac{\partial f_k}{\partial u} = G_k^T(x_k, \lambda_k, u_k, v_k) = 0, \tag{10}$$

where $k = 0, 1, \ldots, K - 1$. Here F_k and G_k denote n-dimensional and m-dimensional functions, respectively, and T denotes transposition. Constraints (4) and (9) can be written in a more compact form

$$\begin{bmatrix} x_{k+1} \\ \lambda_{k+1}^T \end{bmatrix} = \begin{bmatrix} f_k(y_k, u_k, v_k) \\ F_k(y_k, u_k, v_k) \end{bmatrix}, \quad k = 0, 1, \ldots, K - 1, \tag{11}$$

where $[x^T, \lambda]^T$ represents now the new state variable.

Necessary optimality conditions for the Stackelberg solution are then obtained as the necessary optimality conditions for the discrete optimal control problem with the cost functional J_2, constraints (10) and (11), where u and v are the controls, and with boundary conditions

$$x_0 = \text{given}, \quad \lambda_K = -\frac{\partial}{\partial x} g^1(x_K). \tag{12}$$

Under Assumptions 3.1–3.2 the results of [9], again without the convexity requirements, or of [10] can be applied, but from the practical reasons let us make one additional assumption. The terminal condition for λ_K in (12) must be clearly treated as a state constraint. However, in this case we are not allowed to put multiplier -1 in the definition of the corresponding Hamiltonian as it was possible in (7). On the other hand, the case with this multiplier equal to zero is pathological and of little interest from both practical and computational point of view. Hence, let us impose the following "normality" assumption, which is not restrictive from the practical view-point.

Assumption 3.3. For the discrete optimal control problem with the cost functional J_2 and constraints (10)–(12) the corresponding Hamiltonian can be written as

$$\mathscr{H}_{k+1}(x, \lambda, u, v) = -h_k^2(x, u, v) + \mu_{k+1} f_k(x, u, v) + \\ + \nu_{k+1} F_k(x, \lambda, u, v), \quad k = 0, 1, \ldots, K-1,$$

where μ_k and ν_k are n-dimensional row-vector multipliers.

Now we can rather easily apply the mentioned results of [9–10] to this optimal control problem. After some straightforward manipulations, omitted here for the sake of brevity, we obtain the desired necessary optimality conditions for a open-loop Stackelberg solution of multistage games. These conditions are summarized in the following theorem, where we use the notation

$$H_{k+1}^2(x, u, v) = -h_k^2(x, u, v) + \mu_{k+1} f_k(x, u, v), \quad k = 0, 1, \ldots, K-1.$$

Theorem 3.1. Consider a two-player nonzero-sum multistage game (4) and (5), where Player 2 is the leader. Further suppose that Assumptions 3.1–3.3 are fulfilled and that the strategy pair $(\tilde{u}, \tilde{v})$ is a Stackelberg solution of this game. The corresponding trajectory let us denote as $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_K$.

Then there exist row-vector multipliers λ_k, μ_k, ν_k belonging to E^n, $k = 0, 1, \ldots, K$, and $\zeta_k \in E^m$, $k = 0, 1, \ldots, K-1$, such that the following conditions are satisfied:

(a) $$\lambda_k = \frac{\partial H_{k+1}^1}{\partial x}, \quad \lambda_K = -\frac{\partial g^1}{\partial x};$$

(b) $$\mu_k = \frac{\partial H_{k+1}^2}{\partial x} + \nu_{k+1} \frac{\partial F_k}{\partial x} + \zeta_k \frac{\partial G_k}{\partial x}, \quad \mu_K = -\frac{\partial g^2}{\partial x} + \nu_K \frac{\partial^2 g^1}{\partial x^2};$$

(c) $$\nu_{k+1} = \nu_k \left(\frac{\partial f_k}{\partial x}\right)^T - \zeta_k \left(\frac{\partial f_k}{\partial u}\right)^T, \quad \nu_0 = 0;$$

(d) $$\frac{\partial H_{k+1}^1}{\partial u} = 0;$$

(e) $$\frac{\partial H_{k+1}^2}{\partial u} + \nu_{k+1} \frac{\partial F_k}{\partial u} + \zeta_k \frac{\partial G_k}{\partial u} = 0;$$

(f) $$\frac{\partial H_{k+1}^2}{\partial v} + \nu_{k+1} \frac{\partial F_k}{\partial v} + \zeta_k \frac{\partial G_k}{\partial v} = 0,$$

where always $k = 0, 1, \ldots, K-1$ and all expressions are evalutated for $\tilde{x}_k, \lambda_k, \tilde{u}_k$ and $\tilde{v}_k$.

