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On Dimensioning of Samples in Testing Hypotheses

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The aim of this paper is to find upper bounds for the number of independent observations, which are necessary in order to test the probability measure P against Q with given error probabilities. Geometrical considerations concerning the risk set of the testing problem lead to such bounds. A further bound is obtained by use of the central limit theorem. An example shows the applicability of the results.

1. INTRODUCTION

Let P and Q be two probability measures on a measurable space $(\Omega, \mathfrak{A})$ and $(\alpha, \beta) : 0 < \alpha, \beta; \alpha + \beta < 1$ a level vector.

Let us consider n independent identical (P resp. Q) distributed observations in order to test P against Q. I.e. let us consider the testproblems (P^n, Q^n), $n \in \mathbb{N}$.

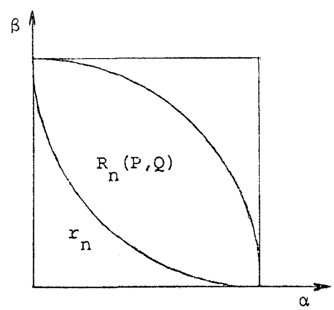

Fig. 1.

Then our interest is concentrated on the number of observations, which are necessary to bound the error probabilities of the first resp. the second kind from above by (α, β).

The most instructive formalization of this problem is in terms of the risk set (cf. [1]).

Definition 1. $R_n(\mathsf{P}, \mathsf{Q}) = R_1(\mathsf{P}^n, \mathsf{Q}^n) := \mathrm{co}\,\{(\mathsf{P}^n(A), 1 - \mathsf{Q}^n(A)) : A \in \bigotimes_{i=1}^{n} \mathfrak{A}\}$ is called risk set of the test problem $(\mathsf{P}^n, \mathsf{Q}^n)$. ("co" stands for convex hull.)

The lower boundary of this set, to be understood as a function of the level α,

$$r_n(\alpha) := \min\,\{y : (\alpha, y) \in R_n(\mathsf{P}, \mathsf{Q})\}$$

is called n-th risk function (see Fig. 1).

Remark 1. $r_n(\alpha)$ is the error probability of the second kind for an optimal test. In the case of strict convexity of $r_n(\alpha)$ in α

$$r_n(\alpha) = 1 - \mathsf{Q}^n(A_k)\,,$$

where the optimal test is characterized by

$$A_k = \{(\omega_1, \ldots, \omega_n) : \prod_{i=1}^{n} q(\omega_i) > k \prod_{i=1}^{n} p(\omega_i)\}\,,$$

where $k = D_+\, r_n(\alpha)$. (D_+ representing the absolute value of the right-hand-side derivative and p, q the Radon-Nikodym derivatives of P resp. Q with respect to a dominating σ-finite measure μ.)

Remark 2. For us the most interesting properties of the (convex) risk function are:

$$1 - \alpha \geqq r_n(\alpha) \geqq 0 \quad \forall \alpha \in [0, 1]\,, \quad \forall n \in \mathbb{N}\,,$$

where – for fixed n – equality holds true in the first case for one $\alpha \in (0, 1)$ (and hence for all $\alpha \in [0, 1]$)

$$\text{iff} \quad \mathsf{P} = \mathsf{Q}$$

and in the second – for fixed n – for all $\alpha \in [0, 1]$

$$\text{iff} \quad \mathsf{P} \perp \mathsf{Q}\,.$$

Furthermore $r_n(\alpha) \downarrow 0$ $\forall \alpha \in (0, 1]$ iff $\mathsf{P} \neq \mathsf{Q}$.

Now the sample-size in question can be expressed by

$$N_{\alpha,\beta} = \min\,\{n : r_n(\alpha) \leqq \beta\}\,.$$

Geometric properties of the risk set and standard estimations are basic for a (rough) lower and upper for $N_{\alpha,\beta}$.

