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Discrete Linear Model Following Systems

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This paper investigates the problem of designing a dynamic compensation for a discrete linear plant so that the transfer matrix of the resulting feedback system coincides with a prespecified model transfer matrix.

A new formulation of this problem is developed in which only output, not state, dynamic feedback is used. A special attention is given to stable model following systems. A necessary and sufficient condition for a solution of the model following problem to exist is given. This condition is explicit and, when satisfied, leads to a simple procedure for constructing a solution. Also the class of admissible models for a given plant is completely and explicitly characterized.

I. INTRODUCTION

This paper investigates the problem of designing a dynamic compensation for a discrete linear plant so that the tansfer matrix of the resulting feedback system coincides with a prespecified model transfer matrix.

A continuous version of this problem was considered by Erzberger [2], Wolovich [7], Wang and Desoer [6], Moore and Silverman [4], Morse [5], and others. In [4, 6, 7] it is called the *exact model matching* while in [2, 5] the *model following problem*. The common approach is to use either static or dynamic state feedback to achieve the specified model transfer matrix. Implicit in this approach is the assumption that the state of the plant is measurable. It is more realistic, however, to assume that only output of the plant is available for measurement.

This idea motivated a conceptually new formulation of the model following problem in which only output, not state, dynamic feedback is used. The structure of such a model following system seems, however, as general as that using the state dynamic feedback.

In applications, the model following system is often required to be stable. Stable model following systems have obtained relatively little attention so far, the best available result being due to Morse [5]. In our formulation both types of model following problem can be directly solved in a unified way.

Let $\mathfrak{F}$ be a field. Denote $\mathfrak{F}\{z^{-1}\}$ the domain of *causal rational functions* over $\mathfrak{F}$, i.e., the set of rational functions $\boldsymbol{a}$ which admit the representation

(1) $$\boldsymbol{a} = \alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \dots, \quad \alpha_k \in \mathfrak{F}$$

and denote $\mathfrak{F}^+\{z^{-1}\}$ the domain of *stable rational functions* over $\mathfrak{F}$, i.e., the set of elements (1) for which the sequence $\{\alpha_0, \alpha_1, \alpha_2, \dots\}$ converges to zero with respect to a given valuation in $\mathfrak{F}$.

The set of elements (1) with only a finite number of nonzero coefficients forms the domain $\mathfrak{F}[z^{-1}]$ of *polynomials* in z^{-1} over $\mathfrak{F}$. A polynomial $a \in \mathfrak{F}[z^{-1}]$ is said to be *causal* if $1/a \in \mathfrak{F}\{z^{-1}\}$ and it is said to be *stable* if $1/a \in \mathfrak{F}^+\{z^{-1}\}$. For $a, b \in \mathfrak{F}[z^{-1}]$ we write $a \sim b$ to denote that a divides b and simultaneously b divides a.

Let us write $\mathfrak{F}_{l,m}$, $\mathfrak{F}_{l,m}[z^{-1}]$, $\mathfrak{F}^+_{l,m}\{z^{-1}\}$ and $\mathfrak{F}_{l,m}\{z^{-1}\}$ for the sets of $l \times m$ matrices over $\mathfrak{F}$, $\mathfrak{F}[z^{-1}]$, $\mathfrak{F}^+\{z^{-1}\}$ and $\mathfrak{F}\{z^{-1}\}$ and speak respectively of matrices, polynomial matrices, stable and causal rational matrices. The $m \times m$ identity matrix will be denoted by I_m.

A discrete, linear, constant, n dimensional, m input, l output system defined over $\mathfrak{F}$ is associated with a quadruple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ of matrices $\mathbf{A} \in \mathfrak{F}_{n,n}$, $\mathbf{B} \in \mathfrak{F}_{n,m}$, $\mathbf{C} \in \mathfrak{F}_{l,n}$, and $\mathbf{D} \in \mathfrak{F}_{l,m}$. For $n = 0$ matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ disappear; only the $\mathbf{D}$ remains.

The polynomial $\det(\mathbf{I}_n - z^{-1}\mathbf{A}) \in \mathfrak{F}[z^{-1}]$ for $n > 0$ or the polynomial 1 for $n = 0$ will be referred to as the *pseudocharacteristic polynomial* of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$. Note that a system is stable if and only if its pseudocharacteristic polynomial is stable.

