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On the Convergence of the Dynamic Stochastic Approximation Method for Stochastic Non-Linear Multidimensional Dynamic Systems

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Some generalization of Dupač's and Katsuji Uosaki dynamic stochastic approximations have been worked out to the multidimensional case. Sufficient conditions for the convergence in the mean square and with probability one to the true state vector of a non-linear stochastic dynamic system in the case where the trend is deterministic are given. Convergence within bound is proved for the random trend case. The estimation seems to be of practical use in optimal control and non-linear filtering.

1. INTRODUCTION

The stochastic approximation method originated by Robbins and Monro in their pioneer paper [5], has been firstly applied by Dupač [2], to the dynamic trend case, where the root or the maximum (minimum) of the regression function moves in a specified but not completely known manner. He discussed in his papers [2], [3] only the cases where the movement of the root (one-dimensional) or the maximum (multidimensional) can be expressed by a certain linear function of its present location, and where the trend is deterministic.

Katsuji Uosaki discussed in his paper [4] some generalization of Dupač's work in the dimensional case only where the movement of the root can be expressed by a specified non-linear function of its present location.

Pearson [6] obtained also convergence conditions for scalar system with non-linear dynamics and linear regression function.

Albert and Gardner [1] provided convergence conditions for scalar system with non-linear dynamics and non-linear regression function.

For the multidimensional case, however, the question of convergence has not been studied yet.

In this paper, we shall be concerned with the dynamic stochastic approximation for non-linear multidimensional dynamic systems. In Section 4, the convergence of the approximations to the moving root of the non-linear regression function in the mean square and with probability one is proved for the case, where the trend is expressed by a certain deterministic non-linear function of the present location. In Section 5, we show also that this procedure makes the mean square error bounded in the case, where the random components are involved in the trend.

Our method of proof differs from that of Katsuji Uosaki and is somewhat closer in spirit to that of Dupač [3].

In this paper all the relations between random variables are meant with probability one.

2. DESCRIPTION OF THE PROCEDURE

Let $\mathbf{R}^k$ be a real k-dimensional vector space. If x and y are two vectors in $\mathbf{R}^k$, we denote their inner product by (x, y) and their norms by $\|x\|$ and $\|y\|$, respectively.

Let $\mathbf{M}_n(x)$, $n = 1, 2, \ldots, x \in \mathbf{R}^k$, be a (unknown) k-vector function. Suppose that the equation

$$\mathbf{M}_n(x) = \alpha \tag{2.1}$$

has a single root θ_n, for $n = 1, 2, \ldots$, which is unknown and is to be estimated. In our case, we assume that the root θ_n moves in such a manner that

$$\theta_{n+1} = g_n(\theta_n) + v_n \tag{2.2}$$

where $g_n(x)$ is in general a non-linear k-vector measurable function (known) defined for all $x \in \mathbf{R}^k$ and v_n in an unknown k-vector (random or non random).

Let $a_n, n = 1, 2, \ldots,$ be positive numbers. Let $\mathbf{X}_1$ be an arbitrary random variable; define for $n = 1, 2, \ldots,$

$$\mathbf{X}_{n+1} = \mathbf{X}_n^* + a_n(\alpha - \mathbf{Y}_n^*)\,, \tag{2.3}$$

where $\mathbf{X}_n^* = g_n(\mathbf{X}_n)$ and $\mathbf{Y}_n^*$ is a random variable, such that

$$\mathbf{Y}_n^* = \mathbf{W}_n + \mathbf{M}_{n+1}(\mathbf{X}_n^*)\,. \tag{2.4}$$

3. CONDITIONS

Conditions on the regression function $\mathbf{M}_n(x)$

M 1: There exist two positive numbers A and B such that

$$\|\mathbf{M}_n(x) - \alpha\| \leqq A\|x - \theta_n\| + B\,.$$

M 2: For all $\delta > 0$, we have

$$\inf_{n \in N} \inf_{\|x-\theta_n\| > \delta} \frac{(x - \theta_n, \mathbf{M}_n(x) - \alpha)}{\|x - \theta_n\|} > 0 ,$$

where $N = 1, 2, \ldots$.

