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Problem of Averaging in Digital Measurements

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The averaging is one of the basic methods in measured data processing. However, in digital measurements the mean value of the quantized signal differs from the analog input mean due to the finite ADC resolution and random error distribution. In this paper the practical limitations of the increasing the number of samples and conditions of ACD resolution improvement by means of averaging are being examined in terms of information theory. Two fundamental types of averaging are discussed by making use of the characteristic function and E-uncertainty which, in addition to definitions given in [3], is likely to be good means to evaluate different measuring processes.

List of symbols

x analog random variable

X_n discrete equidistant values of random variable

$w(x)$ continuous probability distribution, i.e. the probability of x having a value between $x - \mathrm{d}x/2$ and $x + \mathrm{d}x/2$:

$$\int_{-\infty}^{\infty} w(x)\,\mathrm{d}x = 1 \tag{1}$$

$p(n)$ probability in discrete points X_n

$$\sum_{n=\nu_L}^{\nu_P} p(n) = 1\,; \quad p(n) = 0 \quad \text{for} \quad n < \nu_L \quad \text{and} \quad n > \nu_P\,; \tag{2}$$
$$-\infty < \nu_L < \nu_P < \infty$$

the discrete function $p(n)$ is said to be uninterrupted if there are no zero-probability points between any two non-zero-probability points.

Δ interval width in quantization (resolution of ADC)

v continuous real variable

$\Theta(v)$ characteristic function corresponding to $w(x)$

$T(l)$ discrete characteristic function

M number of samples (readings, observations) to be used to calculate the mean value

$m_{n,M}$ statistical n-th order moment of distribution of the mean stated from M samples:

(3) $$m_{n,M} = \int_{-\infty}^{\infty} x^n \, w_M(x) \, dx$$

σ standard deviation

h differential entropy [1]

$H_M(\Delta)$ residual E-(entropy)-uncertainty of error distribution of the mean stated from M samples

$K_{A,M}$ asymmetry coefficient of distribution

$K_{E,M}$ excess coefficient of distribution

(4) $$S_n(M) = \sum_{j=1}^{M} k_j^n \, ; \quad S_0(M) = M$$

$$\mathrm{i} = \sqrt{-1}$$

lb y binary logarithm of y (taken to the base 2)

Some abbreviations

ADC analog-to-digital converter

BC "box car" distribution

CD constant distribution

G Gaussian (normal) distribution

RPHS random phase harmonic signal (distribution)

LSB least significant bit

1. FUNDAMENTAL RELATIONS

The differential entropy [1] of the random variable probability distribution is defined as

(5) $$h = -\int_{-\infty}^{\infty} w(x) \, . \, \mathrm{lb}\,(w(x)) \, dx \geqq 0 \, .$$

If Δ equals the resolution (LSB weight) of ADC (quantization), the discrete distribution

(6) $$p(n) = \int_{X_n - \Delta/2}^{X_n + \Delta/2} w(x) \, dx$$

is characterized, if uninterrupted, by E(entropical) uncertainty

$$H(\Delta) = -\sum_n p(n) \,.\, \mathrm{lb}\,(p(n)) \geqq 0\,. \tag{7}$$

It will be seen from the result of this paper that if $H(\Delta) \geqq 3$ bits (i.e. Δ is very small) the E-uncertainty may be simply expressed by

$$H(\Delta) = h - \mathrm{lb}\,\Delta\,. \tag{8}$$

Note. $H_0(\Delta)$ denotes the "a priori" E-uncertainty of the signal to be measured and $H_M(\Delta)$ is the residual (a posteriori) E-uncertainty of the mean value calculated from $M \geqq 1$ values quantized by an ADC. The information gained by measuring and averaging M samples equals $H_0 - H_M$. Throughout this paper the value to be measured (i.e. the input mean) is supposed to stay perfectly constant during the process of measurement and the random interference to be superimposed on it. The error statistical parameters are presumed to be stationary.

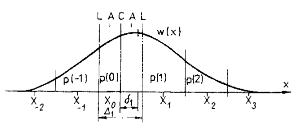


Fig. 1. Continuous probability distribution and division of X axis.