Remark 3.1. Looking through the Theorem 3.1 it is not very difficult to see that the Assumption 3.1 can be somewhat released, e.g., function g^2 can be only continuously differentiable, etc.

Remark 3.2. Strictly speaking the Stackelberg strategy pair $(\tilde{u}, \tilde{v})$ need not be the unique one, however, all such solutions must satisfy the stated Theorem 3.1. It follows that the mapping u^0 is then multivalued. On the other hand, the values of both cost functionals can be different, in general, for various Stackelberg solutions and the question of the appropriate choice arises. Also certain a priori provisions must be postulated to overcome existing uncertainty in decision-making. Some further details in this respect can be found in [7], where two-level optimization problems were studied.

Remark 3.3. It is easy to see that the conditions (a)–(f) of Theorem 3.1 represent a discrete nonlinear two-point boundary-value problem. To solve this problem in its general setting some numerical iterative methods must be applied, see [11].

4. LINEAR MULTISTAGE GAMES WITH QUADRATIC COST FUNCTIONALS

Fairly deep results can be obtained if we assume that the system equations (4) are linear and the cost functionals (5) quadratic. In this special case it is possible to derive an analytic scheme for the computation of open-loop Stackelberg strategies. Relations (4) and (5) are then replaced by

$$x_{k+1} = Ax_k + B_1 u_k + B_2 v_k\,, \quad k = 0, 1, \dots, K-1\,, \tag{13}$$

$$J_i = \tfrac{1}{2} x_K^T M_i x_K + \tfrac{1}{2} \sum_{k=0}^{K-1} (x_k^T Q_i x_k + u_k^T R_{i1} u_k + v_k^T R_{i2} v_k)\,, \quad i = 1, 2\,. \tag{14}$$

Again we assume that the initial state x_0 is given. Dimensions of x, u, and v are the same as in preceding sections. In this way also all dimensions of various matrices in (13) and (14) are determined.

The just described multistage game is supposed to be autonomous, i.e., its parameters (various matrices) do not vary with k. The only reason for such assumption is to avoid resulting notational complexity without any substantial gain. Not loosing any generality we may also assume that the matrices M_i, Q_i, R_{ij}, $i, j = 1, 2$ are symmetric. Further let us assume that the matrix A is regular, M_i, Q_i, R_{ij}, $j = 1, 2$, $i \neq j$ positive semidefinite and R_{ii}, $i = 1, 2$ positive definite. Then also Assumptions 3.1–3.2 are satisfied. The "normality" requirement in Assumption 3.3 would need a longer analysis of the corresponding quadratic programming problem so as to check this fact. However, this is not the purpose of this paper and therefore we simply assume that Assumption 3.3 is fulfilled, i.e., the studied problem is meaningful.

The necessary optimality conditions of the stated Theorem have now a simple form and result in the solution of the following discrete linear two-point boundary-value problem:

$$\left.\begin{aligned}
x_{k+1} &= Ax_k + B_1u_k + B_2v_k ,\\
\lambda_k &= -x_k^TQ_1 + \lambda_{k+1}A ,\\
\mu_k &= -x_k^TQ_2 + \mu_{k+1}A + \nu_{k+1}A^{-T}Q_1 + \\
&\quad + \zeta_kB_1^TA^{-T}Q_1 ,\\
\nu_{k+1} &= \nu_kA^T - \zeta_kB_1^T ,
\end{aligned}\right\} k = 0, 1, \ldots, K-1, \tag{15}$$

where

$$\left.\begin{aligned}
u_k &= R_{11}^{-1}B_1^T\lambda_{k+1}^T ,\\
v_k &= R_{22}^{-1}B_2^T\mu_{k+1}^T ,\\
\zeta_k &= (\mu_{k+1}B_1 - \lambda_{k+1}B_1R_{11}^{-1}R_{21})\,R_{11}^{-1} ,
\end{aligned}\right\} k = 0, 1, \ldots, K-1, \tag{16}$$

and with boundary conditions

$$\begin{aligned}
x_0 &= \text{given} ,\\
\lambda_K &= -x_K^TM_1 ,\\
\mu_K &= -x_K^TM_2 + \nu_KM_1 ,\\
\nu_0 &= 0 .
\end{aligned} \tag{17}$$

Let us recall that λ_k, μ_k and ν_k are by definition the row-vectors. Therefore transposition must be used when necessary. In our notation $A^{-T} = (A^{-1})^T = (A^T)^{-1}$.