The transfer matrix $S = \mathbf{D} + z^{-1}\mathbf{C}(\mathbf{I}_n - z^{-1}\mathbf{A})^{-1}\mathbf{B}$ of the system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is always a causal rational matrix; it is a stable rational matrix for a stable system. A pair of polynomial matrices $P_1 \in \mathfrak{F}_{l,l}[z^{-1}]$, $Q_2 \in \mathfrak{F}_{l,m}[z^{-1}]$ is said a *left coprime factorization* of S if $S = P_1^{-1}Q_2$ and P_1, Q_2 are left coprime; similarly a pair of polynomial matrices $Q_1 \in \mathfrak{F}_{l,m}[R^{-1}]$, $P_2 \in \mathfrak{F}_{m,m}[z^{-1}]$ is a *right coprime factorization* of S if $S = Q_1P_2^{-1}$ and Q_1, P_2 are right coprime. It is easy to see [3] that $\det P_1 \sim \det P_2$ is equal to the pseudocharacteristic polynomial of a minimal realization of S.

III. FEEDBACK SYSTEM

Prior to the formulation of the model following problem we shall study some properties of the feedback system characterized by the equations

(2) $$y_1 = Te_3, \quad y_2 = Re_1, \quad y_3 = He_2$$
$$e_1 = w_1 - y_1, \quad e_2 = w_2 + y_2, \quad e_3 = w_3 + y_3.$$

Here $\boldsymbol{H} \in \mathfrak{F}_{l,m}\{z^{-1}\}$ is the transfer matrix of a given plant $\mathscr{H}$ and the $\boldsymbol{R} \in \mathfrak{F}_{m,q}\{z^{-1}\}$ and $\boldsymbol{T} \in \mathfrak{F}_{q,l}\{z^{-1}\}$ are transfer matrices of two compensators $\mathscr{R}$ and $\mathscr{T}$. Without any real loss of generality we assume that $\mathscr{H}$, $\mathscr{R}$, and $\mathscr{T}$ are minimal realizations of $\boldsymbol{H}, \boldsymbol{R}$, and $\boldsymbol{T}$, respectively. The inputs to the system are $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}_3$ and the outputs of the system are $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.

The transfer matrix of this feedback system has the form

$$\boldsymbol{F} = \begin{bmatrix} (I_q + \boldsymbol{THR})^{-1} & -\boldsymbol{TH}(I_m + \boldsymbol{RTH})^{-1} & -\boldsymbol{T}(I_l + \boldsymbol{HRT})^{-1} \\ \boldsymbol{R}(I_q + \boldsymbol{THR})^{-1} & (I_m + \boldsymbol{RTH})^{-1} & -\boldsymbol{RT}(I_l + \boldsymbol{HRT})^{-1} \\ \boldsymbol{HR}(I_q + \boldsymbol{THR})^{-1} & \boldsymbol{H}(I_m + \boldsymbol{RTH})^{-1} & (I_l + \boldsymbol{HRT})^{-1} \end{bmatrix} =$$

$$= \begin{bmatrix} I_q & 0 & \boldsymbol{T} \\ -\boldsymbol{R} & I_m & 0 \\ 0 & -\boldsymbol{H} & I_l \end{bmatrix}^{-1} .$$

The identities

$$\begin{aligned} \boldsymbol{T}(I_l + \boldsymbol{HRT})^{-1} &= (I_q + \boldsymbol{THR})^{-1}\,\boldsymbol{T} \\ \boldsymbol{R}(I_q + \boldsymbol{THR})^{-1} &= (I_m + \boldsymbol{RTH})^{-1}\,\boldsymbol{R} \\ \boldsymbol{H}(I_m + \boldsymbol{RTH})^{-1} &= (I_l + \boldsymbol{HRT})^{-1}\,\boldsymbol{H} \end{aligned}$$

can be directly verified.

