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Suggestion of a Cooperative Market Model

MILAN MAREŠ

This paper deals with a free exchange market. It suggests a modification of such market model which reflects the cooperation between agents. Also a modified equilibrium model for such market is defined, and analogies of fundamental theorems about connections between market equilibrium and game-theoretical solutions are introduced.

1. INTRODUCTION

The market models investigated in known literature deal with the bargaining between agents during the exchange process. Moreover, they use the apparatus of cooperative game-theory, especially the concept of core, and also many important results about mutual connections between markets and cooperative games are introduced there. However, in their nature, these models are not cooperative in the sense that agents do not correlate their behaviour in order to maximize some common profit, and to increase their individual income by means of rational dealing in such group profit.

There exists, already, a mathematical branch, namely cooperative games theory, in which many strong results concerning cooperative behaviour were obtained. This paper brings a suggestion how to modify the market and equilibrium model in order to introduce in it the cooperation in an explicite form. There are many models of cooperation in game theory. The one of them which was chosen for being investigated here is the model of games with side-payments. It is a simple model but, consequently, also very lucid one, giving a lot of elegant results proper for direct application. It is not difficult to see that also others, more complicated, game models without side-payments could be used for an analogical transposition into market theory, and that many fundamental results would keep valid. Nevertheless, it would be connected with introducing and developing of much more complicated and extensive formal aparatus, which was the main reason why it was not done here.

The cooperative games with side-payments which are mentioned in this paper are familiar for all who are interested in game-theory, but, may be, some of their principal ideas could be rather strange for other readers. For them the main principles of these games are briefly mentioned here.

First of all we have to accept the transferability of utility in such games. It means that the players may to transmit their utility among partners in the some coalition. Then, also, any rational coalition proceeds as one collective player with one common strategy tending to maximize the total utility of all its members. Then the final obtained utility is re-distributed among players in such way that all reasonable individual demands are respected. In this way the profit of all members of coalition is as big as possible in the coalition, and it is, moreover, greater than their individual guaranted profit achievable without cooperation.

The same attitude is applied in this paper concerning the cooperative market. We assume that agents may form coalitions and cooperate inside them. They may gather their goods and money together and realize the exchange process with anti-coalition using all the common goods. Also inside the coalition they exchange the real goods (not money) in order to maximize the common profit. The cooperation inside coalition is so strong that the exchange process among coalition partners is not limited by prices of goods. At the end of the exchange process they, by means of money, re-distribute the common profit among them in such way that everyone gets at least the utility which he had before cooperation. This model of agents behaviour causes a modification of the equilibrium. Any equilibrium in this cooperative market must respect not the possibilities of single agents (their budget-sets and initial commodity bundles) but, it has to reflect the possibilities of some class of coalitions. Namely, it will be the class of coalitions which are in some sense admissible in the given market situation; i.e. which are allowed (by anti-trust laws, e.g.) or which may appear according to usual manners of behaviour in the considered market.

The procedure described above is acceptable and understandable but it requests the fulfilling of a few conditions. The main of them is that there must exist some general representant of utility. We shall call it "money". The game-theoretical results concerning cooperative games with side-payments were obtained under the assumption that the general representant of utility has the same common utility for all players and that it is linear. Simply, the game-theoretical results were obtained under the assumption that the utility of money is the same for all players and that it is equal to their nominal value, i.e. to their quantity. For the game-theoretical investigation, where the direct economical application is not so immediate, this assumption makes no troubles. However, economical models deal with rather more general idea about the utility of money, and the game-theoretical simple assumption can cause some embarassments.

Nevertheless, we keep the assumption about the linear and common utility of money even in this paper. There are the following reasons for such decision. This paper should point at some ways of finding more connections between cooperative games and markets, and not to modify a wide part of games theory, what would be necessary in the other case. It is also probable that most of results which could be obtained for the presented model keep valid even for rather generalized utilities of money (e.g. linear but not the same for all players). Only the preparatory formal tools proving such results would be much more complicated and their detailed elaboration could cover the main purpose of this work.