The probability distribution $w_2(x)$ of the mean of two readings (i.e. samples) having the same $w_1(x)$ may be proved to be

$$w_2(x) = 2\int_{-\infty}^{\infty} w_1(x + \xi)\, w_1(x - \xi)\, \mathrm{d}\xi \tag{9}$$

However, the utility of this formula is limited to some special cases even if it can be extended for averaging more samples. The characteristic function seems to be more convenient to form the theoretical basis of averaging.

The well known relation between $w(x)$ and the characteristic function [1] is described by integral transforms as follows

$$\Theta(v) = \int_{-\infty}^{\infty} w(x)\,.\,\mathrm{e}^{ivx}\,\mathrm{d}x \tag{10}$$

and

$$w(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta(v)\,\mathrm{e}^{-ivx}\,\mathrm{d}v \tag{11}$$

which may be simplified for symmetrical distributions to

(12) $$\Theta_R(v) = 2\int_0^\infty w_R(x) \,.\, \cos(vx)\, dx$$

and

(13) $$w_R(x) = \frac{1}{\pi}\int_0^\infty \Theta_R(v) \,.\, \cos(vx)\, dv$$

where $w_R(x) = w_R(-x)$.

If the shape of $w(x)$ is changed by a constant coefficient k to $(1/k)\, w(x/k)$, then h changes to $h + \text{lb}\, k$ and $\Theta(v)$ to $\Theta(kv)$.

It is very useful to express $\Theta_M(v)$ by an array the coefficients of which are composed of statistical moments of $w_M(x)$ as follows

(14) $$\Theta_M(v) = 1 + \sum_{n=1}^{\infty} \frac{m_{n,M}}{n!} (iv)^n$$

2. THE SUM AND MEAN VALUE CALCULATIONS

The characteristic function of a sum of $M \geqq 1$ independent random variables equals the product of their characteristic functions

(15) $$\Theta_S(v) = \prod_{n=1}^{M} \Theta_n(v)\,.$$

The "weighted sum" results in

(16) $$\Theta_S(v) = \prod_{n=1}^{M} \Theta_1(k_n v)\,,$$

where $\Theta_n(v) = \Theta_1(k_n v)$ for $n = 1, 2, \ldots, M$ since $w_n(x) = (1/k_n)\, w_1(x/k_n)$.

The mean value is stated by dividing the sum by $S_1(M)$ which yields

(17) $$w_M(x) = S_1(M) \,.\, w_S(S_1(M) \,.\, x)$$

and the weighted mean characteristic function is

(18) $$\Theta_M(v) = \Theta_S\left(\frac{v}{S_1(M)}\right) = \prod_{n=1}^{M} \Theta_1\left(\frac{k_n v}{S_1(M)}\right).$$

Statistical moments of $w_M(x)$ are listed in Tab. 1.

Tab. 1. Formulas for statistical parameters of mean value in terms of the number of samples ($H_1(\Delta_1) \geqq 3$ bits).

	General expressions	SD averaging $k_0 = 1, \quad M < \infty$	CF averaging $k_l = k_0^l, \quad 0 < k_0 < 1, \quad M \to \infty$
$m_{2,M}$	$\frac{S_2(M)}{S_1^2(M)} m_{2,1}$	$\frac{m_{2,1}}{M} = \left(\frac{\sigma_1}{\sqrt{M}}\right)^2$	$\frac{1-k_0}{1+k_0} m_{2,1} = \left(\left(\frac{1-k_0}{1+k_0}\right)^{1/2} \sigma_1\right)^2$
$m_{3,M}$	$\frac{S_3(M)}{S_1^3(M)} m_{3,1}$	$\frac{m_{3,1}}{M^2}$	$\frac{(1-k_0)^2}{1+k_0+k_0^2} m_{3,1}$
$m_{4,M}$	$\frac{S_4(M)}{S_1^4(M)} m_{4,1} + 6 \frac{m_{2,1}^2 k_0^2}{S_1^4(M)} \sum_{l=1}^{M-1} S_2(l) k_0^{2(l-1)}$	$\frac{m_{4,1}}{M^3} + 3m_{2,M}^2 \frac{M-1}{M}$	$\frac{(1-k_0)^2 \left[(1-k_0) m_{4,1} + 6m_{2,1}^2 \frac{k_0^2}{1+k_0}\right]}{(1+k_0)(1+k_0^2)}$
$K_{A,M}$	$\frac{m_{3,M}}{\sqrt{m_{2,M}^3}}$	$\frac{K_{A,1}}{\sqrt{M}}$	$\frac{1-k_0^2}{1+k_0+k_0^2} \left(\frac{1+k_0}{1-k_0}\right)^{1/2} K_{A,1}$
$K_{E,M}$	$\frac{m_{4,M}}{m_{2,M}^2} - 3$	$\frac{K_{E,1}}{M}$	$\frac{1-k_0^2}{1+k_0^2} K_{E,1}$