Let A_1, B_2 and B_1, A_2 be left and right coprime factorizations of $\boldsymbol{H}$, let R_1, S_2 and S_1, R_2 be left and right coprime factorizations of $\boldsymbol{R}$, and let T_1, U_2 and U_1, T_2 be left and right coprime factorization of $\boldsymbol{T}$. Then matrices

$$F_1 = \begin{bmatrix} T_1 & 0 & U_2 \\ -S_2 & R_1 & 0 \\ 0 & -B_2 & A_1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & A_1 \end{bmatrix}$$

form a left coprime factorization of $\boldsymbol{F}$ while matrices

$$G_1 = \begin{bmatrix} R_2 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & T_2 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} R_2 & 0 & U_1 \\ -S_1 & A_2 & 0 \\ 0 & -B_1 & T_2 \end{bmatrix}$$

form a right coprime factorization of $\boldsymbol{F}$. Since $\mathscr{H}$, $\mathscr{R}$, and $\mathscr{T}$ are minimal realizations of $\boldsymbol{H}, \boldsymbol{R}$, and $\boldsymbol{T}$, the feedback system studied is a minimal realization of $\boldsymbol{F}$, see [1]. As a result, the pseudocharacteristic polynomial c of the feedback system is

(3) $$c \sim \det F_1 \sim \det F_2 .$$

To obtain explicit expression for c we define polynomial matrices $S_\alpha \in \mathfrak{F}_{m,q}[z^{-1}]$ and $T_\beta \in \mathfrak{F}_{m,m}[z^{-1}]$ such that

(4) $$S_2 T_1^{-1} = T_\beta^{-1} S_\alpha , \quad \det T_1 \sim \det T_\beta .$$

Lemma 1. Let

$$C = T_\beta R_1 A_2 + S_\alpha U_2 B_1 . \tag{5}$$

Then $c \sim \det C$.

Proof. Applying the determinant formula for block matrices to F_1, we obtain

$$\begin{aligned}\det F_1 &= \det T_1 \det \begin{bmatrix} R_1 & S_2 T_1^{-1} U_2 \\ -B_2 & A_1 \end{bmatrix} \\ &= \det T_1 \det A_1 \det (R_1 + S_2 T_1^{-1} U_2 A_1^{-1} B_2) \\ &= \det T_1 \det A_1 \det (R_1 + T_\beta^{-1} S_\alpha U_2 B_1 A_2^{-1}) \\ &= \det T_1 \det A_1 \det A_2^{-1} \det T_\beta^{-1} \det (T_\beta R_1 A_2 + S_\alpha U_2 B_1) \\ &\sim \det C \end{aligned}$$

□

on using (4) and (5). Our claim follows from (3).

There are five more expressions for the pseudocharacteristic polynomial c; however, they are superfluous for our purposes. They arise naturally in the stability theory of feedback system (2).

IV. CAUSAL MODEL FOLLOWING PROBLEM

Roughly speaking, the problem of interest is to design a dynamic compensation for a given discrete linear plant so that the transfer matrix of the resulting feedback system is the same as a prespecified model transfer matrix.

Let the plant $\mathscr{H}$ be characterized by the equation

$$y_3 = \boldsymbol{H} e_2 , \tag{6}$$

where $\boldsymbol{H} \in \mathfrak{F}_{l,m}\{z^{-1}\}$ is the transfer matrix of $\mathscr{H}$.

Consider the feedback law

$$\boldsymbol{R}^{-1} \boldsymbol{e}_2 = \boldsymbol{P}\boldsymbol{v} - \boldsymbol{T}\boldsymbol{y}_3 , \tag{7}$$

where $\boldsymbol{P} \in \mathfrak{F}_{m,p}\{z^{-1}\}$, $\boldsymbol{R} \in \mathfrak{F}_{m,m}\{z^{-1}\}$, and $\boldsymbol{T} \in \mathfrak{F}_{m,l}\{z^{-1}\}$ are transfer matrices of three compensators $\mathscr{P}$, $\mathscr{R}$, and $\mathscr{T}$, respectively, and the $\boldsymbol{v}$ is a new (command) input.

Application of this law to the given plant results in a system $\mathscr{M}$ described by (2) with input $\boldsymbol{w}_1 = \boldsymbol{P}\boldsymbol{v}$ and output $\boldsymbol{e}_3$, the remaining inputs $\boldsymbol{w}_2$ and $\boldsymbol{w}_3$ being zero. Note that the tranfer matrix of this system $\mathscr{M}$ given in (6) and (7) is

$$\boldsymbol{M} = \boldsymbol{H}\boldsymbol{R}(\boldsymbol{I}_m + \boldsymbol{T}\boldsymbol{H}\boldsymbol{R})^{-1} \boldsymbol{P} .$$

The precise formulation of our problem is as follows.