M 3: There exists a number m such that

$$(x - \theta_n, \mathbf{M}_n(x) - \alpha) \geqq m\|x - \theta_n\| \, \|\mathbf{M}_n(x) - \alpha\| ,$$

where $\frac{1}{\sqrt{2}} < m \leqq 1$.

M 4: For all $\delta > 0$

$$\inf_{n \in N} \inf_{\|x-\theta_n\| > \delta} \|\mathbf{M}_n(x) - \alpha\| > 0 .$$

M 5: There exists a number $d_1 > 0$ such that

$$(x - \theta_n, \mathbf{M}_n(x) - \alpha) \geqq \mathrm{d}_1 \|x - \theta_n\|^2 .$$

Conditions on the positive sequence $[a_n]$

A 1:

$$\sum_{n=1}^{\infty} a_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^2 < \infty .$$

Conditions on the function $g_n(x)$

G 1: There exists a sequence of positive numbers $[\gamma_n]$ independent of x and y such that

$$\|g_n(x) - g_n(y)\| \leqq \gamma_n \|x - y\| \quad \text{for all} \quad x, y \in \mathbf{R}^k .$$

G 2:

$$\sum_{n=1}^{\infty} (\gamma_n - 1)^+ < \infty ,$$

where z^+ means $(z + |z|)/2$.

G 3:

$$(\gamma_n - 1)^+ = o(a_n) .$$

G 4: For some $K_5 > 0$ and for all sufficiently large n, we have

$$\gamma_n \leqq 1 - K_5 < 1 .$$

Conditions on v_n

V 1: v_n is non-random and

$$\|v_n\| = o(a_n) .$$

V 2: v_n is a random variable, such that

$$\mathsf{E}(v_n|\mathbf{X}_1, \dots, \mathbf{X}_n, v_1, \dots, v_{n-1}) \leqq v_n^* ,$$
$$\mathsf{E}(\|v_n\|^2|\mathbf{X}_1, \dots, \mathbf{X}_n, v_1, \dots, v_{n-1}) \leqq d^2 < \infty$$

for some constants v_n^*, satisfying

$$v_n^* < 2K_5$$

for sufficiently large n.

Conditions on $\mathbf{W}_n$

W 1:
$$\mathsf{E}(\mathbf{W}_n|\mathbf{X}_1, \dots, \mathbf{X}_n) = 0 ,$$
$$\mathsf{E}(\|\mathbf{W}_n\|^2|\mathbf{X}_1, \dots, \mathbf{X}_n) \leqq \sigma^2 < \infty .$$

W 2:
$$\mathsf{E}(\mathbf{W}_n|\mathbf{X}_1, \dots, \mathbf{X}_n, v_1, \dots, v_n) = 0 ,$$
$$\mathsf{E}(\|\mathbf{W}_n\|^2|\mathbf{X}_1, \dots, \mathbf{X}_n, v_1, \dots, v_n) \leqq \sigma^2 < \infty .$$

Conditions on the initial estimate $\mathbf{X}_1$

B 1:
$$\mathsf{E}\|\mathbf{X}_1 - \theta_1\|^2 < \infty .$$

4. ASYMPTOTIC CONVERGENCE IN THE CASE WHERE THE TREND IS NON-RANDOM

Theorem 4.1. If the conditions **M 1**, **M 2**, **A 1**, **G 1**, **G 2**, **V 1**, **W 1** and **B 1** hold, then

$$\lim_{n\to\infty} \|\mathbf{X}_n - \theta_n\| = 0 \tag{4.1}$$

with probability one and

$$\lim_{n\to\infty} \mathsf{E}\|\mathbf{X}_n - \theta_n\|^2 = 0 . \tag{4.2}$$

Proof. The theorem can be proved by invoking Dvoretzky theorems generalized by Venter [25].