In measurements the following two fundamental types of averaging procedures are being used:

SD: averaging the set of M data the weight of all of them being

$$k_n = k_1 = 1 \quad \text{and} \quad S_n(M) = M$$

and the characteristic function

$$\Theta_M(v) = \Theta_1^M\left(\frac{v}{M}\right). \tag{19}$$

CF: averaging by continuous filtering; in a digital low-pass filter of the "band" j the weights are $k_n = k^{jn} = k_0^n$ and $M \to \infty$ as the filter operation is in steady state (exponential averaging). In this case

$$S_j(\infty) = \frac{1}{1 - k_0} = \frac{1}{1 - k^j} \tag{20}$$

and it follows that

$$\Theta_\infty(v) = \prod_{n=1}^{\infty} \Theta_1(k_0^n \, . \, (1 - k_0)\, v)\,. \tag{21}$$

4. CALCULATION BY MEANS OF MOMENTS

The procedure of successive multiplication of arrays (14) is described in Appendix, where

$$a_{1,n} = \frac{m_{n,1}}{n!}\, \mathrm{i}^n \tag{22}$$

and $A_{M,0} = \Theta_M(v)$.

Resulting moments of the probability distribution of the mean value are

$$m_{n,M} = \frac{n!}{\mathrm{i}^n S_1^n(M)}\, a_{M,n} \tag{23}$$

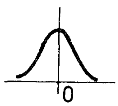
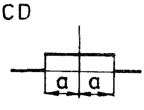
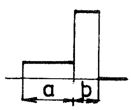
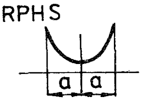
The 1st order moment

$$m_{1,M} = \frac{a_{M,1}}{\mathrm{i}\, . \, S_1(M)} = \frac{a_{1,1}}{\mathrm{i}} = m_{1,1} \tag{24}$$

remains without change as it is well known from experience. Therefore no generality loss occurs by setting

$$m_{1,M} = 0\,, \quad a_{M,1} = 0\,, \quad M = 1, 2, 3, \ldots\,. \tag{25}$$

Tab. 2. Properties of most important distribution types ($H_1(\Delta_1) \gtrsim 3$ bits).

