KYBERNETIKA — VOLUME 14 (1978), NUMBER 5

Stability in the Stochastic Programming

VLASTA KAŇKOVÁ

It is generally known that the question of stability of an optimization problem is very important. The problem of stability in the stochastic linear programming is discussed in [1], where the stability is considered in the space of random variables. In this paper, first we shall generalize the results of [1] to the case of nonlinear stochastic programming. To attain this, we shall introduce the Hausdorff distance of sets. Further, we shall consider the stability in the space of distribution functions. Especially, we shall consider the case of the two-stage stochastic programming.

At the end of this paper we shall introduce an approximation method for solution of some problems of the two-stage stochastic nonlinear programming. A similar method was first introduced for the case of two-stage linear programming by Kall [5].

1. INTRODUCTION

Investigation of stability of an optimization problem means the study changes of the optimal value and optimal solution for "small" variations of parameters. Since, in many practical cases, parameters (or their probability laws) are not exactly known, it is very important that "small" variations of the parameters (or their probability laws) caused "small" variations of the optimum too.

In the literature, this problem is discussed separately for different types of mathematical programming. For nonlinear programming, for example, it is considered in [3], [4]. The case of stochastic linear programming is discussed in [1]. Bereanu in [1] considers a problem of linear programming with coeficients dependent on a random parameter ξ. If we denote the optimal value of the problem of the linear programming (with random parameter ξ) by $v(\xi)$ and if $\{\xi_N\}_{N=1}^{\infty}$ is a sequence of random elements for which $\operatorname{p\,lim}_{N\to\infty} \xi_N = \xi$, then Bereanu found in [1] conditions under which $\operatorname{p\,lim}_{N\to\infty} v(\xi_N) = v(\xi)$.

In this paper first we prove similar assertion for the nonlinear case. The results of this first part we shall use for finding conditions of the stability of problems of the two-stage stochastic programming. In this case the stability will be considered in the space of distribution functions. If we denote the distribution function of random parameters in the two-stage stochastic programming by $G(z)$ and if $I(G)$ is an optimal value of this problem, then we shall find condition under which the following implication holds: $\lim_{N\to\infty} G_N(z) = G(z) \Rightarrow \lim_{N\to\infty} I(G_N) = I(G)$. Here, the convergence of distribution functions is in the sense of Loève. The stability is always considered in the space of distribution functions, if the optimal solution is found with respect to the mathematical expectation.

At the end of this paper we shall introduce a new possibility of an approximate solution of some problems of the two-stage stochastic programming. The approximate solutions are found using concave programming.

2. SOME AUXILIARY ASSERTIONS

First we shall present some assertions useful in the sequel.

Let E_n, $n \geqq 1$ denote n-dimensional Euclidean space, ϱ_n the Euclidean metric in E_n. The Hausdorff distance between two subsets in E_n is defined in the following way.

Definition 1. If $X', X'' \subset E_n$, $n \geqq 1$ are two non-empty sets then the Hausdorff distance of these sets $\Delta_n(X', X'')$ is defined by

$$(1) \qquad \Delta_n(X', X'') = \max\left[\delta_n(X', X''), \delta_n(X'', X')\right],$$
$$\delta_n(X', X'') = \sup_{\mathbf{x}'\in X'} \inf_{\mathbf{x}''\in X''} \varrho_n(\mathbf{x}', \mathbf{x}'').$$

Let, further, $Z \subset E_s$, $X \subset E_n$, $s, n \geqq 1$ be non-empty sets, $X(\mathbf{z})$ a mapping of Z into the space of non-empty closed subsets of X. We shall define a uniform continuity of this mapping.

Definition 2. $X(\mathbf{z})$ is uniformly continuous mapping if for every $\varepsilon > 0$, there exists $\delta > 0$ such that the implication

$$\mathbf{z}, \mathbf{z}' \in Z, \quad \varrho_s(\mathbf{z}, \mathbf{z}') < \delta \Rightarrow \Delta_n(X(\mathbf{z}), X(\mathbf{z}')) < \varepsilon$$

is valid. (We usually leave the subscripts in symbols Δ_n, ϱ_s.)