From (2.3) and (2.4) we have

$$\mathbf{X}_{n+1} - \theta_{n+1} = g_n(\mathbf{X}_n) - g_n(\theta_n) - v_n - a_n(\mathbf{M}_{n+1}(g_n(\mathbf{X}_n)) - \alpha) - a_n\mathbf{W}_n .$$

Let us define

$$\mathbf{T}_n = g_n(\mathbf{X}_n) - g_n(\theta_n) - v_n - a_n(\mathbf{M}_{n+1}(g_n(\mathbf{X}_n)) - \alpha) , \tag{4.3}$$

$$\mathbf{Z}_n = a_n\mathbf{W}_n . \tag{4.4}$$

From **W 1** we have

$$\text{(4.5)} \qquad \mathsf{E}(\mathbf{Z}_n | \mathbf{X}_1, \ldots, \mathbf{X}_n) = 0 \,,$$
$$\mathsf{E}(\|\mathbf{Z}_n\|^2 | \mathbf{X}_1, \ldots, \mathbf{X}_n) = a_n^2 \, \mathsf{E}(\|\mathbf{W}_n\|^2 | \mathbf{X}_1, \ldots, \mathbf{X}_n) \leqq a_n^2 \sigma^2 \,,$$

then from **A 1** we conclude that

$$\text{(4.6)} \qquad \sum_{n=1}^{\infty} \mathsf{E}(\|\mathbf{Z}_n\|^2 | \mathbf{X}_1, \ldots, \mathbf{X}_n) < \infty \,.$$

From **M 2**, it follows that for every sequence $\varrho_n > 0$, $\varrho_n \to 0$, bounded by sufficiently small number, there exists a sequence $\eta_n > 0$, $\eta_n \to 0$ such that

$$\text{(4.7)} \qquad \frac{(x - \theta_n, \mathbf{M}_n(x) - \alpha)}{\|x - \theta_n\|} > \varrho_n$$

for all $\|x - \theta_n\| > \eta_n$, $n = 1, 2, \ldots$.

Let us choose ϱ_n such that

$$\text{(4.8)} \qquad \sum_{n=1}^{\infty} a_n \varrho_{n+1} = \infty \,, \quad \|v_n\| = o(a_n \varrho_{n+1}) \,.$$

Let the corresponding η_n be chosen in such a way that they satisfy

$$\text{(4.9)} \qquad \frac{a_n}{\varrho_{n+1}} = o(\eta_n) \,.$$

Thus for $\|g_n(\mathbf{X}_n) - g_n(\theta_n) - v_n\| \leqq \eta_n$ the equality (4.3) can be written as

$$\|\mathbf{T}_n\| \leqq \|g_n(\mathbf{X}_n) - g_n(\theta_n) - v_n\| + a_n \|\mathbf{M}_{n+1}(g_n(\mathbf{X}_n)) - \alpha\| \,.$$

Using **M 1**, we obtain

$$\|\mathbf{T}_n\| \leqq \|g_n(\mathbf{X}_n) - g_n(\theta_n) - v_n\| \, (1 + a_n A) + a_n B \,;$$

thus for all $n \geqq N_1$

$$\text{(4.10)} \qquad \|\mathbf{T}_n\| \leqq 2\eta_n \,.$$

For $\|g_n(\mathbf{X}_n) - g_n(\theta_n) - v_n\| > \eta_n$, let us define

$$\text{(4.11)} \qquad \mathbf{G}_n = g_n(\mathbf{X}_n) - g_n(\theta_n) - v_n \,,$$

$$\text{(4.12)} \qquad \mathbf{F}_n = \mathbf{M}_{n+1}(g_n(\mathbf{X}_n)) - \alpha \,,$$

$$\text{(4.13)} \qquad \cos \psi_n = \frac{(\mathbf{F}_n, \mathbf{G}_n)}{\|\mathbf{F}_n\| \, \|\mathbf{G}_n\|} \,.$$

Hence the equality (4.3) can be written as

$$\|T_n\|^2 = (\|G_n\| - a_n\|F_n\| \cos\psi_n)^2 + a_n^2\|F_n\|^2 \sin^2\psi_n \leqq$$
$$\leqq (\|G_n\| - a_n\|F_n\| \cos\psi_n)^2 + a_n^2\|F_n\|^2 =$$
$$= (\|G_n\| - a_n\|F_n\| \cos\psi_n)^2 \left(1 + \frac{a_n^2\|F_n\|^2}{(\|G_n\| - a_n\|F_n\| \cos\psi_n)^2}\right).$$