Theorem 1.

$$U_{\alpha,\beta}^{(1)} := \max\left(1, \left[\!\left[\frac{\ln\left(\max\left(\frac{1-\beta}{\alpha}, \frac{1-\alpha}{\beta}\right)\right)}{\ln\frac{1}{d}}\right]\!\right]\right)$$

is an upper bound and

$$L_{\alpha,\beta} := \max\left(1, \left[\!\left[\frac{\ln\frac{1}{\alpha+\beta}}{\ln\frac{1}{b_1}}\right]\!\right]\right)$$

a lower bound for $N_{\alpha,\beta}$.
Thereby:

$$d := \min\{H_\gamma(\mathsf{P}, \mathsf{Q}),\ \gamma \in [0, 1]\}$$

$$H_\gamma(\mathsf{P}, \mathsf{Q}) = \int_{\{p.q>0\}} p^\gamma . q^{1-\gamma}\, d\mu \quad \text{and}$$

$$b_k = \int \min(k . p, q)\, d\mu$$

($[\![x]\!]$ marks the smallest integer greater or equal to x).

Proof. Let us ignore the trivial cases $\mathsf{P} = \mathsf{Q}$ and $\mathsf{P} \perp \mathsf{Q}$.
Twice the Bayes risk with respect to the prior distribution $(\frac{1}{2}, \frac{1}{2})$ is

$$\text{(1)} \qquad \inf\{(\mathsf{P}(A) + 1 - \mathsf{Q}(A)),\ A \in \mathfrak{A}\} = 1 - \int_{\{q>p\}} (q - p)\, d\mu =$$

$$= \int \min(p, q)\, d\mu \leqq \int_{\{p.q>0\}} p^\gamma . q^{1-\gamma}\, d\mu .$$

The latter follows from the inequality

$$\min(a, b) \leqq a^\gamma . b^{1-\gamma} \quad a, b \geqq 0, \quad \gamma \in [0, 1] .$$

Thus

$$b_1(\mathsf{P}^n, \mathsf{Q}^n) := \int \min\left(\prod_{i=1}^n p(\omega_i), \prod_{i=1}^n q(\omega_i)\right) d\mu^n(\omega_1, \ldots, \omega_n) \leqq$$

$$\leqq \min\{H_\gamma(\mathsf{P}^n, \mathsf{Q}^n),\ \gamma \in [0, 1]\} = d^n$$

(see Fig. 2). Now taking also into account that

$$b_1(\mathsf{P}^n, \mathsf{Q}^n) - \alpha \ (\leqq d^n - \alpha)$$

 is supporting line for $r_n(\alpha)$ (cf. Remark 1 and (1)) and the convexity of $r_n(\alpha)$ we derive

$$\max\left(1 - \left(\frac{1}{d}\right)^n \alpha,\ d^n - d^n\alpha\right) \geqq r_n(\alpha) .$$

Now the upper bound is an immediate consequence.

$$\min(a \,.\, b,\ c \,.\, d) \geqq \min(a, c) \,.\, \min(b, d) \quad a, b, c, d \geqq 0$$

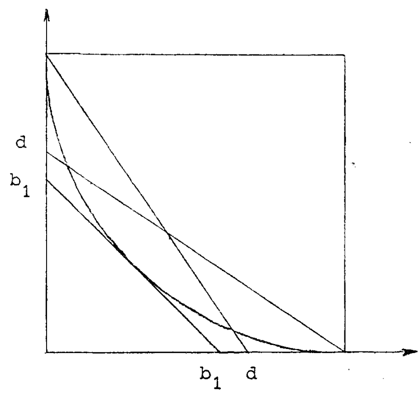

Fig. 2.

implies $b_1(P^n, Q^n) \geqq b_1^n$. Thus $b_1^n - \alpha \leqq b_1(P^n, Q^n) - \alpha \leqq r_n(\alpha)$. The latter inequality is again due to the fact that

$$b_1(P^n, Q^n) - \alpha$$

is supporting line for $r_n(\alpha)$. Therefore

$$L_{\alpha,\beta} = \min\{n : b_1^n - \alpha \leqq \beta\}$$

is a lower bound for $N_{\alpha,\beta}$.

Remark 3. In the same way one can derive the sharper lower bound

$$\max_{k \geqq 0} \min\{n : b_k^n - k^n\alpha \leqq \beta\} .$$

The difficulty of its computation, however, causes that this bound is of less importance.