2. COOPERATIVE MARKET

It is useful to introduce a few auxiliary symbols used during the paper. Let m be a positive integer, then R, R_+, R^m and R_+^m denote the sets of all real numbers, non-negative real numbers, m-dimensional real-valued vectors and m-dimensional real-valued vectors with non-negative coordinates, respectively. If $\mathcal{M}$ is a class of sets then we denote

$$\bigcup \mathcal{M} = \{i : \exists(K \in \mathcal{M})\ i \in K\},$$

and we say that $\mathcal{M}$ is a partition of some set M if $\bigcup \mathcal{M} = M$ and $K \cap L = \emptyset$ for $K, L \in \mathcal{M}$, $K \neq L$. Further, we denote

$$\langle \mathcal{M} \rangle = \{K \in \mathcal{M} : \textit{if } \mathcal{L} \subset \mathcal{M},\ \mathcal{L} \textit{ is a partition of } K, \textit{ then } \mathcal{L} = \{K\}\}.$$

Let us consider the following objects: a non-empty finite set I, continuous mappings U_i, $i \in I$, from R_+^m into R, and real-valued vectors $\boldsymbol{a}^i = (\boldsymbol{a}_0^i, \boldsymbol{a}_1^i, \ldots, \boldsymbol{a}_m^i) \in R \times R_+^m$, $i \in I$. Then the quadruple

$$(1) \qquad \boldsymbol{m} = (I, R \times R_+^m, (U_i)_{i \in I}, (\boldsymbol{a}^i)_{i \in I})$$

is called a *market*. Set I is the set of *agents*, we shall call them also *players*. We suppose that there is *money* and m kinds of regular *goods* in the considered market. Each real-valued vector $\boldsymbol{x} = (x_0, x_1, \ldots, x_m) \in R \times R_+^m$ represents x_0 units of money and x_j, $j = 1, \ldots, m$, units of other goods. Vectors $\boldsymbol{a}^i \in R \times R_+^m$, $i \in I$, represent the *initial distribution* of money and goods among players. The mappings $U_i : R_+^m \to R$, $i \in I$, express the *utilities of goods* for players.

We denote by $\mathcal{K}$ the class of all non-empty subsets of I. Elements from $\mathcal{K}$ are called *coalitions*. The mappings $u_i : R \times R_+^m \to R$, $i \in I$, such that

$$u_i(x_0, x_1, \ldots, x_m) = x_0 + U_i(x_1, \ldots, x_m),$$
$$i \in I, (x_0, x_1, \ldots, x_m) \in R \times R_+^m,$$

are called *utility functions* of players. The assumption of linear utility of money, used here, was already discussed in Introduction. It is rather strong but useful for further explanation, and it is used, e.g., even in some parts of [4].

In order to simplify further notation, we denote the amounts of money and goods belonging to players as

$$X = (x^i)_{i \in I}, \quad x^i \in R \times R^m_+ ,$$

and

(2) $$\mathbf{X} = \{X = (x^i)_{i \in I} : x^i \in R \times R^m_+, \ i \in I\} ,$$

(3) $$\mathbf{X}_K = \{X \in \mathbf{X} : \sum_{i \in K} x^i \leqq \sum_{i \in K} a^i\} , \quad K \in \mathscr{K} .$$

All goods, even money, have their *prices*. We denote them by $p = (p_0, p_1, \ldots, p_m)$, and by $\boldsymbol{P}$ we denote the set of all price vectors. We suppose that $\boldsymbol{P} \subset R^{m+1}_+$ and always $p_j > 0, j = 0, 1, \ldots, m$.