	$w(x)$	$\Theta(v)$	h
G 	$\dfrac{e^{-x^2/2\sigma_1^2}}{\sqrt{(2\pi)}\,\sigma_1}$	$e^{-\sigma_1^2 v^2/2}$	$\mathrm{lb}\,\sigma_1 + \frac{1}{2}\,\mathrm{lb}\,2\pi e = h_G$ $(\frac{1}{2}\,\mathrm{lb}\,2\pi e = 2{\cdot}047)$
CD  $a > 0$	$w(x) = \dfrac{1}{2a}$ for $\lvert x\rvert < a$; $w(x) = 0$ for $\lvert x\rvert \geqq a$	$\dfrac{\sin(va)}{va}$	$\mathrm{lb}\,a + 1 = \mathrm{lb}\,\sigma_1 + 1{\cdot}793 = h_G - 0{\cdot}254$
BC  $a > 0$, $b > 0$	$w(x) = \dfrac{b}{a(a+b)}$ for $-a < x < 0$; $w(x) = \dfrac{a}{b(a+b)}$ for $0 \leqq x < b$; else $w(x) = 0$	$\dfrac{2}{v(a+b)} \cdot \left[\dfrac{b}{a}\,e^{-iv(a/2)} \sin\dfrac{va}{2} + \dfrac{a}{b}\,e^{+iv(b/2)} \sin\dfrac{vb}{2}\right]$	$\mathrm{lb}\,(a+b) - \dfrac{a-b}{a+b}\,\mathrm{lb}\,\dfrac{a}{b}$
RPHS  $a > 0$	$w(x) = \dfrac{1}{\pi\sqrt{(a^2 - x^2)}}$ for $\lvert x\rvert < a$ $w(x) = 0$ for $\lvert x\rvert \geqq a$	$I_0(va) = \sum_{j=0}^{\infty} \dfrac{(-1)^j \left(\dfrac{va}{2}\right)^{2j}}{(j!)^2}$ [1]	$\mathrm{lb}\,a + 0{\cdot}776 = \mathrm{lb}\,\sigma_1 + 1{\cdot}276 = h_G - 0{\cdot}771$

in consequence of which all the probability distributions are presumed to be centered throughout the rest of this paper. Then the formula (A.6) is simplified to

$$a_{M,n} = a_{1,n}k_0^{n(M-1)} + a_{M-1,n} + \sum_{l=2}^{n-2} a_{M-1,l}a_{1,n-l}k_0^{(n-l)(M-1)} \tag{26}$$

which is by making use of (22) further formed to

$$m_{n,M} = \frac{m_{n,1}k_0^{n(M-1)} + S_1^n(M-1)\,m_{n,M-1}}{S_1^n(M)} + \tag{27}$$

Tab. 2. (continuation)

$m_{2,1}$	$m_{3,1}$	$m_{3,1}$	$K_{A,1}$	$K_{E,1}$
σ_1^2	0	$3\sigma_1^4$	0	0
$\frac{a^2}{3}$ $\left(\sigma_1 = \frac{a}{\sqrt{3}}\right)$	0	$\frac{a^4}{5}$	0	$-1{\cdot}200$
$\frac{ab}{3}$ $\left(\sigma_1 = \left(\frac{ab}{3}\right)^{1/2}\right)$	$-\frac{ab(a-b)}{4}$	$\frac{ab(a^2+b^2-ab)}{5}$	$-\frac{(a-b)\sqrt{27}}{4\sqrt{ab}}$	$\frac{9\left(\frac{a}{b}+\frac{b}{a}\right)-24}{5}$
$\frac{a^2}{2}$ $\left(\sigma_1 = \frac{a}{\sqrt{2}}\right)$	0	$\frac{3a^4}{8}$	0	$-1{\cdot}500$

$$+ \frac{1}{S_1^n(M)} \sum_{l=2}^{n-2} \binom{n}{l} S_1^l(M-1)\, m_{l,M-1} m_{n-l,1} k_0^{(n-l)(M-1)} .$$

The formulas for three most important moments and coefficients corresponding to the resulting distribution due to the extent and type of averaging are given in Tab. 1. It may be seen that $K_{E,M}$ and $K_{A,M}$ are decreasing to zero as the number of samples increases and the shape of $w_M(x)$ is approaching the Gaussian function in accordance with the central limit theorem [1].

The formulas in Tab. 1 are valid for $H_M(\Delta_1) \geqq 3$ bits only. If $H_M(\Delta_1) < 3$ bits the bin width is not more negligible in comparison to the distribution extent. The mean value calculations must be performed by numerical methods, either by a discrete form of (9) or by means of discrete characteristic function. Especially beneath the 2-bit level the E-uncertainty is strongly dependent on the input mean position within Δ_1, Fig. 2, and on rounding or truncating method used in handling the results.

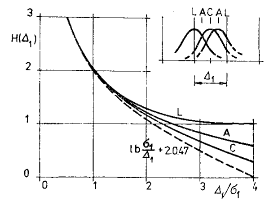


Fig. 2. E-uncertainty of Gaussian distribution for different relative positions of the value to be measured.