To solve the problem (15)–(17) let us assume that

$$\begin{bmatrix}\lambda_k^T\\ \mu_k^T\end{bmatrix} = \begin{bmatrix}P_k & W_k\\ N_k & S_k\end{bmatrix}\begin{bmatrix}x_k\\ \nu_k^T\end{bmatrix}, \quad k = 0, 1, \ldots, K, \tag{18}$$

where the $n \times n$-matrices P_k, N_k, W_k, S_k are to be determined. Such technique is often used when dealing with discrete linear two-point boundary-value problems. Therefore let us state only the final results, which can be obtained after inserting (18) in (15)–(17). Then the open-loop Stackelberg solution $(\tilde{u}, \tilde{v})$ of linear-quadratic multistage game with Player 2 as a leader can be written as follows:

$$\begin{bmatrix}\tilde{u}_k\\ \tilde{v}_k\end{bmatrix} = \begin{bmatrix}R_{11}^{-1}B_1^T & 0\\ 0 & R_{22}^{-1}B_2^T\end{bmatrix}\begin{bmatrix}P_{k+1} & W_{k+1}\\ N_{k+1} & S_{k+1}\end{bmatrix}\Gamma_k\begin{bmatrix}x_0\\ 0\end{bmatrix}, \quad k = 0, 1, \ldots, K-1, \tag{19}$$

where

$$\left.\begin{aligned}
P_k &= A^T(P_{k+1}F_k^1 + W_{k+1}F_k^3) - Q_1 ,\\
N_k &= A^T(N_{k+1}F_k^1 + S_{k+1}F_k^3) - Q_2 ,\\
W_k &= A^T(P_{k+1}F_k^2 + W_{k+1}F_k^4) ,\\
S_k &= A^T(N_{k+1}F_k^2 + S_{k+1}F_k^4) + Q_1 ,
\end{aligned}\right\} k = 0, 1, \ldots, K-1, \tag{20}$$

with the final conditions

(21) $$P_K = -M_1\,,\quad N_K = -M_2\,,\quad W_K = 0\,,\quad S_K = M_1\,.$$

For convenience, the further explained notation was introduced (index k ranges always from 0 to $K - 1$):

(22) $$\Phi_k = \begin{bmatrix} \tilde{F}_k^1 & \tilde{F}_k^2 \\ \tilde{F}_k^3 & \tilde{F}_k^4 \end{bmatrix}, \quad \Phi_k^{-1} = \begin{bmatrix} F_k^1 & F_k^2 \\ F_k^3 & F_k^4 \end{bmatrix},$$

(23) $$\begin{aligned} \tilde{F}_k^1 &= A^{-1}(-B_1R_{11}^{-1}B_1^TP_{k+1} - B_2R_{22}^{-1}B_2^TN_{k+1} + I)\,, \\ \tilde{F}_k^2 &= A^{-1}(-B_1R_{11}^{-1}B_1^TW_{k+1} - B_2R_{22}^{-1}B_2^TS_{k+1})\,, \\ \tilde{F}_k^3 &= A^{-1}(-B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1^TP_{k+1} + B_1R_{11}^{-1}B_1^TN_{k+1})\,, \\ \tilde{F}_k^4 &= A^{-1}(-B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1^TW_{k+1} + B_1R_{11}^{-1}B_1^TS_{k+1} + I)\,, \end{aligned}$$

(24) $$\Gamma_{k+1} = \Phi_k^{-1}\Gamma_k\,,\quad k = 0, 1, \ldots, K-1\,,\quad \Gamma_0 = I\,.$$

Matrices Φ_k and Γ_k are $2n \times 2n$-dimensional and matrices $\tilde{F}_k^i$, F_k^i, $i = 1, 2, 3, 4$, are $n \times n$-dimensional. In fact, matrices F_k^i were introduced to simplify the notation. Finally, by symbol I we denote $n \times n$-dimensional unit matrix.

Now we see that the backward computation of (20) takes place first starting from (21). All required quantities are continuously evaluated using (22) and (23) provided that the inversion indicated in (22) exists. Matrices P_k, N_k, W_k, S_k and Φ_k^{-1} are stored during the backward run. Then in the forward run we determine Γ_k according to (24), which, in turn, enables us a parallel evaluation of Stackelberg strategies $\tilde{u}_k$ and $\tilde{v}_k$ realizing relation (19). However, these strategies can be also determined indirectly [12] by solving the original discrete linear two-point boundary-value problem (15)–(17).