Let $Z, X, X(\mathbf{z})$ and a function $g(\mathbf{x}, \mathbf{z})$ fulfil the following conditions:

(i) $Z \subset E_s$, $X \subset E_n$, $s, n \geqq 1$ are non-empty sets,

(ii) $X(\mathbf{z})$ is a uniformly continuous mapping of Z into a space of non-empty closed subsets of X,

(iii) $g(\mathbf{x}, \mathbf{z})$ is a uniformly continuous real valued function defined on $X \times Z$,

(iv) $\sup\{g(\mathbf{x}, \mathbf{z}) \mid \mathbf{x} \in E_n, \mathbf{x} \in X(\mathbf{z})\} < +\infty$ for every $\mathbf{z} \in Z$.

Lemma 1. If the conditions (i), (ii), (iii), (iv) are fulfilled then

$$\varphi(\mathbf{z}) = \sup \{g(\mathbf{x}, \mathbf{z}) \mid \mathbf{x} \in E_n,\ \mathbf{x} \in X(\mathbf{z})\} \tag{2}$$

is a uniformly continuous function on Z.

Proof. Let $\varepsilon > 0$ be arbitrary. From the assumptions (ii), (iii) it follows that for a $\delta > 0$

$$\begin{aligned} &\mathbf{z}, \mathbf{z}' \in Z\,, \quad \varrho(\mathbf{z}, \mathbf{z}') < \delta\,, \quad \mathbf{x}(\mathbf{z}) \in X(\mathbf{z}) \Rightarrow \\ &\Rightarrow \text{ex.} \quad \mathbf{x}(\mathbf{z}') \in X(\mathbf{z}') \quad \text{such that} \quad |g(\mathbf{x}(\mathbf{z}), \mathbf{z}) - g(\mathbf{x}(\mathbf{z}'), \mathbf{z}')| < \varepsilon\,. \end{aligned} \tag{3}$$

Further, from the assumption (iv), we have

$$g(\mathbf{x}_N(\mathbf{z}), \mathbf{z}) \leqq \varphi(\mathbf{z}) \leqq g(\mathbf{x}_N(\mathbf{z}), \mathbf{z}) + \frac{1}{N}$$

for some $\mathbf{x}_N(\mathbf{z}) \in X(\mathbf{z})$ and every $N = 1, 2, \ldots, \mathbf{z} \in E_n$. It is easy to see that

$$\mathbf{z}, \mathbf{z}' \in Z\,, \quad \varrho(\mathbf{z}, \mathbf{z}') < \delta \Rightarrow \varphi(\mathbf{z}) \leqq \varphi(\mathbf{z}') + \frac{1}{N} + \varepsilon$$

holds for every $N = 1, 2, \ldots$. In the same way we also can show that

$$\mathbf{z}, \mathbf{z}' \in Z\,, \quad \varrho(\mathbf{z}, \mathbf{z}') < \delta \Rightarrow \varphi(\mathbf{z}') \leqq \varphi(\mathbf{z}) + \frac{1}{N} + \varepsilon$$

holds for every $N = 1, 2, \ldots$. Since $\varepsilon > 0$ was arbitrary, the lemma is proved. □

Remark 1. The assumption (iv) of the Lemma 1 is fulfilled if either $g(\mathbf{x}, \mathbf{z})$ is a bounded function or X is a compact set.

Let now $\mathscr{F}_s$ be the space of s-dimensional distribution functions (it is the space of all functions satisfying the necessary and sufficient conditions to be distribution function of an s-dimensional random vector).

In $\mathscr{F}_1$ Loève [8] defined the metric $d_1(G, G')$

$$\begin{aligned} d_1(G, G') = \inf \{h \mid h \in E_1,\ G(z - h) - h \leqq G'(z) \leqq G(z + h) + h \\ \text{for every} \quad z \in E_1\}\,. \end{aligned}$$

We analogously define a metric d_s in $\mathscr{F}_s$, for $s \geqq 1$, by

$$\begin{aligned} d_s(G, G') = \inf \{h \mid h \in E_1,\ G(\mathbf{z} - \mathbf{h}) - h \leqq G'(\mathbf{z}) \leqq G(\mathbf{z} + \mathbf{h}) + h \\ \text{for every} \quad z \in E_s\}\,, \end{aligned}$$

$G, G' \in \mathscr{F}_s$, $\mathbf{h} = (h, \ldots, h) \in E_s$.

Lemma 2. $(\mathscr{F}_s, d_s)$ is for every $s \geqq 1$ a metric space.

Proof. To prove the lemma we have to prove the three axioms of metric. It is easy to see the validity of the axiom 1.