The total development of the residual E-uncertainty vs. the number of samples is given in Fig. 3 for several typical forms of $w_1(x)$. The probability distribution of the

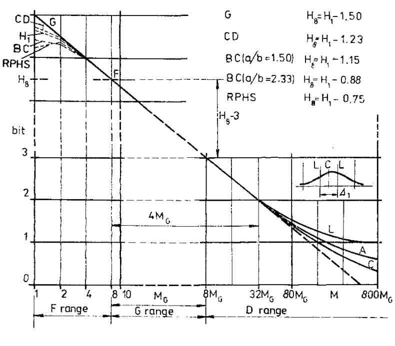


Fig. 3. E-uncertainty decreasing vs. number of samples used to calculate the mean value. The original discrete distribution at $M = 1$ is presumed to be uninterrupted.

random error stated at one sample (i.e. reading or observation) tends to approach a Gaussian form and its E-uncertainty decreases as the number M of samples is getting greater. There are three ranges of E-uncretainty course to be seen. The original generally shaped distribution $w_1(x)$ or better $p_1(n)$ which is presumed to be uninterrupted, changes to nearly Gaussian distribution at the end of F-range (forming) which represents the mean value calculation from up to 8 samples. The E-uncertainty decrement $H_1 - H_8$ is summarized in Fig. 3 for several distribution types.

In practice most of additive errors have symmetrical distribution ($K_{A,1} = 0$) unless these are unfavourably affected by finite extent of the data set or by spectral properties of signals to be averaged. Further it was experienced that $|K_{E,1}| \leqq 1{\cdot}6$ and $|K_{A,1}| \leqq 0{\cdot}9$ are limits to calculate with which yield $|K_{E,8}| \leqq 0{\cdot}2$ and $|K_{A,8}| \leqq 0{\cdot}32$. The latter values have no significant influence on H_8 at the entry of the G-range (Gaussian) where the one-bit decrement of E-uncertainty starts to correspond to multiplying M by 4 until the level of 3 bits is reached so that

$$M_G = 4^{H_8 - 3}\,. \tag{28}$$

The 2-bit uncertainty level is then reached approximately by $32M_G$ samples. By comparing the formulas for $m_{2,M}$ in Tab. 1 the filter feed-back constant on CF-averaging (eq. (20)) is stated to be

$$k_0 = k^j = \frac{M-1}{M+1}\,. \tag{29}$$

The rest of the E-uncertainty course for $M > 8M_G$ is denoted as D-range (deforming). As a result of numerical calculations the dependence of $H_M(\Delta_1)$ on the position of the input mean within Δ_1 was stated and plotted in Figs. 2 and 3.

5. IMPROVING THE RESOLUTION OF QUANTIZATION

In averaging the mean value may be rounded and located into fractional subintervals

$$\Delta_{q+1} = \Delta_1 \cdot 2^{-q}\,, \quad q = 0, 1, 2, \ldots, \tag{30}$$

in order to improve the resolution of A/D conversion by software interpolation [4]; if $q = 0$ then there is no change in resolution. The errors resulting from this procedure may be well understood on the basis of Fig. 1 where the constant value to be measured stands at the distance δ_1 from the interval centre C. For $M \to \infty$ the result of SD-averaging is

$$\delta_\infty = \sum_{n=\nu_L}^{\nu_P} n \int_{(n-1/2)\Delta_1}^{(n+1/2)\Delta_1} w_1(x - \delta_1)\,\mathrm{d}x \tag{31}$$

see eq. (2), too. The expression (31) was evaluated for several types of discrete distributions $p_1(n)$. It was stated that for symmetrical distribution (i.e. $K_{A,1} = 0$) the maximum of $|\delta_1 - \delta_\infty|$ occurs for input mean being situated in A – position whereas if $K_{A,1} \neq 0$ the maximum is found for C-position. The survey of the maxima in graphical form is given in Fig. 4 vs ratio of standard deviation σ_1 to Δ_1. The error given here results in the relative shifts of subinterval boundaries inside every original interval Δ_1 as stated in [4]. Nevertheless it should be born in mind that even if by theory

$$q_{\max} = \text{lb } M \tag{32}$$

the q-bit resolution improvement is accompanied by an E-uncertainty increase of $\frac{1}{2}$ lb M, see eq. (8), in averaging M samples the random error of which is of Gaussian type. It follows therefore, that q should be less than $\frac{1}{2}$ lb M in order to maintain the E-uncertainty decreasing at the same time. Further limitations are due to the distribution form.