In the whole paper we assume that prices form row vectors from $\boldsymbol{P}$ and that quantities of money and goods form the column vectors from $R \times R^m_+$, so that the scalar product px, $p \in \boldsymbol{P}$, $x \in R \times R^m_+$, may be considered. Then we denote

(4) $$\mathbf{A}_K(p) = \{X \in \mathbf{X} : \sum_{i \in K} px^i \leqq \sum_{i \in K} pa^i\} , \quad p \in \boldsymbol{P} , \quad K \in \mathscr{K} .$$

Lemma 1. Let $K \in \mathscr{K}$, $p \in \boldsymbol{P}$, and let us denote u_K the mapping from $\mathbf{X}$ into R such that

$$u_K(X) = \sum_{i \in K} u_i(x^i) , \quad X = (x^i)_{i \in I} \in \mathbf{X} .$$

then there exist the maxima

$$\max \{u_K(X) : X \in \mathbf{X}_K\} \quad \text{and} \quad \max \{u_K(X) : X \in \mathbf{A}_K(p)\} .$$

Proof. Let us denote for any $X = (x^i)_{i \in I} \in \mathbf{X}$ by

(5) $$X^* = (x^{*i})_{i \in I}$$

the quantities of goods, for which

$$x^{*i} \in R^m_+ , \quad x^{*i}_j = x^i_j , \quad j = 1, \ldots, m ; \quad \text{i.e.} \quad x^i = (x^i_0, x^{*i}) . \quad i \in I .$$

Then we write for any $K \in \mathscr{K}$, $p \in \boldsymbol{P}$

(6) $$\mathbf{X}^*_K = \{X^* = (x^{*i})_{i \in I} : \exists (X = (x^i)_{i \in I} \in \mathbf{X}_K) \, \forall (i \in K) , \ x^i = (x^i_0, x^{*i})\} ,$$

(7) $$\mathbf{A}^*_K(p) = \{X^* = (x^{*i})_{i \in I} : \exists (X = (x^i)_{i \in I} \in \mathbf{A}_K(p)) \, \forall (i \in K) , \ x^i = (x^i_0, x^{*i})\} ,$$

and

(8) $$u_K(X) = \sum_{i \in K} x^i_0 + U_K(X^*) , \quad \text{where} \quad U_K(X^*) = \sum_{i \in K} U_i(x^{*i}) , \quad X \in \mathbf{X} ,$$

u_K may be considered as a continuous mapping from $R_+^{(mk)}$ into R, where k is number of agents in K. For any $X \in \mathbf{X}_K$ is

(9) $$\sum_{i \in K} x_0^i \leqq \sum_{i \in K} a_0^i$$

and

(10) $$\sum_{i \in K} x_j^i \leqq \sum_{i \in K} a_j^i\,, \quad x_j^i \geqq 0\,, \quad \text{for all} \quad j = 1, \ldots, m\,, \quad i \in K\,.$$

It means that there exists

$$\max \{U_K(X^*) : X^* \in \mathbf{X}_K^*\}\,,$$

and, according to (8) and (9), also the maximum of u_K on $\mathbf{X}_K$. Analogously, for any $X \in \mathbf{A}_K(\boldsymbol{p})$,

(11) $$\sum_{i \in K} \boldsymbol{p}\boldsymbol{x}^i = p_0 \sum_{i \in K} x_0^i + \sum_{j=1}^{m} p_j (\sum_{i \in K} x_j^i) \leqq \sum_{i \in K} \boldsymbol{p}\boldsymbol{a}^i\,,$$

so that

(12) $$\sum_{i \in K} x_0^i \leqq \frac{1}{p_0} \sum_{i \in K} \boldsymbol{p}\boldsymbol{a}^i\,, \quad \text{where} \quad p_0 > 0\,.$$

Further, for $X \in \mathbf{A}_K(\boldsymbol{p})$ also

(13) $$\sum_{j=1} p_j \sum_{i \in K} x_j^i \leqq \sum_{i \in K} \boldsymbol{p}\boldsymbol{a}^i - p_0 \sum_{i \in K} x_0^i\,,$$

as follows from (11). The positivity of p_j, $j = 0, 1, \ldots, m$, and the assumption $x_j^i \geqq 0$, $j = 1, \ldots, m$, imply that $\mathbf{A}_K^*(\boldsymbol{p}) \cap R_+^{(mk)}$ is a compact subset of $\mathbf{X}^*$, and the maximum of U_K on $\mathbf{A}_K^*(\boldsymbol{p})$ exists. This, together with (8) and (12), proves the existence of the maximum of u_K on $\mathbf{A}_K(\boldsymbol{p})$.