Causal Model Following Problem (CMFP). Given a plant $\mathscr{H}$, which is a minimal realization of $\boldsymbol{H} \in \mathfrak{F}_{l,m}\{z^{-1}\}$, and a model transfer matrix $\boldsymbol{M} \in \mathfrak{F}_{l,p}\{z^{-1}\}$.

Find compensators $\mathscr{P}$, $\mathscr{R}$, and $\mathscr{T}$ which are minimal realizations of $\boldsymbol{P} \in \mathfrak{F}_{m,p}\{z^{-1}\}$, $\boldsymbol{R} \in \mathfrak{F}_{m,m}\{z^{-1}\}$, and $\boldsymbol{T} \in \mathfrak{F}_{m,l}\{z^{-1}\}$, respectively, such that $\boldsymbol{M} = \boldsymbol{HR}(I_m + \boldsymbol{THR})^{-1}\,\boldsymbol{P}$.

□

Note that the formulation is completely general and avoids any restrictive assumptions on both plant and model.

In the following we state a necessary and sufficient condition for the existence of a solution (i.e., a triple $\boldsymbol{P}, \boldsymbol{R}, \boldsymbol{T}$) to CMFP. This condition is explicit, extremely simple and, when satisfied, leads to a procedure for constructing a solution.

Write, as before, A_1, B_2 and B_1, A_2 for left and right coprime factorizations of $\boldsymbol{H}$, R_1, S_2 and S_1, R_2 for left and right coprime factorizations of $\boldsymbol{R}$, and T_1, U_2 and U_1, T_2 for left and right coprime factorizations of $\boldsymbol{T}$.

Theorem 1. There exists a solution to CMFP if and only if

(8) $$\boldsymbol{M} = B_1 \boldsymbol{L}$$

for some $\boldsymbol{L} \in \mathfrak{F}_{m,p}\{z^{-1}\}$.

Proof. If $\boldsymbol{P}, \boldsymbol{R}$, and $\boldsymbol{T}$ is a solution to CMFP, then

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{HR}(I_m + \boldsymbol{THR})^{-1}\,\boldsymbol{P} \\ &= \boldsymbol{H}(I_m + \boldsymbol{RTH})^{-1}\,\boldsymbol{RP} \\ &= \boldsymbol{H}(I_m + R_1^{-1}S_2T_1^{-1}U_2B_1A_2^{-1})^{-1}\,\boldsymbol{RP} \\ &= \boldsymbol{H}(I_m + R_1^{-1}T_\beta^{-1}S_\alpha U_2B_1A_2^{-1})^{-1}\,\boldsymbol{RP} \\ &= B_1A_2^{-1}A_2(T_\beta R_1A_2 + S_\alpha U_2B_1)^{-1}\,T_\beta R_1R_1^{-1}S_2\boldsymbol{P} \\ &= B_1C^{-1}T_\beta\boldsymbol{P}\,. \end{aligned}$$

Hence (8) is true for $\boldsymbol{L} = C^{-1}T_\beta\boldsymbol{P}$. Now det C, being a pseudocharacteristic polynomial, is causal and, therefore, $\boldsymbol{L}$ is causal rational matrix.

Now suppose that (8) holds and we shall construct a solution. Solve the equation

(9) $$XA_2 + YB_1 = I_m$$

for polynomial matrices X, Y such that det X is causal. Then

(10) $$\boldsymbol{P} = \boldsymbol{L}\,, \quad \boldsymbol{R} = X^{-1}\,, \quad \boldsymbol{T} = Y$$

is a solution to CMFP. Indeed,

$$\begin{aligned} \boldsymbol{M} &= B_1\boldsymbol{L} \\ &= B_1(XA_2 + YB_1)^{-1}\,\boldsymbol{L} \\ &= B_1A_2^{-1}(I_m + X^{-1}YB_1A_2^{-1})^{-1}\,X^{-1}\boldsymbol{L} \\ &= \boldsymbol{H}(I_m + \boldsymbol{RTH})^{-1}\,\boldsymbol{RP} \\ &= \boldsymbol{HR}(I_m + \boldsymbol{THR})^{-1}\,\boldsymbol{P}\,. \end{aligned}$$

□

The sufficiency part of the above proof suggests a simple procedure for constructing a solution in the form (10). There is another even simpler solution to CMFP, namely $P = A_2L$, $R = I_m$, $T = 0$. Otherwise speaking, whenever CMFP is solvable, the desired M can be achieved by a feedforward compensation only.