An essential improvement of the upper bound $U_{\alpha,\beta}^{(1)}$ of $N_{\alpha,\beta}$ is

$$U_{\alpha,\beta}^{(2)} := \max(1, [\![\hat{U}_{\alpha,\beta}]\!]) ,$$

where

$$\hat{U}_{\alpha,\beta} := \min_{\gamma\in[0,1]} \frac{\gamma \ln \alpha^{-1} + (1-\gamma)\ln \beta^{-1} - S(\gamma)}{\ln \dfrac{1}{H_\gamma(\mathsf{P}, \mathsf{Q})}}$$

and $S(\gamma) = -(\gamma \ln \gamma + (1-\gamma)\ln(1-\gamma))$ is the entropy of the auxiliary distribution $(\gamma, 1-\gamma)$.

This bound is based on the convexity of the risk function, which therefore can be understood as the envelope of its supporting lines.

In applying

$$b_k \leqq k^\gamma H_\gamma(\mathsf{P}, \mathsf{Q})$$

these supporting lines are replaced by parallel auxiliary lines, lying above the former.

Because of $H_\gamma(\mathsf{P}^n, \mathsf{Q}^n) = H_\gamma^n(\mathsf{P}, \mathsf{Q})$ the envelope of the auxiliary lines, which is bounding $r_n(\alpha)$ from above, is much easier to handle than the risk function (for detail cf. [3]).

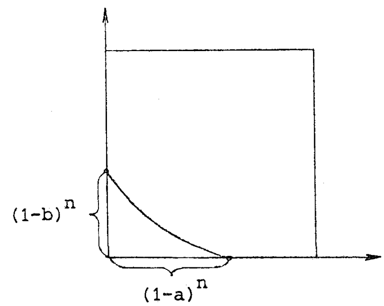

Fig. 3.

2. AN UPPER BOUND FOR "GAUSS-NEAR" DISTRIBUTIONS

A further upper bound makes use of the central limit theorem. Therefore we restrict our interest to the real line (i.e. $\Omega = \mathbb{R}$, $\mathfrak{A} = \mathscr{B}_1$).

Furthermore, singularities of the measures P and Q can be excluded from the following because of

Remark 1. In the case of singularities of P and Q, i.e. for

$$A_\infty = \{p = 0,\ q > 0\}\,,\quad A_0^c = \{p > 0,\ q = 0\}\,,$$
$$b = \mathsf{Q}(A_\infty) > 0 \quad \text{and/or} \quad a = \mathsf{P}(A_0^c) > 0\,,$$

the risk function $r_n(\alpha)$ is of a form as sketched in Fig. 3.

Transformation of this function by

$$(\alpha, \beta) \to \left(\frac{\alpha}{(1-a)^n}, \frac{\beta}{(1-b)^n}\right)$$

(the trivial case $\mathsf{P} \perp \mathsf{Q}$ can be excluded) results in the n-th risk function of the test problem of the conditional distributions $\mathsf{P}(\cdot \mid A_\infty^c \cap A_0)$ and $\mathsf{Q}(\cdot \mid A_\infty^{c\,\prime} \cap A_0)$. Therefore the problem can be reduced to a test problem of distributions without singularities.

In this section we will choose our tests in terms of the statistic

$$T_n(x_1, \ldots, x_n) = \frac{\sum_{i=1}^{n} x_i}{n}.$$

Remark 2. In general we loose information this way, which means: the tests based on T_n are not best tests or equivalently

$$R(\mathsf{P}^n, \mathsf{Q}^n) \supset R(\mathsf{P}^n(T_n), \mathsf{Q}^n(T_n)),$$

except: T_n is a sufficient statistic with respect to (P, Q) or equivalently $R(\mathsf{P}^n, \mathsf{Q}^n) = R(\mathsf{P}^n(T_n), \mathsf{Q}^n(T_n))$. The latter can be seen from

$$\left\{\prod_{i=1}^{n} \frac{q}{p}(x_i) > k\right\} = \left\{\sum_{i=1}^{n} \ln \frac{q}{p}(x_i) > \ln k\right\} = \{c_1(n)\, T_n(x_1, \ldots, x_n) + c_2(n) > \ln k\}$$

taking into account the definition of sufficiency of T_n and the fact that a best test is of the form

$$\left\{\prod_{i=1}^{n} \frac{q}{p}(x_i) > k\right\}.$$

From the first equality in the above chain it can also be seen that one does not loose any information with the statistic

$$\hat{T}_n(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} \ln \frac{q}{p}(x_i)$$

