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Optimal Control under Discrete Observation of Continuous Stochastic Systems with Time Delay

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The linear-quadratic problem is considered. The observations are supposed to be indirect and affected by noise. The solution of filtration and smoothing problems which is needed for the optimal control is given by a recursive system of deterministic differential equations.

PRELIMINARIES

We consider a system described by the linear stochastic differential equation with time delay

$$(1) \qquad \mathrm{d}x(t) = [A_0(t) \, . \, x(t) + \int_{-h}^{0} A_1(t, \tau) \, . \, x(t + \tau) \, \mathrm{d}\tau + A_2(t) \, . \, x(t - h) + B(t) \, . \, u(t)] \, \mathrm{d}t + G(t) \, . \, \mathrm{d}w(t) \quad \text{for} \quad t \in [t_0, T]$$

with a deterministic bounded and measurable initial condition $x_0(\tau)$; $\tau \in [t_0 - h; t_0]$ where:

$x(t)$ is the n-dimensional state vector of the system,
$u(t)$ is the p-dimensional control function,
$w(t)$ is the n-dimensional Wiener process with covariance $I \, . \, t$,
A_0, A_1, A_2, B, G are matrix coefficients of appropriate types which are continuous in their domains.

We suppose that we have an increasing sequence $\{t_k\}_{k=1}^{m} \subset (t_0, T)$ of observation times and that q-dimensional results of observations are described by the equations

$$(2) \qquad z(t_k) = C_k \, . \, x(t_k) + e_k$$

where $\{C_k\}_{k=1}^{m}$ is a sequence of matrices of type $q \times n$ and $\{e_k\}_{k=1}^{m}$ is a sequence of independent normal random variables with zero mean and covariances E_k; $k =$

$= 1, \in.., m$. Further we suppose that the random variables $w(t)$ and e_k are independent for any $t \in [t_0, T]$ and $k = 1, \ldots, m$.

For $t \in [t_0, T]$ we define a σ-algebra $\mathscr{F}_t$ by

$$\mathscr{F}_t = \begin{cases} \{\Omega, \emptyset\} & \text{for} \quad t \in [t_0, t_1), \\ \sigma\{z(t_1); \ldots, z(t_k)\} & \text{for} \quad t \in [t_k, t_{k+1}). \end{cases}$$

The set of admissible control functions contains all nonanticipative (with respect to the system $\{\mathscr{F}_t\}$) functions satisfying the condition

$$\int_{t_0}^{T} E\|u(t)\|^2 \, \mathrm{d}t < \infty$$

where $\|.\|$ stands for the Euclidean norm.

Our aim is to minimize the loss function

$$L(x_0, u) = E \int_{t_0}^{T} c(s, x(s), u(s)) \, \mathrm{d}s,$$

where

$$c(s, x(s), u(s)) = x'(s) \,.\, Q_1(s) \,.\, x(s) + u'(s) \,.\, Q_2(s) \,.\, u(s)$$

$Q_1(s)$; $Q_2(s)$ are continuous matrix functions with nonnegative definite and positive definite values, respectively.

OPTIMAL CONTROL

We can write the equation (1) in the simplified form

$$\text{(1a)} \qquad \mathrm{d}x(t) = \left[\int_{-h}^{0} A(t, \tau) \,.\, x(t + \tau) \, \mathrm{d}\mu(\tau) + B(t) \,.\, u(t)\right] \mathrm{d}t + G(t) \, \mathrm{d}w(t),$$

where we put

$$A(t, \tau) = \begin{cases} A_0(t) & \text{for} \quad \tau = 0, \\ A_1(t, \tau) & \text{for} \quad \tau \in (-h, 0), \\ A_2(t) & \text{for} \quad \tau = -h, \end{cases}$$

and

$$\mu(M) = \lambda(M) + \operatorname{card}(M \cap \{-h, 0\})$$

for any Lebesgue measurable subset M of $[-h. 0]$ (λ is the standard Lebesgue measure).

Let $\{\mathscr{F}_t\}_{t\in[t_0,T]}$ be a nondecreasing system of σ-algebras generated by observations. We put

$$\text{(3)}\quad \hat{x}_t(\tau) = E[x(t+\tau)/\mathscr{F}_t] \quad \tilde{x}_t(\tau) = x(t) - \hat{x}_t(\tau) \qquad \text{for} \quad \tau \in [-h, 0]$$
$$\hat{x}(t) = \hat{x}_t(0) \qquad \tilde{x}(t) = \tilde{x}_t(0) = x(t) - \hat{x}(t)$$

The following solution of the optimal control problem for any admissible observation rule was developed in [4] by methods similar to those used in [3] and [2].