The state of market in any moment of the exchange process can be described in a sufficient way by the amount of goods owned by agents, and by the prices which are prescribed to them. So, we call the pair

(14) $$(X, \boldsymbol{p})\,, \quad X \in \mathbf{X}\,, \quad \boldsymbol{p} \in \mathbf{P}\,,$$

a *market state*, and we study its properties in the considered market.

3. EQUILIBRIUM

The main goal of this section is to introduce a modified concept of equilibrium. Such modification ought to fulfill two principal properties. First – it should represent the possibility of collaboration among agents, i.e. it should reflect the coalitions forming. Second – it should be possible to modify that equilibrium in some degree into stronger or weaker forms according to the actual properties of the given market.

These two properties can be realized in such way that the modified equilibrium does not respect the demands of all single agents, but it does so for some class of their coalitions. That class may be wider or very limited, and it may contain "small" coalitions or the large ones., up to the situation in any given market.

The concept of equilibrium, suggested in the following definition, fulfills the conditions formulated above.

Definition 1. Let $\mathcal{M} \subset \mathcal{K}$, $X = (x^i)_{i\in I} \in \mathbf{X}$, $p \in \boldsymbol{P}$. We call the market state $(X, \boldsymbol{p})$ an *$\mathcal{M}$-equilibrium* if $X \in \mathbf{X}_I$, and for any $K \in \mathcal{M}$

$$X \in \mathbf{A}_K(p)\,, \tag{15}$$

$$\sum_{i\in K} u_i(x^i) = \max\left\{\sum_{i\in K} u_i(y^i) : Y = (y^i)_{i\in I} \in \mathbf{A}_K(p)\right\}. \tag{16}$$

In any market $\boldsymbol{m}$ there exists at least one non-empty class $\mathcal{M} \subset \mathcal{K}$ for which the $\mathcal{M}$-equilibrium does exist, as follows from the next theorem.

Theorem 1. If $\mathcal{M} = \{I\}$ is one-element class of coalitions containing exactly the coalition of all players, and if $\boldsymbol{p} \in \boldsymbol{P}$ is an arbitrary price vector then there always exists $X \in \mathbf{X}$ such that $(X, \boldsymbol{p})$ is an $\mathcal{M}$-equilibrium.

Proof. It follows immediately from (3) and (4) that $\mathbf{X}_I \cap \mathbf{A}_I(p) \neq \emptyset$. If we keep the notation introduced in proof of Lemma 1, we may use (6) and (7), as well as (10), (13) (where we substitute $K = I$). It means that $\mathbf{X}_I^* \cap \mathbf{A}_I^*(p)$ is a compact subset of $R_+^{(mn)}$, where n is the number of agents in I. Consequently, the continuous function $U_I : \mathbf{X}_I^* \to R$ defined by (8) has its maximum on the set $\mathbf{X}_I^* \cap \mathbf{A}_I^*(p)$. The desired assertion follows immediately from this fact and from relations (8), (9) and (12) for $K = I$ and

$$u_I(X) = \sum_{i\in I} u_i(x^i) = \sum_{i\in I} x_0^i + U_I(X^*) = \sum_{i\in I} x_0^i + \sum_{i\in I} U_i(x^{*i})\,.$$

Lemma 2. Let $\boldsymbol{p} \in \boldsymbol{P}$, $K \in \mathcal{K}$, and let $\mathcal{J} \subset \mathcal{K}$ be a partition of K. Then for all $X = (x^i)_{i\in I} \in \mathbf{A}_K(p)$

$$\sum_{i\in K} u_i(x^i) \leqq \sum_{J\in\mathcal{J}} \max\left\{\sum_{i\in J} u_i(y_i) : Y = (y^i)_{i\in I} \in \mathbf{A}_J(p)\right\}. \tag{17}$$