As the equivalence

$$G(\mathbf{z} - \mathbf{h}) - h \leqq G'(\mathbf{z}) \leqq G(\mathbf{z} + \mathbf{h}) + h \quad \text{for all} \quad \mathbf{z} \in E_s \Leftrightarrow$$
$$\Leftrightarrow G'(\mathbf{z} - \mathbf{h}) - h \leqq G(\mathbf{z}) \leqq G'(\mathbf{z} + \mathbf{h}) + h \quad \text{for all} \quad \mathbf{z} \in E_s$$

implies $d_s(G, G') = d_s(G', G)$, it remains to prove the triangular inequality only. But if for some h_1, h_2

$$G(\mathbf{z} - \mathbf{h}_1) - h_1 \leqq G''(\mathbf{z}) \leqq G(\mathbf{z} + \mathbf{h}_1) + h_1 \quad \text{for all} \quad z \in E_s\,,$$
$$G''(\mathbf{z} - \mathbf{h}_2) - h_2 \leqq G'(\mathbf{z}) \leqq G''(\mathbf{z} + \mathbf{h}_2) + h_2 \quad \text{for all} \quad z \in E_s\,,$$

we also have

$$G(\mathbf{z} - \mathbf{h}_2 - \mathbf{h}_1) - h_1 - h_2 \leqq G'(\mathbf{z}) \leqq G(\mathbf{z} + \mathbf{h}_1 + \mathbf{h}_2) + h_1 + h_2 \quad \text{for all} \quad \mathbf{z} \in E_s\,.$$

From this it already follows the validity of the triangular inequality. □

Lemma 3. If $G, G_N \in \mathscr{F}_s$, $s \geqq 1$, $N = 1, 2, \ldots$ then

1. $\lim_{N\to\infty} \sup \{|G_N(\mathbf{z}) - G(\mathbf{z})| \mid \mathbf{z} \in E_s\} = 0 \Rightarrow \lim_{N\to\infty} G_N(\mathbf{z}) = G(\mathbf{z})$ at all points of continuity of the function $G(\mathbf{z})$,
2. $\lim_{N\to\infty} d_s(G, G_N) = 0 \Rightarrow \lim_{N\to\infty} G_N(\mathbf{z}) = G(\mathbf{z})$ at all points of continuity of the function $G(\mathbf{z})$.

Proof. The proof of 1. is trivial. To prove 2. we denote

$$\bar{\mathbf{h}}_N = (\bar{h}_N, \ldots, \bar{h}_N) \in E_s\,, \quad \bar{h}_N = h_N + \lambda_N \quad \text{where} \quad h_N = d_s(G, G_N)\,,$$

$N = 1, 2, \ldots$ and λ_N, $N = 1, 2, \ldots$ are arbitrary constants satisfying the conditions $\lambda_N \to 0$, $(N \to +\infty)$, $\lambda_N > 0$. As from the definition it follows

$$G(\mathbf{z} - \bar{\mathbf{h}}_N) - \bar{h}_N \leqq G_N(\mathbf{z}) \leqq G(\mathbf{z} + \bar{\mathbf{h}}_N) + \bar{h}_N \quad \text{for all} \quad \mathbf{z} \in E_s\,,$$

we can easily obtain

$$G(\mathbf{z}) \leqq \lim_{N\to\infty} G_N(\mathbf{z}) \leqq G(\mathbf{z})$$

at all points of continuity of the function $G(\mathbf{z})$. □

In this part we shall generalize the results of [1].

Let $(\Omega, \mathscr{S}, P)$ be probability space, $\xi = \xi(\omega)$, $\xi_N = \xi_N(\omega)$, $N = 1, 2, \ldots$ s-dimensional random vectors defined on $(\Omega, \mathscr{S}, P)$. If we substitute $\xi(\omega)$ instead of $\mathbf{z}$ in $g(\mathbf{x}, \mathbf{z})$ and $X(\mathbf{z})$, we obtain a function and a mapping depending on the random vector ξ.

We can introduce the general stochastic optimization problem as a problem of finding

$$\sup \{g(\mathbf{x}, \xi(\omega)) \mid \mathbf{x} \in E_n,\ \mathbf{x} \in X(\xi(\omega))\}\,. \tag{4}$$

Let the random vectors $\xi(\omega)$, $\xi_N(\omega)$, $N = 1, 2, \ldots$ fulfil the condition

$$P\{\omega \mid \xi(\omega) \in Z\} = 1 = P\{\omega \mid \xi_N(\omega) \in Z\}\,, \quad N = 1, 2, \ldots\,. \tag{v}$$

We shall formulate the main result of this part.