V. STABLE MODEL FOLLOWING PROBLEM

In applications we often require model following systems which are stable. A theory of such systems is developed here; it parallels the theory of (possibly unstable but causal) model following systems and also leads to a simple direct procedure for obtaining a solution.

Stable Model Following Problem (SMFP). Given a plant $\mathscr{H}$, which is a minimal realization of $H \in \mathfrak{F}_{l,m}\{z^{-1}\}$, and a model transfer matrix $M \in \mathfrak{F}_{l,p}\{z^{-1}\}$.

Find compensators $\mathscr{P}$, $\mathscr{R}$, and $\mathscr{T}$ which are minimal realizations of $P \in \mathfrak{F}_{m,p}\{z^{-1}\}$, $R \in \mathfrak{F}_{m,m}\{z^{-1}\}$, and $T \in \mathfrak{F}_{m,l}\{z^{-1}\}$, respectively, such that the resulting system $\mathscr{M}$ is stable and $M = HR(I_m + THR)^{-1} P$. □

We can now state and prove a solvability condition.

Theorem 2. There exists a solution to SMFP if and only if

$$M = B_1L \tag{11}$$

for some $L \in \mathfrak{F}^{+}_{m,p}\{z^{-1}\}$.

Proof. If P, R and T is a solution to SMFP, then, as in the proof of Theorem 1, (11) holds for $L = C^{-1}T_\beta P$. Now $\mathscr{M}$ stable implies that P is a stable rational matrix and that det C is a stable polynomial. Therefore, L is a stable rational matrix.

Now suppose that (11) holds. Then (10) is a solution to SMFP. Indeed, P is now a stable rational matrix. Moreover, matrices X, I_m form a left coprime factorization of R and matrices I_m, Y form a left coprime factorization of T. Hence $\det(I_mXA_2 + I_mYB_1) = 1$ is a pseudocharacteristic polynomial of the feedback part of $\mathscr{M}$. As a result, the model following system is stable and $M = HR(I_m + THR)^{-1} P$ as shown in the proof of Theorem 1. □

Theorem 2 leads to a direct constructive procedure for obtaining a solution to SMFP, if one exists. What is particularly interesting is the complete similarity of the synthesis procedures for CMFP and SMFP. The only difference is in the matrix L implied by the solvability condition.

Note that the pseudocharacteristic polynomial c_M of the model following system $\mathscr{M}$ is the product of the pseudocharacteristic polynomials of $\mathscr{P}$ and of the remaining feedback part of $\mathscr{M}$. The above described procedure makes the latter polynomial equal to 1 and, hence, we have

$$c_M \sim \det L_1 \sim \det L_2\,,$$

where L_1 and L_2 are denominator matrices from left and right coprime factorizations of $\boldsymbol{L}$.

It is to be noted that for a *stable* plant $\mathscr{H}$ we have also the immediate feedforward solution $\boldsymbol{P} = A_2\boldsymbol{L}$, $\boldsymbol{R} = I_m$, $\boldsymbol{T} = O$ for SMFP, like for a causal plant and CMFP.

VI. COMPUTATIONAL ALGORITHMS

The synthesis procedure described in this paper consists of three steps, namely computing a right coprime factorization B_1, A_2 of $\boldsymbol{H}$, then extracting B_1 from $\boldsymbol{M}$, and solving a matrix equation $XA_2 + YB_1 = I_m$ for X, Y with $\det X$ causal. These steps can be efficiently algorithmized as shown below.

Computing a right coprime factorization B_1, A_2 of $\boldsymbol{H}$.