Lemma. The optimal control is given by the formula

$$\text{(4)}\qquad u(t) = K_0(t) \,.\, \hat{x}(t) + \int_{-h}^{0} K_1(t,\tau) \,.\, \hat{x}_t(\tau)\, d\tau$$

where

$$\text{(4a)}\quad K_0(t) = -Q_2^{-1}(t) \,.\, B'(t) \,.\, W_0(t) \,; \quad K_1(t,\tau) = -Q_2^{-1}(t) \,.\, B'(t) \,.\, W_1(t,\tau) \,.$$

The matrix functions $W_0(t)$; $W_1(t,\tau)$ can be obtained as a part of the unique solution of the known system of three Riccati type equations (cf. [1], [3]).

FILTRATION

To obtain the optimal control function we have to determine the function $\hat{x}_t(\tau)$, $t \in [t_0, T]$; $\tau \in [-h, O]$. We return to the discrete observation case and put

$$\xi_k(t) = E[x(t)/\mathscr{F}_{t_k}] \,, \quad \eta_k(t) = x(t) - \xi_k(t) \,,$$
$$\pi_k(t,s) = \operatorname{cov}[\eta_k(t); \eta_k(s)]$$
$$\text{for} \quad k = 0, 1, \ldots, m \,; \quad t \in [t_k - h; t_{k+1}] \quad (\text{where } t_{m+1} = T) \,.$$

Further we put

$$\tilde{z}(t_k) = z(t_k) - C_k \,.\, \xi_{k-1}(t_k) \,; \quad F_k = [C_k \,.\, \pi_{k-1}(t_k, t_k) \,.\, C_k' + E_k]^+$$

Remark 1. Notice that for $t \in [t_k; t_{k+1}]$ $k = 0, 1, \ldots, m$ and $\tau \in [-h, 0]$ the following equation holds

$$\hat{x}_t(\tau) = \xi_k(t + \tau) \,.$$

Definition. a. Random vectors x_i of dimension n_i, $i = 1, \ldots, p$ are called jointly Gaussian if the joined random variable

$$(x_{1,1}, \ldots, x_{1,m_1}, x_{2,1}, \ldots, x_{2,n_2}, \ldots, x_{P,1}, \ldots, x_{P,n_P})$$

is Gaussian.

b. A measurable n-dimensional stochastic process $x(t)$; $t \in [t', t'']$ is called Gaussian if any $p = 1, 2, \ldots$ and $s_1, \ldots, s_p \in [t', t'']$ the random variables $x(s_1), \ldots, x(s_p)$ are jointly Gaussian.

Proposition 1. a. The functions $\xi_k(t)$, fulfil on $[t_k, t_{k+1}]$ the differential equation

$$\text{(5)} \qquad \frac{\mathrm{d}\xi_k(t)}{\mathrm{d}t} = \int_{-h}^{0} A(t, \tau) \,.\, \xi_k(t + \tau)\, \mathrm{d}\mu(\tau) + B(t) \,.\, u(t)$$

with the initial condition

$$\text{(5a)} \qquad \xi_k(s) = \xi_{k-1}(s) + \Delta_k(s) \quad \text{for} \quad k \geqq 1\,; \quad s \in [t_k - h; t_k]$$

and

$$\text{(5b)} \qquad \xi_0(s) = x_0(s) \quad \text{for} \quad s \in [t_0 - h; t_0]\,,$$

where

$$\text{(6)} \qquad \Delta_k(s) = E[\eta_{k-1}(s)/\tilde{z}(t_k)] = \pi_k(s, t_k) \,.\, C_k' \,.\, F_k[z(t_k) - C_k \,.\, \xi_{k-1}(t_k)]\,.$$

b) The functions $\eta_k(t)$ fulfil on $[t_k; t_{k+1}]$ the differential equation

$$\text{(7)} \qquad \mathrm{d}\eta_k(t) = \left[\int_{-h}^{0} A(t, \tau) \,.\, \eta_k(t + \tau)\, \mathrm{d}\mu(\tau)\right] \mathrm{d}t + G(t)\, \mathrm{d}w(t)$$

with the initial condition

$$\text{(7a)}' \qquad \eta_k(s) = \eta_{k-1}(s) - \Delta_k(s) \quad \text{for} \quad k \geqq 1; s \in [t_k - h; t_k]$$

$$\text{(7b)} \qquad \eta_0(s) = 0 \qquad \text{for} \quad s \in [t_0 - h; t_0]\,.$$