Proof. Let there exists $X = (x^i)_{i\in I} \in \mathbf{A}_K(p)$ such that (17) is not true. We can redistribute the money among members of K constructing $Z = (z^i)_{i\in I}$ such that

$$z^i = x^i\,, \qquad i \in I - K\,,$$
$$z_j^i = x_j^i\,, \qquad i \in K\,,\ j = 1, \ldots, m\,,$$

$$z_0^i = x_0^i + d_i\,, \quad i \in K\,, \quad d_i \in R\,,$$

where d_i are such that

$$\sum_{i \in K} d_i = 0\,,$$

(18) $$\sum_{i \in J} u_i(z^i) < \max \left\{ \sum_{i \in J} u_i(y^i) : Y = (y^i)_{i \in I} \in \mathbf{A}_J(p) \right\}$$

for all $J \in \mathscr{J}$. Then $Z \in \mathbf{A}_K(p)$, as

$$\sum_{i \in K} p z^i = \sum_{i \in K} p x^i\,,$$

and there exists at least one $J \in \mathscr{J}$ such that $Z \in \mathbf{A}_J(p)$, as

$$\sum_{J \in \mathscr{J}} \sum_{i \in J} p z^i = \sum_{i \in K} p z^i \leqq \sum_{i \in K} p a^i = \sum_{J \in \mathscr{J}} \sum_{i \in J} p a^i\,.$$

This contradicts to (18) and, consequently, (17) holds for all $x \in \mathbf{A}_K(p)$.

Lemma 3. Let $p \in P$, $\mathscr{M} \subset \mathscr{K}$, $\langle \mathscr{M} \rangle$ be a partition of I. Let $X \in \mathbf{A}_J(p)$ for all $J \in \langle \mathscr{M} \rangle$. Then $X \in \mathbf{A}_K(p)$ for all $K \in \mathscr{M}$.

Proof. The statement follows immediately from

$$\sum_{i \in K} p x^i = \sum_{J \in \langle \mathscr{M} \rangle, J \subset K} \sum_{i \in J} p x^i \leqq \sum_{J \in \langle \mathscr{M} \rangle, J \subset K} \sum_{i \in J} p a^i = \sum_{i \in K} p a^i\,.$$

Lemma 4. Let $p \in X$, $\mathscr{M} \subset \mathscr{K}$, $\langle \mathscr{M} \rangle$ be a partition of I, and let $X = (x^i)_{i \in I} \in \mathbf{X}$. If

$$\sum_{i \in J} u_i(x^i) = \max \left\{ \sum_{i \in J} u_i(y^i) : Y = (y^i)_{i \in I} \in \mathbf{A}_J(p) \right\}$$

for all $J \in \langle \mathscr{M} \rangle$ then also

$$\sum_{i \in K} u_i(x^i) = \max \left\{ \sum_{i \in K} u_i(y^i) : Y = (y^i)_{i \in I} \in \mathbf{A}_K(p) \right\}$$

for all $K \in \mathscr{M}$.

Proof. If we consider the class $\mathscr{J} = \{J \in \mathscr{K} : J \subset K,\ J \in \langle \mathscr{M} \rangle\}$, $K \in \mathscr{K}$, then Lemma 2 immediately implies that

$$\max \left\{ \sum_{i \in K} u_i(y^i) : Y \in \mathbf{A}_K(p) \right\} \leqq$$
$$\leqq \sum_{J \in \langle \mathscr{M} \rangle, J \subset K} \max \left\{ \sum_{i \in J} u_i(y^i) : Y \in \mathbf{A}_J(p) \right\},$$

and the statement is proved.

After having proved these lemmas we may introduce a few of simple results concerning the equilibrium defined above. Some further results concerning that equilibrium are given in [7], too.

We denote by $\mathscr{I}$ the set of all one-element coalitions in the considered market; in symbols

$$\mathscr{I} = \{\{i\}\}_{i\in I}. \tag{19}$$

The concept of equilibrium investigated e.g. in [4] is identical with an $\mathscr{I}$-equilibrium in the terminology of this paper.