(i.e. $R(\mathsf{P}^n, \mathsf{Q}^n) = R(\mathsf{P}^n(\hat{T}_n), \mathsf{Q}^n(\hat{T}_n))$).

Furthermore the test problem $(\mathsf{P}^n(\hat{T}_n), \mathsf{Q}^n(\hat{T}_n))$ has monotone (increasing) likelihood ratio and

$$\mathsf{E}_\mathsf{P}(\hat{T}_n) < 0 < \mathsf{E}_\mathsf{Q}(\hat{T}_n).$$

Thus it is sometimes convenient to replace the test problem (P, Q) by $(\mathsf{P}(\hat{T}_1), \mathsf{Q}(\hat{T}_1))$.

Remark 3. It is obvious, that tests based on the statistic T_n are powerful only in the case, when

$$T_n \text{ is “near” sufficiency .}$$

In the latter case, however, best tests are of the form

$$\{T_n > t\}\,, \quad t \in \mathbb{R}\,.$$

Taking into consideration that the transformation

$$T_n \to T_n - \xi$$

does not cause any change of the corresponding risk sets, we will consider the problem under the

Assumptions (A):

$$\Omega = S \subset \mathbb{R}\,, \quad \mathfrak{A} = \mathscr{S} \subset S \cap \mathscr{B}_1\,,$$

P and Q are equivalent (i.e. mutual absolute continuous) probability measures on $(S, \mathscr{S})$ with the means $\xi = 0$, $\eta > 0$ and the variances $0 < \sigma^2 < \infty$, $0 < \tau^2 < \infty$.

Parallel with P and Q we consider the Gaussian distributions $\mathsf{G}(\xi, \sigma^2)$ and $\mathsf{G}(\eta, \tau^2)$.

Applying the central limit theorem we have

$$\left.\begin{aligned} \mathsf{P}^n\left(\frac{T_n - \xi}{\sigma/\sqrt{n}}\right) &\to \mathsf{G}(0,1) \\ \mathsf{Q}^n\left(\frac{T_n - \eta}{\tau/\sqrt{n}}\right) &\to \mathsf{G}(0,1) \end{aligned}\right\} \text{ weakly}\,.$$

More precisely, we make use of a *Berry-Esseen*-type result, derived by *Zolotarev* in [4] (which is expressed here for P):

$$(2) \qquad \sup_{x \in R} \left|F_{\mathsf{P}^n(T_n)}(x) - F_{\mathsf{G}(\xi,\sigma^2/n)}(x)\right| = \sup_{x \in R} \left|F_n(x) - \Phi(x)\right| \leqq c \,.\, \mu_n(P) \,.\, n^{-1/2}$$

where $\mu_n(\mathsf{P}) = \min\left(\max\left(\varkappa, \varkappa^{n/(3n+1)}\right), \max\left(\varkappa_0, \varkappa_0^{n/(n+1)}\right), v_0\right)$ with

$$\varkappa = 3\int x^2 |F(x) - \Phi(x)|\,\mathrm{d}x\,,$$

$$\varkappa_0 = \int \max(1, 3x^2)\,|F(x) - \Phi(x)|\,\mathrm{d}x\,,$$

$$v_0 = \int \max(1, |x|^3)\,|\mathrm{d}(F - \Phi)|\,,$$

$$c = 1{\cdot}88\,,$$

$F_{\mathsf{P}^n(T_n)}, F_{\mathsf{G}(\xi,\sigma^2/n)}$	denoting the distribution-function of	$\mathsf{P}^n(T_n), \mathsf{G}\left(\xi, \frac{\sigma^2}{n}\right)$
F_n, F, Φ		$\mathsf{P}^n\left(\frac{T_n - \xi}{\sigma/\sqrt{n}}\right), \quad \frac{\chi - \xi}{\sigma}, \quad \mathsf{G}(0, 1)$.