Theorem 1. If the conditions (i), (ii), (iii), (iv), (v) are fulfilled then

1. $\operatorname{p}\lim_{N\to\infty} \xi_N(\omega) = \xi(\omega) \Rightarrow \operatorname{p}\lim_{N\to\infty} \psi_N(\omega) = \psi(\omega)\,,$
2. $\lim_{N\to\infty} \xi_N(\omega) = \xi(\omega)$ a. s. $\Rightarrow \lim_{N\to\infty} \psi_N(\omega) = \psi(\omega)$ a. s.,

where

$$\begin{aligned} \psi_N(\omega) &= \sup \{g(\mathbf{x}, \xi_N(\omega)) \mid \mathbf{x} \in E_n,\ \mathbf{x} \in X(\xi_N(\omega))\} = \\ &= \varphi(\xi_N(\omega))\,, \quad N = 1, 2, \ldots\,, \\ \psi(\omega) &= \sup \{g(\mathbf{x}, \xi(\omega)) \mid \mathbf{x} \in E_n,\ \mathbf{x} \in X(\xi(\omega))\} = \\ &= \varphi(\xi(\omega))\,. \end{aligned} \tag{5}$$

Proof. Since $\varphi(\mathbf{z})$ is uniformly continuous (Lemma 1) we can easily obtain measurability of the functions $\psi(\omega)$, $\psi_N(\omega)$, $N = 1, 2, \ldots$ defined on $(\Omega, \mathscr{S}, P)$ and, further, that for arbitrary $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\{\omega \mid \varrho_s(\xi_N(\omega), \xi(\omega)) < \delta\} \subset \{\omega \mid |\psi_N(\omega) - \psi(\omega)| < \varepsilon\} \quad \text{for} \quad N = 1, 2, \ldots\,.$$

But from this it follows the assertion 1.

Analogously we can obtain the assertion 2. □

If $\mathbf{A}$ is an $(m \times n)$ matrix with non-negative elements and non-zero columns and and if we denote $E_n^+ = \{\mathbf{x} \mid \mathbf{x} \in E_n,\ \mathbf{x} \geqq \mathbf{0}\}$, we can formulate a corollary of the Theorem 1.

Corollary 1. Let $X = E_n^+$, $Z = E_s^+$ and the mapping $X(\mathbf{z})$ fulfil the condition

$$X(\mathbf{z}) = \{\mathbf{x} \mid \mathbf{x} \in E_n^+,\ \mathbf{A}\mathbf{x} \leqq \mathbf{z}\} \quad \text{for} \quad \mathbf{z} \in E_s^+\,. \tag{6}$$

If the assumptions (iii), (iv), (v) hold then the assertion of Theorem 1 is valid.

To prove the Corollary we must prove the next lemma.

Lemma 4. If $X(\mathbf{z})$ fulfils the condition (6) then $X(\mathbf{z})$ is a uniformly continuous mapping of E_s into the space of non-empty compact subsets of E_n^+.

Proof. From the theory of the linear programming it is easy to see that (under the assumptions of the Lemma 4) for $\mathbf{z} \in E_s^+$ the sets $X(\mathbf{z})$ are closed convex bounded polyhedra. The number of the extremal points of every set $X(\mathbf{z})$ is finite. Considering every extremal point as a function of the vector $\mathbf{z}$ we know from the theory of the parametric linear programming that this function is linear. From this and from the convex analysis we obtain the assertion of Lemma 4. □

Proof of Corollary 1. The assertion of the Corollary 1 follows from Lemma 4 and Theorem 1. □

It is easy to see that Theorem 1 has mostly theoretical importance. The condition (ii) is generally very complicated. On the contrary the Corollary 1 is more suitable for applications.

4. STABILITY IN THE SPACE OF THE DISTRIBUTION FUNCTIONS

Till now we considered the optimal solution as a function of random variables. This approach is suitable if we know realizations of the random variables at the decision moment. But in many cases we have to decide without any knowledge of the random variables realizations. In this case we have to determine a deterministic equivalent problem of our stochastic problem. One of the possibilities is to take the mathematical expectation of the optimization function. In this case when random elements figure in the conditions, we add a penalty function (to the optimized function). This approach is used for example in Dupačová [2], Williams [9]. Generally we consider instead of the problem (4), i.e. instead of the problem of finding

$$\sup \{g(\mathbf{x}, \xi(\omega)) \mid \mathbf{x} \in E_n,\ \mathbf{x} \in X(\xi(\omega))\}$$