Theorem 2. Let $\mathscr{M} \subset \mathscr{K}$, $X \in \mathbf{X}$, $\boldsymbol{p} \in \boldsymbol{P}$. If $(X, \boldsymbol{p})$ is an $\mathscr{I}$-equilibrium then it is also an $\mathscr{M}$-equilibrium.

Proof. $\mathscr{I}$ is a partition of I and $\mathscr{I} = \langle\mathscr{K}\rangle$. If $(X, \boldsymbol{p})$ is an $\mathscr{I}$-equilibrium then Lemma 3 and Lemma 4 imply that it is a $\mathscr{K}$-equilibrium and, by Definition 1, it is also an $\mathscr{M}$-equilibrium for all $\mathscr{M} \subset \mathscr{K}$.

Corollary 1. If $\mathscr{I} \subset \mathscr{M} \subset \mathscr{K}$, $X \in \mathbf{X}$, $\boldsymbol{p} \in \boldsymbol{P}$ then $(X, \boldsymbol{p})$ is an $\mathscr{M}$-equilibrium if and only if it is an $\mathscr{I}$-equilibrium.

Corollary 2. If $\mathscr{I} \subset \mathscr{M} \subset \mathscr{K}$, $\mathscr{I} \subset \mathscr{N} \subset \mathscr{K}$ and if $X \in \mathbf{X}$. $\boldsymbol{p} \in \boldsymbol{P}$ then $(X, \boldsymbol{p})$ is an $\mathscr{M}$-equilibrium if and only if it is an $\mathscr{N}$-equilibrium.

Corollary 3. If $\mathscr{I} \subset \mathscr{M} \subset \mathscr{K}$ then any market state $(X, \boldsymbol{p})$ is an $\mathscr{M}$-equilibrium (or, especially, a $\mathscr{K}$-equilibrium) if and only if it is a competitive equilibrium in the sense of [4].

4. COALITION-GAME CONNECTED WITH A MARKET

It was said in the introductory section of this paper that the usual concept of equilibrium, in its nature, is not cooperative. Nevertheless, there exist some features of that model which led to certain results concerning the mutual analogy between market and game-theoretical concepts. Namely, the analogy between equilibrium and game-theoretical core was investigated, and very interesting and useful results were obtained.

The main purpose of this section is to show that these results are of more general nature, and they are true, after some modification, even for our concept of equilibrium. Moreover, the coalition-games theory suggested, except the well-known core, even some other, weaker, solutions. Here, we shall investigate the connection between the concept of $\mathscr{M}$-equilibrium and certain class of such game-theoretical solutions.

First of all we introduce a useful notion. Lemma 1 enables us to define a mapping $\boldsymbol{v} : \mathscr{K} \to R$ such that for any $K \in \mathscr{K}$ is

$$\boldsymbol{v}(K) = \max \{ \sum_{i\in K} u_i(\boldsymbol{x}^i) : X = (\boldsymbol{x}^i)_{i\in I} \in \mathbf{X}_K \} . \tag{20}$$

For this mapping the following statement is true.

Lemma 5. The mapping v defined by (20) is superadditive, it means that $v(K \cup L) \geqq \geqq v(K) + v(L)$ for any $K, L \in \mathscr{K}$, $K \cap L = \emptyset$.

Proof. Let K, $L \in \mathscr{K}$, $K \cap L = \emptyset$, let $X = (x^i)_{i \in I} \in \mathbf{X}_K$ and $Y = (y^i)_{i \in I} \in \mathbf{X}_L$ be such that

$$v(K) = \sum_{i \in K} u_i(x^i), \quad v(L) = \sum_{i \in L} u_i(y^i).$$

Then there exists $Z = (z^i)_{i \in I} \in \mathbf{X}$ such that $z^i = x^i$ for $i \in K$, $z^i = y^i$ for $i \in L$. Moreover, $Z \in \mathbf{X}_{K \cup L}$, as $X \in \mathbf{X}_K$, $Y \in \mathbf{X}_K$ and $K \cap L = \emptyset$. For this Z