5. EXAMPLE

As an illustration let us solve a simple linear-quadratic multistage game from the point of view of Stackelberg solution concept. For comparison also equilibrium and noninferior solution concepts are applied to the same problem. All variables will be scalars and let Player 2 be the leader in this game.

(25) $$x_{k+1} = x_k + u_k + v_k\,,\quad k = 0, 1, \ldots, K-1\,,\quad x_0 = \text{given}\,,$$

(26) $$J_1 = \tfrac{1}{2}x_K^2 + \tfrac{1}{2}\sum_{k=0}^{K-1} u_k^2\,,\quad J_2 = \tfrac{1}{2}x_K^2 + \tfrac{1}{2}\sum_{k=0}^{K-1} v_k^2\,.$$

In this really academic example it is advisable to prefer the general conditions of the stated Theorem 3.1 to the scheme given in the previous section. We omit the obvious manipulations and state briefly only the results. However, let us remark that in this case the Assumption 3.3 is a priori satisfied. Really, if we assume that the problem in question is not normal we obtain that the multipliers μ_k, ν_k, ζ_k are always zero, which contradicts with the general conclusions made in [9]. Thus we have the relations

$$\left.\begin{aligned} \tilde{u}_k &= -\frac{K+1}{K^2+3K+1}\,x_0\,, \\ \tilde{v}_k &= -\frac{1}{K^2+3K+1}\,x_0\,, \end{aligned}\right\} k = 0, 1, \ldots, K-1\,. \tag{27}$$

The corresponding values of the cost functionals (outcomes):

$$\tilde{J}_1 = \frac{1}{2}\frac{(K+1)^3}{(K^2+3K+1)^2}\,x_0^2\,, \quad \tilde{J}_2 = \frac{1}{2}\frac{1}{K^2+3K+1}\,x_0^2\,. \tag{28}$$

Further it is not very hard to show that in this case relations (28) determine the unique Stackelberg solution with Player 2 as a leader. It is enough to realize that J_2 is a quadratic function of the control sequence $v_0, v_1, \ldots, v_{K-1}$ of Player 2 over the so-called "rational reaction set" of Player 1, as defined in [3], with the corresponding matrix being positive definite. Therefore, relation (28), derived using necessary optimality conditions, is the unique Stackelberg solution with Player 2 as a leader of the multistage game given by (25) and (26).

Now let us compute for the studied game also equilibrium and noninferior solutions applying the theory developed in [8]. For the equilibrium solution (u^*, v^*) we then have:

$$u_k^* = v_k^* = -\frac{1}{2K+1}\,x_0\,, \quad k = 0, 1, \ldots, K-1\,, \tag{29}$$

$$J_1^* = J_2^* = \frac{1}{2}\frac{K+1}{(2K+1)^2}\,x_0^2\,. \tag{30}$$

It is a simple exercise to show that

$$\tilde{J}_1 > J_1^*\,, \quad \tilde{J}_2 < J_2^* \tag{31}$$

for general number of stages K. This result is in agreement with the general conclusions of [2–4] which say that, in general, the leader's cost functional will not increase if the Stackelberg solution is enforced instead of the equilibrium one. On the other hand, the follower's cost functional can increase or decrease depending on the particular problem when such comparison is made. In the studied example we see that the Stackelberg solution is not attractive for the follower.

The noninferior solution (vector-valued functional) for the game (25) and (26) consists in fact of the whole family of such solutions. Necessary optimality conditions for this solution type can be found again in [8]. The one-parameter family of noninferior solutions has the following form:

$$(32)\qquad \left.\begin{aligned} \hat{u}_k(\alpha) &= -\frac{1-\alpha}{\alpha(1-\alpha)+K}x_0\,,\\ \hat{v}_k(\alpha) &= -\frac{\alpha}{\alpha(1-\alpha)+K}x_0\,. \end{aligned}\right\}\quad 0 \leqq \alpha \leqq 1\,,\quad k = 0, 1, \ldots, K-1\,.$$

The corresponding values of the cost functionals:

$$(33)\qquad \left.\begin{aligned} \hat{J}_1(\alpha) &= \frac{1}{2}\frac{(1-\alpha)^2(\alpha^2+K)}{[\alpha(1-\alpha)+K]^2}x_0^2\,,\\ \hat{J}_2(\alpha) &= \frac{1}{2}\frac{\alpha^2[(1-\alpha)^2+K]}{[\alpha(1-\alpha)+K]^2}x_0^2\,, \end{aligned}\right\}\quad 0 \leqq \alpha \leqq 1\,.$$