Our strategy in getting an upper bound for the number of observations is explained in the proof of the following lemma where we use the abbreviations:

$\mathsf{P}_n, \mathsf{Q}_n$	denotes	$\mathsf{P}^n(T_n), \mathsf{Q}^n(T_n)$
$\mathsf{G}_{\mathsf{P},n}, \mathsf{G}_{\mathsf{Q},n}$		$\mathsf{G}\left(0, \frac{\sigma^2}{n}\right), \mathsf{G}\left(\eta, \frac{\tau^2}{n}\right)$
ε_n, δ_n		$c \,.\, \mu_n(\mathsf{P}) \,.\, n^{-1/2}, \; c \,.\, \mu_n(\mathsf{Q}) \,.\, n^{-1/2}$

and $r_{\sigma^2,\tau^2,n}(\alpha)$ denotes the risk function of the test problem $(\mathsf{G}(0, \sigma^2/n), \mathsf{G}(\eta, \tau^2/n))$.

Lemma 1. Under the assumptions (A) and the above notation-conveniences the following functions are upper bounds for the risk function of the test problem $(\mathsf{P}^n, \mathsf{Q}^n)$:

(I) $r_{\sigma^2,\sigma^2,n}(\alpha - \varepsilon_n) + \delta_n \quad \forall \alpha \in [\varepsilon_n, 1]$ in the case $\sigma^2 = \tau^2$.

(IIA) $r_{\tau^2,\tau^2,n}(\alpha - \varepsilon_n) + \delta_n \quad \forall \alpha \in [\varepsilon_n, \frac{1}{2} + \varepsilon_n]$

(IIB) $r_{\sigma^2,\tau^2,n}(\alpha - 2\varepsilon_n) + 2\delta_n \quad \forall \alpha \in [2\varepsilon_n, 1]$

(IIA), (IIB): in the case $\sigma^2 < \tau^2$

Proof. First we consider tests of the form

$$1_{\{T_n > t\}}, \quad t \in \mathbb{R}.$$

Taking into account (2) and the analogue for Q we have:

$$\mathsf{P}_n((t, \infty)) \leqq \mathsf{G}_{\mathsf{P},n}((t, \infty)) + \varepsilon_n \quad \text{and}$$
$$\mathsf{Q}_n((-\infty, t]) \leqq \mathsf{G}_{\mathsf{Q},n}((-\infty, t]) + \delta_n$$

and after standardization of $\mathsf{G}_{\mathsf{P},n}$ and $\mathsf{G}_{\mathsf{Q},n}$

$$\alpha' = \mathsf{P}_n((t, \infty)) \leqq 1 - \Phi\left(\frac{t}{\sigma/\sqrt{n}}\right) + \varepsilon_n = \alpha, \tag{3}$$

$$\mathsf{Q}_n((-\infty, t]) \leqq \Phi\left(\frac{t - \eta}{\tau/\sqrt{n}}\right) + \delta_n. \tag{4}$$

In case I ($\sigma^2 = \tau^2$): $\{(1 - \Phi(t/\sigma/\sqrt{n}), \Phi((t - \eta)/\sigma/\sqrt{n})), t \in \mathbb{R}\}$ is already the graph of the risk function $r_{\sigma^2,\sigma^2,n}(\alpha)$ of the test problem $(\mathsf{G}(0, \sigma^2/n), \mathsf{G}(\eta, \sigma^2/n))$.