$$v(K \cup L) = \max \{ \sum_{i \in K \cup L} u_i(\hat{z}^i) : \hat{Z} = (\hat{z}^i)_{i \in I} \in \mathbf{X}_{K \cup L} \} \geqq$$

$$\geqq \sum_{i \in K \cup L} u_i(z^i) = v(K) + v(L).$$

We accept the usual game-theoretical terminology and we call an *imputation* any real-valued vector $\xi = (\xi^i)_{i \in I}$, $\xi^i \in R$, such that there exists $X = (x^i)_{i \in I} \in \mathbf{X}$ for which $\xi^i = u_i(x^i)$ for any $i \in I$. The following auxiliary symbols enable us to simplify the further explanation.

$$\Xi = \{ \xi = (\xi^i)_{i \in I} : \exists (X \in \mathbf{X}) \, \forall (i \in I) \, u_i(x^i) = \xi^i \}, \tag{21}$$

$$\Xi_K = \{ \xi \in \Xi : \exists (X \in \mathbf{X}_K) \, \forall (i \in I) \, u_i(x^i) = \xi^i \}, \quad K \in \mathscr{K}, \tag{22}$$

If $\mathbf{m} = (I, R \times R^m_+, (U_i)_{i \in I}, (\mathbf{a}^i)_{i \in I})$ is a market then the pair $\Gamma_{\mathbf{m}} = (I, v)$, where v is defined by (20), is called a *coalition-game connected with the marked* $\mathbf{m}$, and the mapping v is called the *characteristic function* of the game $\Gamma_{\mathbf{m}}$.

Many concepts and results, well-known in the coalition-games theory, are true even for the game $\Gamma_{\mathbf{m}}$. The close relation between market $\mathbf{m}$ and game $\Gamma_{\mathbf{m}}$ enables us to apply those results also in the investigation of market.

The set of imputations Ω defined by

$$\Omega = \{ \xi : \xi \in \Xi_I, \ \forall (K \in \mathscr{K}) \sum_{i \in K} \xi^i \geqq v(K) \} \tag{23}$$

is called the *core* of the game $\Gamma_{\mathbf{m}}$.

There exists an important relation between equilibrium and core, investigated, e.g., in [4] and generalized in [8], namely the following one. If $(X, \mathbf{p})$, $X = (x^i)_{i \in I}$, $\mathbf{p} \in \mathbf{P}$, is an equilibrium in the classical sense (i.e. it is an $\mathscr{I}$-equilibrium in the terminology of this paper) and if $\xi = (\xi^i)_{i \in I} \in \Xi$ is such that $\xi^i = u_i(x^i)$, $i \in I$, then $\xi \in \Omega$ in the game $\Gamma_{\mathbf{m}}$ connected with considered market $\mathbf{m}$. The concept of $\mathscr{M}$-equilibrium defined in this paper represents a weaker form of equilibrium. Then we may ask whether some weaker form of core can be defined, too. The answer is positive. There are many weaker concepts of game solutions given in the literature, e.g. in [3], [1] or [2]. One of them, suggested by author in [2], is of the form that it substitutes a class $\mathcal{M} \subset \mathcal{K}$ for $\mathcal{K}$ in formula (23). Other solutions are defined in another way, but they can be, at least approximately, described by formulas analogous to (23) with some class $\mathcal{N} \subset \mathcal{K}$ instead of $\mathcal{K}$. None of the solutions mentioned above is completely satisfactory for all cases of actual games appearing in real applications. It means that we may expect the appearance of some new solutions based, may be, also on the principle that they will be defined as sets of imputations analogous to Ω, where the class $\mathcal{K}$ in (23) will be substituted by some $\mathcal{M} \subset \mathcal{K}$. All game-theoretical solutions of this general type are closely related to the market $\mathcal{M}$-equilibria and it is worth formulating their intuitive description by the following exact definition.