Proof. The solution $x(t)$ of equation (1) can be expressed in the form (cf. [5])

$$\text{(8)} \qquad x(t) = \int_{-h}^{0} Y(t, t_k, \tau) \,.\, x(t_k + \tau)\, \mathrm{d}\mu_0(\tau) + \int_{t_k}^{t} X(t, s) \,.\, B(s) \,.\, u(s)\, \mathrm{d}s + {} + \int_{t_k}^{t} X(t, s) \,.\, G(s)\, \mathrm{d}w(s)$$

where $X(t, s)$ is the matrix solution of the equation

$$\text{(1b)} \qquad \frac{\partial X(t, s)}{\partial t} = \int_{-h}^{0} A(t, \tau) \,.\, X(t + \tau, s)\, \mathrm{d}\mu(\tau)$$

subject to the initial conditions $X(t, t) = I$, $X(t, s) = 0$ for $t < s$. The function

$Y(t, t_k, \tau)$ is defined for $t \in [t_k, T]$; $\tau \in [-h, 0]$ by

$$Y(t, t_k, \tau) = \begin{cases} X(t, t_k) & \text{for} \quad \tau = 0\,, \\ X(t, t_k + \tau + h) \,.\, A_2(t_k + \tau + h) + \\ \quad + \displaystyle\int_0^{\tau+h} X(t, t_k + s) \,.\, A_1(t_k + s, \tau - s)\, \mathrm{d}s & \text{for} \quad \tau \in [-h, 0]\,. \end{cases}$$

The measure μ_0 is defined on a Lebesgue measurable subsets of $[-h, 0]$ by

$$\mu_0(M) = \lambda(M) + \operatorname{card}(M \cap \{\emptyset\}) = \mu(M) - \operatorname{card}(M \cap \{-h\})\,.$$

Taking the conditional expectation with respect to $\mathscr{F}_t$ we get for $t \in [t_k; t_{k+1}]$

$$\text{(9a)} \qquad \xi_k(t) = \int_{-h}^{0} Y(t, t_k, \tau) \,.\, \xi_k(t_k + \tau)\, \mathrm{d}\mu_0(\tau) + \int_{t_k}^{t} X(t, s) \,.\, B(s) \,.\, u(s)\, \mathrm{d}s$$

(notice that $u(s)$ is $\mathscr{F}_{t_k}$ measurable for $s \in [t_k; t_{k+1})$), and

$$\text{(9b)} \qquad \eta_k(t) = \int_{-h}^{0} Y(t, t_k, \tau) \,.\, \eta_k(t_k + \tau)\, \mathrm{d}\mu_0(\tau) + \int_{t_k}^{t} X(t, s) \,.\, G(s)\, \mathrm{d}w(s)\,.$$

Differentiating (9a) and (9b) we obtain equations (5) and (7). We can rewrite (9b) in the form

$$\eta_k(t) = y_k(t) + v_k(t) \quad t \in [t_k; t_{k+1}]\,,$$

where

$$\text{(9c)} \qquad y_k(t) = \int_{-h}^{0} Y(t, t_k, \tau)\, \eta_k(t_k + \tau)\, \mathrm{d}\mu_0(\tau)$$

and

$$\text{(9d)} \qquad v_k(t) = \int_{t_k}^{t} X(t, s) \,.\, G(s)\, \mathrm{d}w(s)\,.$$

Considering the stochastic integral on the right-hand side of (9d) as a limit of the integral sums in the sense of the mean square (cf. [2], ch. 3) we deduce that the stochastic process $v_k(t)$, $t \in [t_k, t_{k+1}]$ is Gaussian (cf. [6]. ch. XIII). From (9b) and (7b) we get

$$\eta_0(t) = 0 \qquad \text{for} \quad t \in [t_0 - h; t_0]$$
$$\eta_0(t) = v_0(t) \quad \text{for} \quad t \in [t_0; t_1]\,,$$

hence the process $\eta_0(t)$ is Gaussian on $[t_0 - h; t_1]$. Suppose that for $k \geqq 1$ the process $\eta_{k-1}(t)$ is Gaussian on $[t_k - h; t_k]$. For $k = 1$, $t \in [t_1 - h; t_1]$ we have

$$\xi_1(t) = E[x(t) | \mathscr{F}_{t_1}] = E[x(t) | z(t_1)] = E[(\xi_0(t) + \eta_0(t)) / \tilde{z}(t_1)] =$$
$$= \Delta_1(t) + \xi_0(t)\,.$$