Let $\boldsymbol{H} = B/a$, where $a \in \mathfrak{F}[z^{-1}]$, $B \in \mathfrak{F}_{l,m}[z^{-1}]$. Then $\boldsymbol{H} = BA_m^{-1} = A_l^{-1}B$, where $A_l = aI_l$ and $A_m = aI_m$. The matrices B, A_m already form a right factorization of $\boldsymbol{H}$; however, this factorization is not always right coprime. To ensure right coprimeness, we have to extract the greatest right divisor common to B and A_m.

This is most efficiently done by applying elementary column transformations Q to transform the matrix

$$K = [A_l \; B]$$

to a lower triangular form,

$$KQ = [D_1 \; O],$$

where $D_1 \in \mathfrak{F}_{l,l}[z^{-1}]$. Write

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

for $Q_{12} \in \mathfrak{F}_{l,m}[z^{-1}]$ and $Q_{22} \in \mathfrak{F}_{m,m}[z^{-1}]$. Then $AQ_{12} + BQ_{22} = O$, i.e., $A^{-1}B = -Q_{12}Q_{22}^{-1}$ and matrices Q_{12}, Q_{22} are right coprime since Q is a unimodular matrix. Hence the pair Q_{12}, $-Q_{22}$ is a right coprime factorization of $\boldsymbol{H}$.

It is convenient to obtain the factorization in which B_1 is a (generalized) lower triangular matrix. Therefore, we again apply elementary column transformations E on Q_{12} to get $Q_{12}E = [B_{11} \; O]$, where the matrix B_{11} is formed by the nonzero columns of $Q_{12}E$. Then

$$B_1 = [B_{11} \; O], \quad A_2 = -Q_{22}E.$$

Extracting B_1 from $\boldsymbol{M}$.

Let $B_1 = [B_{11} \; O]$, where $B_{11} \in \mathfrak{F}_{l,r}[z^{-1}]$ is a (generalized) lower triangular matrix and $r = \operatorname{rank} B_1$. Then equation

$$\boldsymbol{M} = B_1\boldsymbol{L} = [B_{11} \; O]\begin{bmatrix} \boldsymbol{L}_1 \\ \boldsymbol{L}_2 \end{bmatrix} = B_{11}\boldsymbol{L}_1 \tag{12}$$

 for L_1 can be written as follows

$$\begin{bmatrix} m_{11} \dots m_{1p} \\ m_{21} \dots m_{2p} \\ \dots \\ m_{l1} \dots m_{lp} \end{bmatrix} = \begin{bmatrix} b_{11} & 0 & \dots 0 \\ b_{21} & b_{22} & \dots 0 \\ & & \dots \\ b_{r1} & b_{r2} & \dots b_{rr} \\ & & \dots \\ b_{l1} & b_{l2} & \dots b_{lr} \end{bmatrix} \begin{bmatrix} x_{11} \dots x_{1p} \\ x_{21} \dots x_{2p} \\ \dots \\ x_{r1} \dots x_{rp} \end{bmatrix}.$$

Due to the form of B_{11}, the elements x_{ij} of L_1 can be computed successively starting with the first row. After computing the last row of L_1, the remaining equations, if $r < l$, must hold identically. Otherwise there is no L to satisfy equation (12).

It remains to check if all elements x_{ij} of L_1 are causal rational functions for CMFP or stable rational functions for SMFP. If not, the respective problem has no solution. The matrix L_2 can be chosen arbitrarily within causal or stable rational matrices.

Solving the equation $XA_2 + YB_1 = I_m$.

This matrix polynomial equation can easily be solved by applying elementary row transformations P to transform the matrix

$$J = \begin{bmatrix} A_2 \\ B_1 \end{bmatrix}$$

to an upper triangular form,

$$PJ = \begin{bmatrix} D_2 \\ O \end{bmatrix},$$

where $D_2 \in \mathfrak{F}_{m,m}[z^{-1}]$. Write

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

for $P_{11} \in \mathfrak{F}_{m,m}[z^{-1}]$, $P_{12} \in \mathfrak{F}_{m,l}[z^{-1}]$, $P_{21} \in \mathfrak{F}_{l,m}[z^{-1}]$, and $P_{22} \in \mathfrak{F}_{l,l}[z^{-1}]$. Then

$$P_{11}A_2 + P_{12}B_1 = D_2$$

$$P_{21}A_2 + P_{22}B_1 = O$$

with D_2 being an upper triangular unimodular matrix (due to right coprimeness of B_1 and A_2). Thus, for an arbitrary matrix $T \in \mathfrak{F}_{m,l}[z^{-1}]$,

(13) $$X = D_2^{-1}P_{11} + TP_{21}$$

$$Y = D_2^{-1}P_{12} + TP_{22}$$

is the general solution of equation (9).