the deterministic problem of finding

$$\sup \{\mathsf{E}[g(\mathbf{x}, \xi(\omega)) - p(\mathbf{x}, \xi(\omega))] \mid \mathbf{x} \in E_n,\ \mathbf{x} \in X\},$$

where the penalty function $p(\mathbf{x}, \mathbf{z})$ fulfils the following conditions

$$p(\mathbf{x}, \mathbf{z}) = 0 \quad \text{if} \quad \mathbf{x} \in X(\mathbf{z}),$$
$$p(\mathbf{x}, \mathbf{z}) > 0 \quad \text{if} \quad \mathbf{x} \notin X(\mathbf{z})$$

(usually $p(\mathbf{x}, \mathbf{z})$ is required to be a convex and continuous function), and where $\mathbf{E}$ denotes the operator of mathematical expectation.

It is easy to see that in this case we have to consider the stability in the space of the distribution functions.

Let $G(\mathbf{z})$, $G_N(\mathbf{z})$, $N = 1, 2, \ldots$ be distribution functions of the random vectors $\xi(\omega)$, $\xi_N(\omega)$, $N = 1, 2, \ldots$ respectively. If $G(\mathbf{z})$, $G_N(\mathbf{z})$, $N = 1, 2, \ldots$ and $g(\mathbf{x}, \mathbf{z})$ fulfil the conditions

(vi) $\lim_{N \to \infty} d_s(G, G_N) = 0$,

(vii) X is a non-empty compact set,

(viii) $g(\mathbf{x}, \mathbf{z})$ is a bounded continuous real valued function,

we can formulate these results of this part.

Theorem 2. Let P be a complete probability measure. If conditions (i), (v), (vi), (vii), (viii) are fulfilled then

$$\lim_{N \to \infty} \max \left\{ \int_{E_s} g(\mathbf{x}, \mathbf{z}) \, \mathrm{d}G_N(\mathbf{z}) \mid \mathbf{x} \in X \right\} = \max \left\{ \int_{E_s} g(\mathbf{x}, \mathbf{z}) \, \mathrm{d}G(\mathbf{z}) \mid \mathbf{x} \in X \right\}.$$

If, further, $g(\mathbf{x}, \mathbf{z})$ is a strictly concave function of $\mathbf{x}$ for every $\mathbf{z} \in Z$, and $\mathbf{x}_0$, $\mathbf{x}_{0N}$, $N = 1, 2, \ldots$ are points at which

$$\max \left\{ \int_{E_s} g(\mathbf{x}, \mathbf{z}) \, \mathrm{d}G(\mathbf{z}) \mid \mathbf{x} \in E_n \right\} = \int_{E_s} g(\mathbf{x}_0, \mathbf{z}) \, \mathrm{d}G(\mathbf{z}),$$

$$\max \left\{ \int_{E_s} g(\mathbf{x}, \mathbf{z}) \, \mathrm{d}G_N(\mathbf{z}) \mid x \in E_n \right\} = \int_{E_s} g(\mathbf{x}_{0N}, \mathbf{z}) \, \mathrm{d}G(\mathbf{z}), \quad N = 1, 2, \ldots$$

then

$$\lim_{N \to \infty} \mathbf{x}_{0N} = \mathbf{x}_0 .$$

Proof. Since we get from Lemma 3 $\lim_{N \to \infty} G_N(\mathbf{z}) = G(\mathbf{z})$ at all points of the continuity of the function $G(\mathbf{z})$, the assertion of the Theorem 3 follows from the Theorem 1 and the Remark 1 of [6].

Remark 2. If we shall assume either $\lim_{N \to \infty} \sup \{ |G_N(\mathbf{z}) - G(\mathbf{z})| \mid \mathbf{z} \in E_s \} \to 0$ or $\lim_{N \to \infty} G_N(\mathbf{z}) \to G(\mathbf{z})$ at all points of continuity of the function $G(\mathbf{z})$ instead of (vi) then the assertion of Theorem 2 is valid too.

The results of this part we shall apply to another special case of the deterministic equivalent problem [5]. We shall consider the problem of the two-stage stochastic nonlinear programming.

5. STABILITY OF PROBLEMS OF THE TWO STAGE STOCHASTIC PROGRAMMING

Let $U \subset E_r$, $r \geqq 1$ be a non-empty set, $F(\mathbf{u}, \mathbf{x}, \mathbf{z})$ a real valued function defined on $U \times X \times Z$, $X(\mathbf{u}, \mathbf{z})$ a mapping of $U \times Z$ into the space of non-empty closed subsets of X.