From **M 1** and (4.9) $\|G_n\| > a_n\|F_n\|$ for $n \geqq N_2$ hence

$$\|T_n\| \leqq \|G_n\| - a_n\|F_n\| \cos\psi_n + \frac{1}{2}\frac{a_n^2\|F_n\|^2}{\|G_n\| - a_n\|F_n\| \cos\psi_n} \leqq$$
$$\leqq \|G_n\| - a_n\|F_n\| \cos\psi_n + \frac{1}{2}\frac{a_n^2\|F_n\|^2}{\|G_n\|}\left(1 + 2a_n\frac{\|F_n\|}{\|G_n\|}\cos\psi_n\right) \leqq$$
$$\leqq \|G_n\| - a_n\|F_n\| \cos\psi_n + \frac{1}{2}\frac{a_n^2\|F_n\|^2}{\|G_n\|}\left(1 + 2Aa_n + \frac{2Ba_n}{\|G_n\|}\right) \leqq$$
$$\leqq \|G_n\| - a_n\|F_n\| \cos\psi_n + a_n^2\left(A^2\|G_n\| + \frac{B^2}{\|G_n\|}\right)\left(1 + 2a_nA + \frac{2Ba_n}{\|G_n\|}\right).$$

Using (4.9), we obtain

$$\|T_n\| \leqq \|G_n\| - a_n\|F_n\| \cos\psi_n + 2a_n^2\left(A^2\|G_n\| + \frac{B^2}{\|G_n\|}\right) = \tag{4.14}$$
$$= \|G_n\|(1 + 2A^2a_n^2) - a_n\left(\frac{(F_n, G_n)}{\|G_n\|} - \frac{2a_nB^2}{\|G_n\|}\right) \leqq$$
$$\leqq \|G_n\|(1 + 2A^2a_n^2) - a_n\varrho_{n+1}\left(1 - \frac{a_n}{\varrho_{n+1}}\frac{2B^2}{\eta_n}\right) \leqq$$
$$\leqq \|G_n\|(1 + 2A^2a_n^2) - \tfrac{1}{2}a_n\varrho_{n+1} \leqq$$
$$\leqq \gamma_n(1 + 2A^2a_n^2)\|X_n - \theta_n\| - (\tfrac{1}{2}a_n\varrho_{n+1} - (1 + 2A^2a_n^2)\|v_n\|)$$
$$\text{for} \quad n > N_3 .$$

Using the 2-nd part of (4.8), we can set

$$\|T_n\| \leqq (1 + \beta_n)\|X_n - \theta_n\| - \frac{a_n\varrho_{n+1}}{4},$$

where $\beta_n = O((\gamma_n - 1)^+ + a_n^2)$.

From **A 1** and **G 2**

$$\sum_{n=1}^{\infty}\beta_n < \infty .$$

Thus from (4.5), (4.6), (4.10), (4.14) and the first part of (4.8) it follows that the conditions of Dvoretzky theorem are satisfied, completing the proof of the theorem.

If the condition **M 2** in Theorem 4.1 is replaced by **M 3** and **M 4**, then the proof can be somehow simplified. Theorem 4.2 shows this.

Theorem 4.2. If the conditions **M 1**, **M 3**, **M 4**, **A 1**, **G 1**, **G 2**, **V 1**, **W 1** and **B 1** hold, then (4.1) and (4.2) follow.

Proof. Define $\boldsymbol{T}_n$, $\boldsymbol{G}_n$, $\boldsymbol{F}_n$, $\boldsymbol{Z}_n$ and ψ_n as in proving the Theorem 4.1.
From **M 4**, it follows that for every sequence $\varrho_n > 0$, $\varrho_n \to 0$, bounded by sufficiently small number, there exists a sequence $\eta_n > 0$, $\eta_n \to 0$, such that

$$\|\boldsymbol{M}_n(x) - \alpha\| > \varrho_n \quad \text{for all} \quad \|x - \theta_n\| > \eta_n\,, \quad n = 1, 2, \ldots \tag{4.15}$$