Therefore $\{(1 - \Phi(t/\sigma/\sqrt{n}) + \varepsilon_n, \Phi((t - \eta)/\sigma/\sqrt{n}) + \delta_n), t \in \mathbb{R}\}$ can be described by

$$r_{\sigma^2,\sigma^2,n}(\alpha - \varepsilon_n) + \delta_n .$$

$\alpha' \leqq \alpha$, the fact that a risk function is decreasing and (4) imply

$$\mathsf{Q}_n((-\infty, t]) \leqq r_{\sigma^2,\sigma^2,n}(\alpha' - \varepsilon_n) + \delta_n .$$

The remainder

$$r_{\mathsf{P}_n,\mathsf{Q}_n}(\alpha') \leqq \mathsf{Q}_n(-\infty, t])$$

is caused by the fact that $1_{\{T_n > t\}}$ is in general not optimal.

In case IIA we have

$$\frac{t}{\sigma/\sqrt{n}} \geqq \frac{t}{\tau/\sqrt{n}}$$

and hence

$$1 - \Phi\left(\frac{t}{\sigma/\sqrt{n}}\right) \leqq 1 - \Phi\left(\frac{t}{\tau/\sqrt{n}}\right) \quad \forall t \geqq 0 .$$

Thus from (3) we get

$$\mathsf{P}_n((t, \infty)) \leqq 1 - \Phi\left(\frac{t}{\tau/\sqrt{n}}\right) + \varepsilon_n \quad \forall t \geqq 0 .$$

Starting from this inequality and using the same considerations as in case I we get the result IIA.

In case IIB we use tests of the form

$$1_{\{s < T_n \leqq t\}^c} \quad s < t ; \quad s, t \in \mathbb{R}$$

(which are optimal for the test problem $(\mathsf{G}_{\mathsf{P},n}, \mathsf{G}_{\mathsf{Q},n})$ $(\sigma^2 < \tau^2)$). Applying (2) we derive:

$$\mathsf{P}_n((s, t]^c) \leqq \mathsf{G}_{\mathsf{P},n}((s, t]^c) + 2\varepsilon_n \quad \text{and}$$

$$\mathsf{Q}_n((s, t]) \leqq \mathsf{G}_{\mathsf{Q},n}((s, t]) + 2\delta_n .$$

The remainder of the proof is the same as in case I.

The resulting bound is expressed in

Theorem 1. Under the assumptions (A) $U^{(3)}_{\alpha,\beta}$ is an upper bound for $N_{\alpha,\beta}$, where in case

(I) $\sigma^2 = \tau^2 : U^{(3)}_{\alpha,\beta} := \min\{n \in \mathbb{N} : \varepsilon_n \leqq \alpha,\ r_{\sigma^2,\sigma^2,n}(\alpha - \varepsilon_n) \leqq \beta - \delta_n\}$

(II) $\sigma^2 < \tau^2 : U^{(3)}_{\alpha,\beta} := \min\{\tilde{U}_{\alpha,\beta}, \tilde{\tilde{U}}_{\alpha,\beta}\}$ with

$$\tilde{U}_{\alpha,\beta} := \min\{n \in \mathbb{N} : \ \varepsilon_n \leqq \alpha , \quad r_{\tau^2,\tau^2,n}(\alpha - \varepsilon_n) \leqq \beta - \delta_n\} ,$$

$$\tilde{\tilde{U}}_{\alpha,\beta} := \min\{n \in \mathbb{N} : 2\varepsilon_n \leqq \alpha , \quad r_{\sigma^2,\tau^2,n}(\alpha - 2\varepsilon_n) \leqq \beta - 2\delta_n\} .$$

Remark 4. The case $\sigma^2 > \tau^2$ is not treated above. It turns into case II, when the distributions P and Q are exchanged. This can be done without difficulty because of the symmetry of the problem.

3. COMPARISON OF THE DIFFERENT BOUNDS BY MEANS OF AN EXAMPLE

The most interesting comparison is that of the upper bounds $U^{(2)}_{\alpha,\beta}$ and $U^{(3)}_{\alpha,\beta}$. According to the slow rate of convergence in the *Beryy-Esseen*-type result

$$\sup_{x \in R} |F_n(x) - \Phi(x)| \leqq c \,.\, \mu_n \,.\, n^{-1/2}$$

with $c > 0$, $0 < \mu_n \uparrow \mu < \infty$ as opposed to the exponential rate, used both in the derivation of $U^{(1)}_{\alpha,\beta}$ and $U^{(2)}_{\alpha,\beta}$, $U^{(3)}_{\alpha,\beta}$ can be a better bound than $U^{(2)}_{\alpha,\beta}$ (and therefore also $U^{(1)}_{\alpha,\beta}$) only as long as the conditions for the application of the estimation technique used in Lemma 2.1 are extremely good.