The general problem of the two-stage stochastic programming can be introduced as a problem to find

$$(7) \qquad \sup \{\mathsf{E}[\sup \{F(\mathbf{u}, \mathbf{x}, \xi(\omega)) \mid \mathbf{x} \in E_n, \ \mathbf{x} \in X(\mathbf{u}, \xi(\omega))\}] \mid \mathbf{u} \in E_r, \ \mathbf{u} \in U\} .$$

(In this paper we shall assume such conditions that all symbols in (7) are meaningful.)

Let U, $F(\mathbf{u}, \mathbf{x}, \mathbf{z})$ and $X(\mathbf{u}, \mathbf{z})$ fulfil the following conditions:

(ix) $U \subset E_r$ is a non-empty compact set,

(x) $F(\mathbf{u}, \mathbf{x}, \mathbf{z})$ is a uniformly continuous function defined on $U \times X \times Z$,

(xi) $X(\mathbf{u}, \mathbf{z})$ is a uniformly continuous mapping of $U \times Z$ into a space of non-empty and closed subsets of X,

(xii) at least one of the following conditions is valid
 a) $F(\mathbf{u}, \mathbf{x}, \mathbf{z})$ is a bounded function,
 b) X, Z are compact sets.

Theorem 3. Let P be a complete probability measure. If the conditions (i), (v), (vi), (ix), (x), (xi), (xii) are fulfilled then

$$\lim_{N \to \infty} I(G_N) = \lim_{N \to \infty} \max \left\{ \int_{E_s} \sup \{F(\mathbf{u}, \mathbf{x}, \mathbf{z}) \mid \mathbf{x} \in X(\mathbf{u}, \mathbf{z})\} \, \mathrm{d}G_N(\mathbf{z}) \mid \mathbf{u} \in U \right\} =$$

$$= \max \left\{ \int_{E_s} \sup \{F(\mathbf{u}, \mathbf{x}, \mathbf{z}) \mid \mathbf{x} \in X(\mathbf{u}, \mathbf{z})\} \, \mathrm{d}G(\mathbf{z}) \mid \mathbf{u} \in U \right\} = I(G) .$$

Remark 3. If we assume either

$$\lim_{N \to \infty} \sup \{|G_N(\mathbf{z}) - G(\mathbf{z})| \mid \mathbf{z} \in E_s\} = 0 , \quad \text{or} \quad \lim_{N \to \infty} G_N(\mathbf{z}) = G(\mathbf{z})$$

at all points of the continuity of the function $G(\mathbf{z})$ instead of (vi), then the assertion of Theorem 3 is valid too.

Proof. From Lemma 1 and Remark 1 we can easily obtain the uniform continuity of the function

$$\varphi_1(\mathbf{u}, \mathbf{z}) = \sup \{F(\mathbf{u}, \mathbf{x}, \mathbf{z}) \mid \mathbf{x} \in X(\mathbf{u}, \mathbf{z})\}$$

defined on $U \times Z$. Further, it follows from (xii) that $\varphi_1(\mathbf{u}, \mathbf{z})$ is a bounded function. Now it is easy to see that the assumptions of Theorem 2 are already fulfilled. This completes the proof of Theorem 3. □

From Theorem 3 it is to see that under rather general conditions the problems of the two-stage stochastic nonlinear programming are stable.

6. APPROXIMATING SOLUTION OF PROBLEMS OF THE TWO-STAGE STOCHASTIC NONLINEAR PROGRAMMING

It is generally known that practically, it is impossible to find out an exact solution of the two-stage stochastic nonlinear programming problems. At this point we can use our former results for finding an appioximate solution. As we shall see, our approximating problems happen to be problems of concave programming.

Let $\boldsymbol{\eta}(\omega)$ be an s-dimensional random vector defined on $(\Omega, \mathscr{S}, P)$ and let $\boldsymbol{\eta}(\omega)$ have a (finite) step distribution function. We denote $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ the points of jumps of this distribution function, it is

$$(8) \qquad P\{\omega \mid \boldsymbol{\eta}(\omega) = \mathbf{z}_i\} = p_i\,, \quad \sum_{i=1}^{k} p_i = 1\,.$$

Let, further, the mapping $X(\boldsymbol{u}, \mathbf{z})$ be defined by

$$(9) \qquad X(\boldsymbol{u}, \mathbf{z}) = \{\mathbf{x} \in X \mid F_i(\boldsymbol{u}, \mathbf{x}, \mathbf{z}) \geqq 0,\ i = 1, 2, \ldots, m\}\,, \quad \boldsymbol{u} \in U,\ \mathbf{z} \in Z\,,$$

where $F_i(\boldsymbol{u}, \mathbf{x}, \mathbf{z})$, $i = 1, 2, \ldots, m$ are real valued continuous functions defined on $U \times X \times Z$. (All other symbols are the same as above.)