Let us choose ϱ_n, such that

$$\sum_{n=1}^{\infty} a_n \varrho_{n+1} = \infty\,, \tag{4.16}$$

$$\|v_n\| = o(a_n \varrho_{n+1})\,. \tag{4.17}$$

Let the corresponding η_n be chosen such that

$$a_n = o(\eta_n)\,. \tag{4.18}$$

For $\|\boldsymbol{G}_n\| \leqq \eta_n$ we have as in proving Theorem 4.1

$$\|\boldsymbol{T}_n\| \leqq 2\eta_n\,. \tag{4.19}$$

For $\|\boldsymbol{G}_n\| > \eta_n$ we can make the following estimate for $\|\boldsymbol{T}_n\|$ in (4.3)

$$\|\boldsymbol{T}_n\| \leqq \big|\, \|\boldsymbol{G}_n\| - a_n\|\boldsymbol{F}_n\| \cos \psi_n \big| + a_n\|\boldsymbol{F}_n\|\, |\sin \psi_n|\,.$$

Using **M 1** and (4.18), we conclude that $\|\boldsymbol{G}_n\| > a_n\|\boldsymbol{F}_n\|$ for large n, therefore, using **M 3** and **G 1**, we can set

$$\begin{aligned}
\|\boldsymbol{T}_n\| &\leqq \|\boldsymbol{G}_n\| - a_n\|\boldsymbol{F}_n\| \cos \psi_n + a_n\|\boldsymbol{F}_n\|\, |\sin \psi_n| = \\
&= \|\boldsymbol{G}_n\| - a_n\|\boldsymbol{F}_n\| \left(\cos \psi_n - (1 - \cos^2 \psi_n)^{1/2}\right) \leqq \\
&\leqq \|\boldsymbol{G}_n\| - a_n\|\boldsymbol{F}_n\| \left(m - (1 - m^2)^{1/2}\right) \leqq \\
&\leqq \|g_n(\boldsymbol{X}_n) - g_n(\theta_n)\| + \|v_n\| - a_n\|\boldsymbol{F}_n\| \left(m - (1 - m^2)^{1/2}\right) \leqq \\
&\leqq \gamma_n\|\boldsymbol{X}_n - \theta_n\| + \|v_n\| - a_n\|\boldsymbol{F}_n\| \left(m - (1 - m^2)^{1/2}\right) \leqq \\
&\leqq [1 + (\gamma_n - 1)^+]\, \|\boldsymbol{X}_n - \theta_n\| + \|v_n\| - a_n\|\boldsymbol{F}_n\| \left(m - (1 - m^2)^{1/2}\right).
\end{aligned}$$

Using (4.17) we can write

$$\|\boldsymbol{T}_n\| \leqq (1 + (\gamma_n - 1)^+)\, \|\boldsymbol{X}_n - \theta_n\| - a_n \varrho_{n+1}\left(m - \delta_n - (1 - m^2)^{1/2}\right), \tag{4.20}$$

where $\delta_n \to 0$ for $n \to \infty$.

For large n and using **M 3**, we have $m - \delta_n \geqq 1/\sqrt{2}$, hence, using (4.16) it follows that

$$\sum_{n=1}^{\infty} a_n \varrho_{n+1}\left(m - \delta_n - (1 - m^2)^{1/2}\right) = \infty .$$

Thus from (4.5), (4.6), (4.19), (4.20) and **G 2**, the conditions of Dvoretzky theorem are satisfied, completing proof of the theorem.

Further, if the condition **M 2** is replaced by **M 5**, then **G 2** can be weakened to **G 3**. Theorem 4.3 shows this.

Theorem 4.3. If **M 1**, **M 5**, **A 1**, **G 1**, **G 3**, **V 1**, **W 1** and **B 1** hold, then (4.1) and (4.2) follow.

Proof. Define $\boldsymbol{T}_n$, $\boldsymbol{F}_n$, $\boldsymbol{G}_n$ and $\boldsymbol{Z}_n$ as in proving Theorem 4.1.
Choose $\eta_n > 0$, $\eta_n \to 0$ for $n \to \infty$, such that

$$a_n = o(\eta_n^2) . \tag{4.21}$$

For $\|\boldsymbol{G}_n\| \leqq \eta_n$, we have as before in proving Theorem 4.1.

$$\|\boldsymbol{T}_n\| \leqq 2\eta_n , \tag{4.22}$$

whereas for $\|\boldsymbol{T}_n\| > 2\eta_n$ we proceed as follows:

Using **M 1** and **M 5**, the equality (4.3) can be written as

$$\|\boldsymbol{T}_n\|^2 = \|\boldsymbol{G}_n\|^2 - 2a_n(\boldsymbol{F}_n, \boldsymbol{G}_n) + a_n^2\|\boldsymbol{F}_n\|^2 \leqq$$
$$\leqq \|\boldsymbol{G}_n\|^2 - 2a_n d_1 \|\boldsymbol{G}_n\|^2 + 2a_n^2(A^2\|\boldsymbol{G}_n\|^2 + B^2) =$$
$$= \|\boldsymbol{G}_n\|^2 \left(1 - 2a_n d_1 + 2A^2 a_n^2 + \frac{2B^2 a_n^2}{\|\boldsymbol{G}_n\|^2}\right).$$

Using (4.21), for large n, we can set

$$\|\boldsymbol{T}_n\|^2 \leqq \|\boldsymbol{G}_n\|^2 (1 - a_n d_1) .$$

Then

$$\|\boldsymbol{T}_n\| \leqq \|\boldsymbol{G}_n\| (1 - \tfrac{1}{2}a_n d_1) \leqq \tag{4.23}$$
$$\leqq \|g_n(\boldsymbol{X}_n) - g_n(\theta_n)\| (1 - \tfrac{1}{2}a_n d_1) + \|v_n\| (1 - \tfrac{1}{2}a_n d_1) \leqq$$
$$\leqq [1 + (\gamma_n - 1)^+] \|\boldsymbol{X}_n - \theta_n\| (1 - \tfrac{1}{2}a_n d_1) + \|v_n\| (1 - \tfrac{1}{2}a_n d_1) .$$

Denoting

$$\beta_n = (\gamma_n - 1)^+ ,$$

we obtain

$$\|T_n\| \leqq (1 + \beta_n - \tfrac{1}{2}a_n d_1) \|X_n - \theta_n\| + \|v_n\| (1 - \tfrac{1}{2}a_n d_1) .$$

Using **G 3**, we obtain

$$\|T_n\| \leqq \|X_n - \theta_n\| - \frac{a_n d_1}{4} \|X_n - \theta_n\| + \|v_n\| .$$

Let us choose a sequence $\varrho_n < 0$, $\varrho_n \to 0$, such that

$$\sum a_n \varrho_n = \infty , \tag{4.24}$$

$$\|v_n\| = o(a_n \varrho_n) . \tag{4.25}$$

Thus for $\|X_n - \theta_n\| \leqq \varrho_n$, we have

$$\|T_n\| \leqq \varrho_n + \|v_n\| \leqq 2\varrho_n \tag{4.26}$$

and for $\|X_n - \theta_n\| > \varrho_n$, we have

$$\|T_n\| \leqq \|X_n - \theta_n\| + \|v_n\| - \frac{a_n d_1}{4} \|X_n - \theta_n\| . \tag{4.27}$$

Using (4.25), the inequality (4.27) can be written as

$$\|T_n\| \leqq \|X_n - \theta_n\| - \frac{a_n \varrho_n}{8} d_1 , \quad \text{for large } n . \tag{4.28}$$

Hence, from (4.22), (4.26) and (4.28), we have

$$\|T_n\| \leqq \max \left(2\eta_{n^*}, 2\varrho_n, \|X_n\| - \theta_n\| - \frac{a_n \varrho_n d_1}{8} \right) . \tag{4.29}$$

Using (4.24) and from (4.5), (4.6), the conditions of Dvoretzky theorem are satisfied, which completes the proof of the theorem.

5. CONVERGENCE OF THE ESTIMATION ERROR WITHIN BOUND

In the case, where the unknown fluctuation v_n is random, the estimation error is allowed to be bounded. Theorem 5.1 shows this case.

Theorem 5.1. If **M 1**, **M 5**, **A 1**, **G 1**, **G 4**, **V 2**, **W 2** and **B 1** hold, then

$$\lim_{n \to \infty} \mathsf{E}\|X_n - \theta_n\|^2 < C , \tag{5.1}$$

where C is a finite constant.

The proof will be omitted, as its one-dimensional version, given by Uosaki [4], can be applied to the multidimensional case as well, with obvious modifications. Let us only point out, that the correct conditioning of expectations is by *both* the X_i's and v_i's, as we did in our **V 2** and **W 2**, not by the X_i's only, as in [4]. Uosaki's assumption on conditional orthogonality of v_n and W_n is then not needed, as it is implied by our **W 2**.

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