All mentioned solution types are schematically depicted in Fig. 1. The shaded region $\mathscr{P}$ represents the set of the all possible outcomes. Stackelberg solution (27) is denoted as the point H_2 and, by the symmetry of the problem in question, H_1

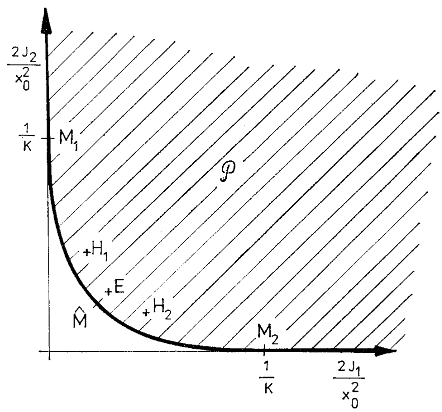

Fig. 1. Comparison of various outcomes for the illustrative example.

corresponds to the case when Player 1 is a leader. Point E represents the equilibrium outcome (30) and curve M_1M_2 depicts the set of noninferior outcomes. This curve is parametrized by parameter α ranging from 0 to 1; points M_1 and M_2 correspond to $\alpha = 1$ and $\alpha = 0$, respectively. For example, choosing $\alpha = 0{\cdot}5$ we obtain the point $\hat{M}$ on the curve M_1M_2 which can be also denoted as a "fair" negotiated solution of the studied game, which is thus attractive to both players.

6. CONCLUSIONS

The so-called Stackelberg solution concept for two-player nonzero-sum games was successfully applied to a general class of multistage games. Using the results of discrete optimal control theory and mathematical programming it was possible to derive a set of necessary optimality conditions for the open-loop Stackelberg solution.

For a special case of linear multistage games with quadratic cost functionals the analytical form of Stackelberg solution was derived. The suggested scheme can be easily implemented to the numerical solution of such problems arising for example in economy, management, etc. Let us finally note that the so-called "feedback" Stackelberg solution for this class of multistage games was studied recently in [13].

(Received May 6, 1978.)

REFERENCES

[1] M. D. Intrilligator: Mathematical Optimization and Economic Theory. Prentice-Hall, Englewood Cliffs, N. J. 1971.

[2] C. I. Chen, J. B. Cruz, Jr.: Stackelberg Solution for Two-Person Games with Biased Information Patterns. IEEE Transactions on Automatic Control *AC-17* (1972), 6, 791—797.

[3] M. Simaan, J. B. Cruz, Jr.: On the Stackelberg Strategy in Nonzero-Sum Games. Journal of Optimization Theory and Applications *11* (1973), 5, 533—555.

[4] M. Simaan, J. B. Cruz, Jr.: Additional Aspects of the Stackelberg Strategy in Nonzero-Sum Games. Journal of Optimization Theory and Applications *11* (1973), 6, 613—626.

[5] J. Doležal: Discrete Optimal Control Problems and Multistage Games. Institute of Information Theory and Automation, Thesis, 1973 (in Czech).

[6] J. Doležal: Open-Loop Hierarchical Solutuion of Multistage Games. Proceedings of the "2nd FORMATOR Symposium", Academia, Prague 1975, 269—282.

[7] J. Doležal: Hierarchical Solution Concept for Static and Multistage Decision Problems with Two Objectives. Kybernetika *12* (1976), 5, 363—385.

[8] J. Doležal: Necessary Optimality Conditions for *N*-Player Nonzero-Sum Multistage Games. Kybernetika *12* (1976), 4, 268—295.

[9] J. Doležal: Necessary Optimality Conditions for Discrete Systems with State-Dependent Control Region. Kybernetika *11* (1975), 6, 423—450.

[10] J. B. Pearson, Jr., R. Sridhar: A Discrete Optimal Control Problem. IEEE Transactions on Automatic Control *AC-11* (1966), 2, 171—174.

[11] J. Doležal: On the Modified Quasilinearization Method for Discrete Two-Point Boundary-Value Problems. Institute of Information Theory and Automation, Research Report No. 788, 1977.

[12] J. Doležal: Optimization of Discrete Linear-Quadratic Systems and Discrete Linear Two-Point Boundary-Value Problems. Institute of Information Theory and Automation, Research Report No. 847, 1977.

[13] B. F. Gardner, Jr., J. B. Cruz, Jr.: Feedback Stackelberg Strategy for a Two Player Game. IEEE Transactions on Automatic Control *AC-22* (1977), 2, 270—271.

Ing. Jaroslav Doležal, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.