For $k \geqq 2$ the Gaussian random variables $\eta_{k-1}(t)$; $t \in [t_k - h; t_k]$ and

$$\tilde{z}(t_k) = z(t_k) - E[z(t_k)/z(t_1), \ldots, z(t_{k-1})] = C_k \, . \, \eta_{k-1}(t_k) + e_k$$

are independent of $z(t_1), \ldots, z(t_{k-1})$. Thus for $t \in [t_k - h; t_k]$ we have

$$\begin{aligned} E[x(t)/\mathscr{F}_{t_k}] &= E[(\xi_{k-1}(t) + \eta_{k-1}(t))/\mathscr{F}_{t_k}] = \\ &= \xi_{k-1}(t) + E[\eta_{k-1}(t)/z(t_1), \ldots, z(t_{k-1}), z(t_k)] = \\ &= \xi_{k-1}(t) + E[\eta_{k-1}(t)/\mathscr{F}_{t_{k-1}}] + E[\eta_{k-1}(t)/\tilde{z}(t_k)] = \xi_{k-1}(t) + \Delta_k(t) \, . \end{aligned}$$

Hence for $k \geqq 1$ equations (5a) and (7a)

$$\begin{aligned} \eta_k(t) &= \eta_{k-1}(t) - E[\eta_{k-1}(t)/\tilde{z}(t_k)] \, ; \\ \xi_k(t) &= \xi_{k-1}(t) + E[\eta_{k-1}(t)/\tilde{z}(t_k)] \, ; \quad t \in [t_k - h; t_k] \, ; \end{aligned}$$

result from the assumption that the process $\eta_{k-1}(t)$ is Gaussian. Equation (6)

$$\begin{aligned} \Delta_k(t) &= E[\eta_{k-1}(t)/\tilde{z}(t_k)] = \\ &= \operatorname{cov}[\eta_{k-1}(t), \tilde{z}(t_k)] \{\operatorname{cov}[\tilde{z}(t_k), \tilde{z}(t_k)]\}^{-1} \, . \, \tilde{z}(t_k) = \\ &= \pi_{k-1}(t, t_k) \, . \, C'_k \, . \, F_k \, . \, [z(t_k) - C_k \, . \, \xi_{k-1}(t_k)] \end{aligned}$$

results from the normal correlation theorem (cf. [6], ch. XIII). The stochastic process $\eta_k(t)$; $t \in [t_k - h, t_k]$ is Gaussian according to equations (7a) and (6). Considering the integral on the right-hand side of (9c) as a limit of integral sums we conclude that the stochastic process $y_k(t)$; $t \in [t_k, t_{k+1}]$ is Gaussian and independent of $v_k(s)$; $s \in [t_k, t_{k+1}]$. Therefore the stochastic process $\eta_k(t) = y_k(t) + v_k(t)$; $t \in [t_k - h, t_{k+1}]$ is Gaussian.

Proposition 2. Functions $\pi_k(t, s)$ fulfil for $s, t \in [t_k; t_{k+1}]$ the following system of differential equations

(10a) $$\frac{\mathrm{d}\pi_k(t, t)}{\mathrm{d}t} = \int_{-h}^{0} A(t, \tau) \, . \, \pi_k(t + \tau, t) \, \mathrm{d}\mu(\tau) + \int_{-h}^{0} \pi_k(t, t + \tau) \, . \, A'(t, \tau) \, \mathrm{d}\mu(\tau) + G(t) \, . \, G'(t) \, ,$$

(10b) $$\frac{\partial \pi_k(t, s)}{\partial t} = \int_{-h}^{0} A(t, \tau) \, . \, \pi_k(t + \tau, s) \, \mathrm{d}\mu(\tau)$$

with the initial conditions

(10c) $$\pi_k(t, s) = \pi_{k-1}(t, s) - \pi'_{k-1}(t, t_k) \, . \, C'_k \, . \, F_k \, . \, C_k \, . \, \pi_{k-1}(t_k, s); \quad t, s \in [t_k - h; t_k]$$