The only problem remaining is to choose a particular solution X, Y such that det X is a causal polynomial; this is effected by an appropriate choice of T in (13). Let

$$\begin{aligned} X &= X_0 + X_1 z^{-1} + \dots \\ D_2^{-1} P_{11} &= M_0 + M_1 z^{-1} + \dots \\ P_{21} &= N_0 + N_1 z^{-1} + \dots \end{aligned}$$

Then $T = T_0$, a constant matrix such that $X_0 = M_0 + T_0 N_0$ is nonsingular, yields the desired particular solution.

Note that when B_1 is divisible by z^{-1} (the plant exhibits a delay), we can always set $T = 0$ since M_0 is already nonsingular.

VII. EXAMPLE

To illustrate the synthesis procedure, consider SMFP for

$$\boldsymbol{H} = \begin{bmatrix} \frac{1}{1 - z^{-1}} & \frac{1}{1 - z^{-2}} \\ \frac{1}{1 - z^{-1}} & 1 \end{bmatrix}, \qquad \boldsymbol{M} = \begin{bmatrix} \frac{4}{2 - z^{-1}} \\ 2 + z^{-1} \end{bmatrix}$$

over the field of reals.

We simply compute a right coprime factorization of $\boldsymbol{H}$, say

$$B_1 = \begin{bmatrix} 1 & 0 \\ 1 - z^{-2} & -z^{-2} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -1 + z^{-1} \\ 1 - z^{-2} & 1 - z^{-2} \end{bmatrix}$$

and check for (11). We obtain

$$\boldsymbol{L} = \begin{bmatrix} \frac{4}{2 - z^{-1}} \\ -\frac{3}{2 - z^{-1}} \end{bmatrix}$$

and, therefore, a solution exists. Since equation (9) is satisfied e.g. by

$$X = \begin{bmatrix} 1 + z^{-1} & 1 \\ -1 - z^{-1} & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} z^2 & 0 \\ 1 - z^{-2} & -1 \end{bmatrix},$$

a solution to SMFP is obtained using (10)

$$\boldsymbol{P} = \begin{bmatrix} \frac{4}{2 - z^{-1}} \\ -\frac{3}{2 - z^{-1}} \end{bmatrix}, \qquad \boldsymbol{R} = \begin{bmatrix} 0 & -\frac{1}{1 + z^{-1}} \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{T} = \begin{bmatrix} z^{-2} & 0 \\ 1 - z^{-2} & -1 \end{bmatrix}.$$

The pseudocharacteristic polynomial of the resulting model following system is $c_M = 2 - z^{-1}$.

VIII. CONCLUDING REMARKS

A new approach to the design of model following systems with dynamic output feedback has been presented. A necessary and sufficient condition for the existence of a solution has been given in the form

$$M = B_1 L \,, \tag{14}$$

where L is to be causal rational matrix for CMFP or a stable rational matrix for SMFP. This condition not only leads to a simple and direct procedure for constructing a solution but also completely and explicitly characterizes the class of achievable model transfer matrices for a given plant.

Even though only a dynamic *output* feedback is used, it seems that the presented approach is as powerful as when a dynamic *state* feedback is allowed. To illustrate this point, consider a single-input single-output plant and model. Then it was shown in [4] that CMFP is solvable by dynamic state feedback if and only if $H^{-1}M$ does not have more zeros than poles, which is equivalent to L being causal in (14). As to SMFP, certain subset of zeros of H is in addition required to be stable [5]. It is plausible that it is the subset of those zeros of H which are not zeros of M, this being equivalent to L stable in (14).

The synthesis procedure described in this paper is also computationally attractive. It consists of three steps, namely computing a right coprime factorization B_1, A_2 of H, then extracting B_1 from M, and solving the matrix polynomial equation $XA_2 + YB_1 = I_m$, all of which can be efficiently algorithmized.

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