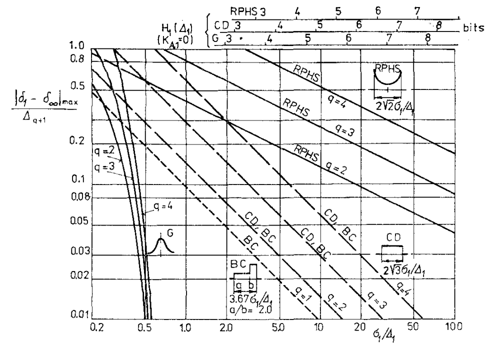


Fig. 4. Maxima of relative deviation of calculated mean for important distribution types and q bit improvement of resolution. Comparison with E-uncertainty for $H_1(\Delta_1) \geqq 3$ bits.

The problem should be well understood by the help of an example illustrated in Fig. 5 : let the analog input signal probability distribution $w_1(x)$ be of CD type having $2a = 2{\cdot}5\Delta_1$ (i.e. $\sigma_1 = 0{\cdot}72\Delta_1$) and the mean $0{\cdot}25\Delta_1$ (A-position); the point $x = 0$ is located in the center of the bin denoted by $n = 0$. The diagram shows $p'(n)$ after quantization, $M = 1$, the discrete output mean is X_0. In order to improve the resolution by $q = 3$ bits further 14 discrete zero-probability points must be artificially added inside the original bins the result of which being a new interrupted discrete probability distribution $p_1(n)$ and the discrete mean $X_{2/8}$ by rounding to Δ_4. After having averaged 8 samples ($M = 2^q = 8$) one gains an uninterrupted distribution $p_8(n)$ characterized by $H_8(\Delta_4) = 3{\cdot}08$ bits and the mean $0{\cdot}20\Delta_1$ which yields a relative deviation $|0{\cdot}25-0{\cdot}20|\ \Delta_1/\Delta_4 = 0{\cdot}40$, see Fig. 4, too. It follows that if the input mean was within $(0{\cdot}188\Delta_1;\ 0{\cdot}237\Delta_1)$ the resulting mean would be rounded erroneously to $X_{1/8}$ due to deviation $0{\cdot}05\Delta_1$. This shift diminishes toward the points L and C of Δ_1.

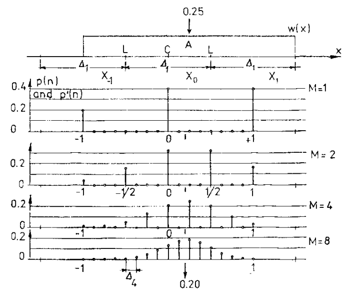


Fig. 5. Illustrative example of ADC resolution improvement.

6. DISCRETE CHARACTERISTIC FUNCTION

If $H_1(\Delta_1) < 3$ bits the simple relations (10) ... (13) are not more valid and it is more advantageous to make use of the discrete transforms

$$T_1(l) = \sum_{n=\nu_{L,1}}^{\nu_{P,1}} p_1(n)\, e^{+i2\pi(nl/N_1)} \tag{33}$$

and

$$p_1(n) = \frac{1}{N_1} \sum_{l=\lambda_{L,1}}^{\lambda_{P,1}} T_1(l)\, e^{-i2\pi(nl/N_1)}, \tag{34}$$

where the number of discrete points $N_1 = 1 + \nu_{P,1} - \nu_{L,1} = 1 + \lambda_{P,1} - \lambda_{L,1}$. Out of the extent of these points the probability equals zero. In a symmetrical case $p_R(-n) = p_R(n)$, $\nu_{P,1} = -\nu_{L,1}$ (N_1 is odd):