Proof. Formula (10b) follows immediately from (7). To obtain (10a) we make use of (7) and the modification of Ito formula for differentiating a composed function (cf. [3]). Let $k = 0, 1, \ldots$. The random variables

$$\eta_k(t) = \eta_{k-1}(t) - E[\eta_{k-1}(t)/\tilde{z}(t_k)] \quad t \in [t_k - h; t_k)$$

are independent of $\tilde{z}(t_k)$ and of $\Delta_k(s) = \pi_k(s, t_k) \,.\, C_k \,.\, F_k \,.\, \tilde{z}(t_k)$; $s \in [t_k - h, t_k]$. Hence

$$\begin{aligned}\pi_k(t, s) &= E[\eta_k(t) \,.\, \eta_k'(s)] = E[(\eta_{k-1}(t) - \Delta_k(t)) \,.\, \eta_k'(s)] = \\ &= E[\eta_{k-1}(t) \,.\, \eta_k'(s)] = E[\eta_{k-1}(t) \,.\, \eta_{k-1}'(s)] - \\ &- E[(\eta_k(t) + \Delta_k(t)) \,.\, \Delta_k'(s)] = \pi_{k-1}(t, s) - E[\Delta_k(t) \,.\, \Delta_k'(s)]' = \\ &= \pi_{k-1}(t, s) - \pi_{k-1}(t, t_k) \,.\, C_k' \,.\, F_k \,.\, C_k \,.\, \pi_{k-1}'(s; t_k)\end{aligned}$$

and formula (10c) holds.

Remark 2. Formula (4) for optimal control can be rewritten in the form

$$u(t) = \int_{-h}^{0} K(t, \tau) \,.\, \xi_k(t + \tau) \, \mathrm{d}\mu_0(\tau) \,, \quad \text{for} \quad t \in [t_k, t_{k+1}] \tag{11}$$

where

$$K(t, \tau) = \begin{cases} K_0(t) \text{ for } \tau = 0 \,, \\ K_1(t, \tau) \text{ for } \tau \in [-h, 0) \end{cases}$$

and K_0 and K_1 are given by (4a).

Equation (5) can then be transformed into the form

$$\frac{\mathrm{d}\xi_k(t)}{\mathrm{d}t} = \int_{-h}^{0} A_c(t, \tau) \,.\, \xi_k(t + \tau) \, \mathrm{d}\mu(\tau) \quad \text{for} \quad t \in [t_k; t_{k+1}] \tag{12}$$

where

$$A_c(t, \tau) = A(t, \tau) + B(t) \,.\, K(t, \tau)$$

Remark 3. The system of equations (10) does not depend on results of observations and can be solved in advance. The linear form of these equations enables us to use contraction mapping arguments for proving uniqueness of the solution $\pi_k(t, s)$.

Remark 4. From equations (12) and (8) we obtain

$$\xi_k(t) = \int_{-h}^{0} Y_c(t, t_k, \tau) \,.\, \xi_k(t_k + \tau) \, \mathrm{d}\mu_0(\tau) = \xi_{k-1}(t) + \Delta_k(t) \tag{13}$$

where ξ_{k-1} is the unique prolongation on $[t_{k-1}, t_{k+1}]$ of the solution ξ_{k-1} of equation (12) with the initial condition

$$\xi_{k-1}(s) ; \quad s \in [t_{k-1} - h; t_{k-1}]$$

and

$$\Delta_k(t) = \int_{-h}^{0} Y_c(t, t_k, \tau) . \Delta_k(t_k + \tau) \, \mathrm{d}\mu_0(\tau)$$

is the solution on $[t_k; t_{k+1}]$ of (12) with the initial condition $\Delta_k(s)$; $s \in [t_k - k; t_k]$ given by equation (6).

Substituting (13) into (11) we get

$$(14) \quad u(t) = \int_{-h}^{0} K(t, \tau) . \xi_{k-1}(t + \tau) \, \mathrm{d}\mu_0(\tau) + \int_{-h}^{0} K(t, \tau) \, \Delta_k(t + \tau) \, \mathrm{d}\mu_0(\tau) = \\ = u_{k-1}(t) + \delta_k(t)$$

where $u_{k-1}(t)$ would be the optimal control without the new information from observation $z(t_k)$ and $\delta_k(t)$ is the correction caused by this information.

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