The fulfillment of these conditions and consequently the quality of the upper bound $U^{(3)}_{\alpha,\beta}$ depend essentially on:

(1) how close the sample mean T_n is to sufficiency;

(2) how close P resp. Q is to the corresponding (auxiliary) Gaussian distribution $G(\xi, \sigma^2)$ resp. $G(\eta, \tau^2)$ in the sense of Zolotarev (cf. (2));
and (related to (1))

(3) how close σ^2 is to τ^2.

In order to illustrate the things mentioned above let us consider the following

Example. $\Omega = R$, $\mathscr{S}_\varepsilon = \mathfrak{A}_\sigma(\{(\varepsilon/2 + (n-1)\varepsilon, \varepsilon/2 + n\varepsilon], n \text{ integer}\})$ with $\varepsilon = 0{\cdot}1$, $0{\cdot}01$; P_ε resp. Q_ε being the conditional distribution of $G(0, 1)$ resp. $G(1, 1)$ with respect to $\mathscr{S}_\varepsilon$ (which is formalizing a round-off procedure).

For the levels we choose $\alpha = \beta = 0{\cdot}1$; $0{\cdot}01$

$\alpha = \beta$	ε	$L_{\alpha,\beta}$	$N_{\alpha,\beta}$	$U^{(1)}_{\alpha,\beta}$	$U^{(2)}_{\alpha,\beta}$	$U^{(3)}_{\alpha,\beta}$
0·1	0·1	4	7	18	13	12
	0·01	4	7	18	13	9
0·01	0·1	9	22	37	32	*)
	0·01	9	22	37	32	25

*) In this case the ε_n's are too great in relation to α.

Remark 1. To get an idea of the percentage $p_{\alpha,\beta} = N_{\alpha,\beta}/U^{(2)}_{\alpha,\beta}$ we consider the test problem $(\mathsf{G}(\xi, \sigma^2), \mathsf{G}(\eta, \sigma^2))$.

From this we get for $\alpha = \beta$

$$\tilde{p}_{\alpha,\beta}(\eta - \xi, \sigma^2) = \frac{\left(\frac{2\sigma \cdot \Phi^{-1}(1-\alpha)}{\eta - \xi}\right)^2}{\hat{U}^{(2)}_{\alpha,\beta}} = \frac{(\Phi^{-1}(1-\alpha))^2}{2 \ln \frac{1}{2\alpha}}$$

observing that

$$\ln \frac{1}{H_\gamma(\mathsf{P}, \mathsf{Q})} = \tfrac{1}{2}\gamma(1-\gamma)\frac{(\eta - \xi)^2}{\sigma^2}.$$

It is interesting that this percentage does not depend on $\eta - \xi$ and σ^2 and that for our example

$$\tilde{N}_{\alpha,\beta} = [\![\tilde{p}_{\alpha,\beta} \cdot \hat{U}^{(2)}_{\alpha,\beta}]\!]$$

coincides with $N_{\alpha,\beta}$.

Generally it would be interesting to get guiding principles for $p_{\alpha,\beta}$ for different classes of test problems. These would be very useful hints for practical purposes.

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REFERENCES

[1] T. S. Ferguson: Mathematical Statistics — A decision theoretic approach. Academic Press, New York—London 1967.

[2] S. Kullback: Information Theory and Statistics. Dover Publications, New York 1968.

[3] F. Österreicher: On dimensioning of tests (1976), to be published in the 'Journal of Combinatorics, Information and System Sciences', Delhi, India.

[4] V. M. Zolotarev: Exactness of an approximation in the central limit theorem. Proceedings of the 2nd Japan—USSR-Symposium on Probability Theory. Springer, Berlin—Heidelberg—New York 1973, 531—543.

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