In [7] the following theorem is proved.

Theorem 4. Let $\boldsymbol{\eta}(\omega)$ and $X(\boldsymbol{u}, \mathbf{z})$ fulfil (8), (9). If $(\boldsymbol{u}(0), \mathbf{x}_1(0), \ldots, \mathbf{x}_k(0))$ is the point at which the value

$$(10) \qquad \sup \{ \sum_{j=1}^{k} p_i\, F(\boldsymbol{u}, \mathbf{x}_j, \mathbf{z}_j) \mid F_i(\boldsymbol{u}, \mathbf{x}_j, \mathbf{z}_j) \geqq 0\,,$$
$$\mathbf{x}_j \in X\,, \quad \boldsymbol{u} \in U\,, \quad i = 1, 2, \ldots, m\,, \quad j = 1, 2, \ldots, k\}$$

is achieved then

$$(11) \qquad \boldsymbol{u}_{\text{opt}} = \boldsymbol{u}(0)\,,$$
$$\mathbf{x}_{\text{opt}} = \mathbf{x}_i(0) \Leftrightarrow \boldsymbol{\eta}(\omega) = \mathbf{z}_i\,, \quad i = 1, 2, \ldots, k\,,$$

where $\boldsymbol{u}_{\text{opt}}$, $\mathbf{x}_{\text{opt}} = \mathbf{x}(\boldsymbol{u}_{\text{opt}}, \mathbf{z})$ fulfil the conditions

$$\sup \{F(\boldsymbol{u}_{\text{opt}}, \mathbf{x}, \mathbf{z}) \mid \mathbf{x} \in X(\boldsymbol{u}, \mathbf{z})\} = F(\boldsymbol{u}_{\text{opt}}, \mathbf{x}_{\text{opt}}, \mathbf{z}_j)\,, \quad j = 1, 2, \ldots, k\,,$$
$$\sup \{\mathsf{E}[\sup \{F(\boldsymbol{u}, \mathbf{x}, \boldsymbol{\eta}(\omega)) \,|\mathbf{x} \in X(\boldsymbol{u}, \boldsymbol{\eta}(\omega))\}]\mid \boldsymbol{u} \in U\} = \mathsf{E}F(\boldsymbol{u}_{\text{opt}}, \mathbf{x}_{\text{opt}}, \boldsymbol{\eta}(\omega))\,.$$

On the other hand, if $\boldsymbol{u}_{\text{opt}}$, $\mathbf{x}_{\text{opt}}$ exist then there exists a solution of the problem (10) and it is given by (11).

Remark 4. If the sets U, X are convex and if for every $\mathbf{z} \in Z$, $F(\mathbf{u}, \mathbf{x}, \mathbf{z})$, $F_i(\mathbf{u}, \mathbf{x}, \mathbf{z})$, $i = 1, 2, \ldots, m$ are concave (continuously) differentiable functions then the problem (10) is a problem of the concave programming with a (continuously) differentiable optimalized function.

We shall use Theorem 4 and the previous Remark of this paper for finding an approximation solution of some problems of the two-stage stochastic programming.

Let $G_N(\mathbf{z})$, $N = 1, 2, \ldots$ be step distribution functions and let $\mathbf{z}_1^N, \ldots, \mathbf{z}_{k_N}^N$ be jump-points of this distribution functions. We shall denote

$$P\{\omega \mid \xi_N(\omega) = \mathbf{z}_j^N\} = p_j^N, \quad j = 1, 2, \ldots, k_N, \quad \sum_{j=1}^{k_N} p_j^N = 1. \tag{12}$$

The problems of finding

$$\sup \{ \sum_{j=1}^{k_N} p_j^N F(\mathbf{u}^N, \mathbf{x}_j^N, \mathbf{z}_j^N) \mid F_i(\mathbf{u}^N, \mathbf{x}_j^N, \mathbf{z}_j^N) \geqq 0, \quad j = 1, 2, \ldots, k_N, \quad i = 1, 2, \ldots, m\}, \quad N = 1, 2, \ldots \tag{13}$$

are (under the conditions of Theorem 4 and Remark 4) problems of the concave programming with a (continuously) differentiable optimalized function.