$$T_{R,1}(l) = p_{R,1}(0) + 2\sum_{n=1}^{\nu_{P,1}} p_{R,1}(n) \cos\left(2\pi \frac{nl}{N_1}\right) \tag{35}$$

and for $\lambda_{P,1} = -\lambda_{L,1}$

$$p_{R,1}(n) = \frac{1}{N_1}\left[T_{R,1}(0) + 2\sum_{l=1}^{\lambda_{P,1}} T_{R,1}(l) \cos\left(2\pi \frac{nl}{N_1}\right)\right]. \tag{36}$$

The sums and mean values are to be calculated by formulas similar to those denoted by (15) ... (18) where $T(l)$ is to be used instead of $\Theta(v)$. However, the non-zero probability range of the sum of M values will be enlarged to N_M points, where

$$(37) \qquad N_M = 1 + \nu_{P,M} - \nu_{L,M} = 1 + M(\nu_{P,1} - \nu_{L,1}) = 1 + M(N_1 - 1).$$

When calculating the mean value we must return back to the original set of discrete points X_n, $(\nu_{L,1} \leqq n \leqq \nu_{P,1})$ completed with additional interpolation points for $q \geqq 0$ (improvement of resolution) if possible. The distribution of the mean rounded to discrete points denoted by non-integer subscript

$$(38) \qquad n = \nu_{L,1} + j + t \,.\, 2^{-q} \quad \text{for} \quad j = 0, 1, 2, \ldots, N_1 - 1 \,,$$

where $t = 0, 1, 2, \ldots, 2^q - 1$, is

$$(39) \qquad p_{M,q}(n) = \frac{1}{N_M} \sum_{j=\mu_L}^{\mu_P} \left(\sum_{l=\lambda_{L,M}}^{\lambda_{P,M}} T_M(l) \,.\, \mathrm{e}^{-i2\pi(jl/N_M)} \right),$$

$$\mu_P = \text{Min}\,\{M\nu_{P,1};\, (n + 2^{-q-1})\, M - 1\}\,,$$

$$\mu_L = \text{Max}\,\{M\nu_{L,1};\, (n - 2^{-q-1})\, M\}$$

In practice the discrete transform procedure will be useful in the range $H_1(\Delta_1) < 3$ bits only where the ratio σ_1/Δ_1 is relatively small, Figs 2 and 3.

7. PRACTICAL RESULTS AND CONCLUSION

The diagram given in Fig. 3 is a good mean to estimate the maximum number of samples to reach the 2-bit level of E-uncertainty by averaging, i.e. $M_{T2} = 32M_G$. If there is no intention to improve the ADC resolution the H_8 may be stated by means of H_1 and of the type of the input random error distribution; M_G is given by (28). The 2-bit level of final E-uncertainty seems to be a practical limit of averaging beyond which the further increasing of M stops to be effective.

In order to illustrate the outlined considerations let us describe an example: a centered constant distribution (CD, Tab. 2) having $a = 32\Delta_1$, $K_{E,1} = -1{\cdot}20$, $\sigma_1 \doteq 18{\cdot}5\Delta_1$ and $H_1 = 6$ bits changes by SD averaging 8 samples to $K_{E,8} = -0{\cdot}15$ and $H_8 = 6{\cdot}0 - 1{\cdot}23 = 4{\cdot}77$ bits so that by (28) $M_G = 4^{1.77} = 11{\cdot}6 \doteq 12$ groups of 8 samples are necessary to reach a 3-bit level, i.e. in total $M_{T2} = 32M_G = 384$ samples to reach a 2-bit level. A digital LP filter having $k_0 = 383/385$ should be optimal by theory to match this averaging problem in CF mode.

The resolution improvement results always in an E-uncertainty augment and it does not bring any profit in information gain unless accompanied by suitable increasing the total number M_{T2} of samples to reach the 2-bit level. As a result of considerations given in this paper the diagram in Fig. 6 was plotted in which M_{T2} is given vs. the original one-sample E-uncertainty provided the maximum relative output mean deviation (Fig. 4) should not exceed 0·5. In practice M_{T2} is limited by a record length in data store or by a time interval of measurement. On the right axis in Fig. 6 there are the LP filter coefficients corresponding to M_{T2} by eq. (29).

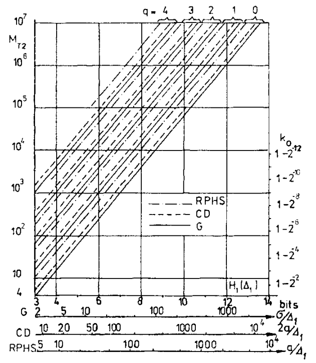


Fig. 6. The total number of samples M_{T2} to reach a 2-bit level and corresponding LP filter coefficients vs the original E-uncertainty (at $M = 1$) and parameters of symmetrical distribution types. The influence of resolution improvement by q bits.

These are limited by digital filter construction, i.e. word length etc. If, for example, $M_{T2} = 10^4$ samples or $k_0 = 1 - 2^{-12}$ the ratio $\sigma_1/\Delta_1 = 100$ of Gaussian random error distribution represents the maximum one-sample E-uncertainty $H_1(\Delta_1) = 8{\cdot}65$ bits which can be reduced to 2-bit level without the possibility of ADC resolution improvement (i.e. $q = 0$). Under the same conditions the E-uncertainty $H_1(\Delta_1) = 4{\cdot}7$ bits offers the improvement possibility of $q = 4$ bits. From the practical point of view it seems that there is no reason for taking $q > 4$ bits into account due to further errors introduced by non-linearity and thermal dependability of AD converter.

In case of Gaussian noise the resulting mean deviation due to resolution improvement is very small even for $\sigma_1/\Delta_1 \geqq 0{\cdot}3$, Fig. 4. For practical purposes ($q \leqq 4$ bits), therefore, good results can be obtained even without adding artificial noise to the input analog signal [4]. On the other hand other distribution types are by far not so advantageous.

The idea of this paper was to show the problems and limitations of quality increasing in measurements by averaging. The approach based on the E-uncertainty is likely to give very simple means for solving many similar problems in preprocessing of data.

The array

(A.1) $$A_{\alpha,\beta}(v) = 1 + \sum_{n=1}^{\infty} a_{\alpha,n} k^{n\beta} v^n$$

is for $M > 1$ and $\beta = 0$:

(A.2) $$A_{M-1,0}(v) = 1 + a_{M-1,1}v + a_{M-1,2}v^2 + a_{M-1,3}v^3 + \dots$$

and for $M = 1$, $\beta = M - 1$

(A.3) $$A_{1,M-1}(v) = 1 + a_{1,1}k^{M-1}v + a_{1,2}k^{2(M-1)}v^2 + a_{1,3}k^{3(M-1)}v^3 + \dots .$$

The components of the product

(A.4) $$A_{M,0}(v) = A_{M-1,0}(v)\, A_{1,M-1}(v)$$

are

(A.5) $$a_{M,n} = \sum_{j=0}^{n} a_{M-1,j} a_{1,n-j} k^{(n-j)(M-1)} .$$

For $a_{n,0} = 1$, $n = 1, 2, 3, \dots$ the expression (A.5) is simplified to

(A.6) $$a_{M,n} = a_{1,n}k^{n(M-1)} + a_{M-1,n} + \sum_{j=1}^{n-1} a_{M-1,j} a_{1,n-j} k^{(n-j)(M-1)}$$

i.e. for example

(A.7) $$a_{M,1} = a_{1,1}k^{M-1} + a_{M-1,1} = a_{1,1}k^{M-1} + a_{1,1}k^{M-2} + a_{M-2,1} = \dots$$

$$\dots = a_{1,1}(k^{M-1} + k^{M-2} + \dots + k + 1) = \frac{1 - k^M}{1 - k}\, a_{1,1} = S_1(M)\, a_{1,1} .$$

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