Since, further, under the conditions of Theorem 3 the following relation is valid

$$\lim_{N \to \infty} \sup \{ \sum_{j=1}^{k_N} p_j^N F(\mathbf{u}^N, \mathbf{x}_j^N, \mathbf{z}_j^N) \mid F_i(\mathbf{u}^N, \mathbf{x}_j^N, \mathbf{z}_j^N) \geqq 0,$$
$$\mathbf{x}_j \in X, \quad j = 1, 2, \ldots, k_N, \quad i = 1, 2, \ldots, m, \quad \mathbf{u} \in U\} =$$
$$= \sup \{\mathbf{E} \sup \{F(\mathbf{u}, \mathbf{x}, \xi(\omega)) \mid \mathbf{x} \in X(\mathbf{u}, \xi(\omega))\} \mid \mathbf{u} \in U\},$$

where $X(\mathbf{u}, \mathbf{z})$ is given by (9), it is easy to see that, under rather general conditions, the problems (13) are suitable approximations to the problem of finding

$$\sup \{\mathbf{E} \sup \{F(\mathbf{u}, \mathbf{x}, \xi(\omega)) \mid \mathbf{x} \in X(\mathbf{u}, \xi(\omega))\} \mid u \in U\}.$$

As it is easy to see that in a general case it is difficult to verify assumption (xi) of Theorem 3, we shall introduce particular case where this assumption is fulfilled.

Let $\mathbf{A}$ be an $(m \times n)$ matrix with non-negative elements and non-zero columns, $\mathbf{h}(\mathbf{u}, \mathbf{z}) = [h_1(\mathbf{u}, \mathbf{z}), \ldots, h_m(\mathbf{u}, \mathbf{z})]$ an m-dimensional non-negative uniformly continuous vector function defined on $U \times Z$. If $X = E_n^+$, $Z = E_s^+$ and the mapping $X(\mathbf{u}, \mathbf{z})$ fulfils the conditions

$$X(\mathbf{u}, \mathbf{z}) = \{\mathbf{x} \mid \mathbf{x} \in E_n^+, \mathbf{A}\mathbf{x} \leqq \mathbf{h}(\mathbf{u}, \mathbf{z})\}, \tag{14}$$

then it follows from Lemma 4 that the condition (xi) is fulfilled.

It would surely be useful to give an estimation of the error of this approximation. Because the aim of this paper was not to find a solution of problems of the two-stage stochastic programming we leave this. Kall [5] considers similar approximation method in the case of two-stage stochastic linear programming. He found for this case an upper bound for the approximation error.

At the end of this paper we shall note the following: if we shall take the sample distribution functions instead of $G_N(\mathbf{z})$, $N = 1, 2, \ldots$ then corresponding results of this paper are valid almost surely.

(Received May 18, 1978.)

REFERENCES

[1] B. Bereanu: Stable Stochastic Linear Programs and Applications. Mathematische Operationsforschung und Statistik *4* (1975), 593—608.

[2] J. Dupačová: Minimax Approach to Stochastic Linear Programming and the Moment Problem. (In Czech.) Ekonomicko-matematický obzor *3* (1977), 279—307.

[3] A. Fiako, G. Mak-Kormik: Nonlinear Programming: Sequential Unconstrained Minimization Techniques. John Wiley and Sons, New York 1968.

[4] J. Guddat: Stability in Convex Quadratic Parametric Programming. Mathematische Operationsforschung und Statistik *2* (1976), 223—246.

[5] P. Kall: Stochastic Linear Programming. Springer-Verlag, Berlin 1976.

[6] V. Kaňková: Optimum Solution of a Stochastic Optimization Problem with Unknown Parameters. Trans. of the Seventh Prague Conference, Prague 1974. Pp. 239—244.

[7] V. Kaňková: Differentiability of the Optimalized Function in a Two Stage Stochastic Nonlinear Programming Problem. (In Czech.) To appear in Ekonomicko-matematický obzor.

[8] M. Loève: Probability Theory. Second Edition, D. van Nostrand Company, New York 1960.

[9] A. C. Williams: A Stochastic Transportion Problem. Oper. Res. *11* (1963), 759—770.

Vlasta Kaňková, prom. mat., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.