Definition 2. Let $\Gamma_{\boldsymbol{m}}$ be a coalition-game, let $\mathcal{M} \subset \mathcal{K}$ and $\xi \in \Xi$. Then we say that ξ is *$\mathcal{M}$-stable* if $\xi \in \Xi_I$ and

$$\sum_{i \in K} \xi^i \geqq v(K) \quad \text{for all} \quad K \in \mathcal{M} .$$

The set of all $\mathcal{M}$-stable imputations in $\Gamma_{\boldsymbol{m}}$ will be denoted by $\Omega_{\mathcal{M}}$.

Remark 1. It is obvious from Definition 2 and (23) that $\Omega = \Omega_{\mathcal{K}}$.

The relation between $\mathcal{M}$-equilibria and $\mathcal{M}$-stable imputations is described by the following result.

Theorem 3. Let $\boldsymbol{m}$ be a market and $\Gamma_{\boldsymbol{m}}$ be the coalition-game connected with $\boldsymbol{m}$. Let $\mathcal{M} \subset \mathcal{K}$, $X = (x^i)_{i \in I} \in \mathbf{X}$, $p \in P$, $\xi = (\xi^i)_{i \in I} \in \mathbf{X}$, and let $\xi^i = u_i(x^i)$ for all $i \in I$. If (X, p) is an $\mathcal{M}$-equilibrium then the imputation ξ is $\mathcal{M}$-stable.

Proof. It follows immediately from (3) and (4) that for all $K \in \mathcal{K}$ is $\mathbf{A}_K(p) \subset \mathbf{X}_K$. If $\mathcal{M} \subset \mathcal{K}$, $X = (x^i)_{i \in I} \in \mathbf{X}$, $p \in P$ and (X, p) is an $\mathcal{M}$-equilibrium then the previous inclusion implies

$$\sum_{i \in K} u_i(x^i) = \max \left\{ \sum_{i \in K} u_i(y^i) : Y \in \mathbf{A}_K(p) \right\} \geqq \max \left\{ \sum_{i \in K} u_i(y^i) : Y \in \mathbf{X}_K \right\} = v(K)$$

for all $K \in \mathcal{M}$. Consequently,

$$\sum_{i \in K} \xi^i \geqq v(K)$$

for all $K \in \mathcal{M}$, and $\xi \in \Xi_I$ as $X \in \mathbf{X}_I$.

The classical result about the relations between core and $\mathcal{I}$-equilibria is an immediate consequence of the previous Theorem 3 and of Theorem 2 as follows from the next statement.

Corollary 4. Let the assumptions of Theorem 3 be fulfilled and let $\mathcal{M} = \mathcal{I}$. If (X, p) is an $\mathcal{I}$-equilibrium then ξ is $\mathcal{K}$-stable; i.e. $\xi \in \Omega_{\mathcal{K}} = \Omega$.

The relations represented by previous Theorem 3 and Corollary 4 are formulated as implications. These implications can not be, generally, turned into equivalences, as there exists a principal difference between market equilibrium and game-theoretical stability. It follows from the difference between (16) and (20), namely from the fact that in case of $\mathcal{M}$-equilibrium and $\mathcal{M}$-stability the common profit of coalitions is maximized over sets $\mathbf{A}_K(p)$ and $\mathbf{X}_K$, $K \in \mathcal{M}$, respectively.

However, there exist special assumptions under which the complementary implication to Theorem 3 can be proved.

Theorem 4. If $p \in P$, $\mathcal{I} \subset \mathcal{M} \subset \mathcal{K}$, if

$$\max \{u_i(y^i) : Y = (y^i)_{i \in I} \in \mathbf{X}_{\{i\}}\} = \max \{u_i(y^i) : Y = (y^i)_{i \in I} \in \mathbf{A}_{\{i\}}(p)\}$$

for all $i \in I$, and if $X = (x^i)_{i \in I} \in \mathbf{X}_I$, $\xi = (\xi^i)_{i \in I} \in \Xi$, $\xi^i = u_i(x^i)$, $i \in I$, then (X, p) is an $\mathcal{M}$-equilibrium if ξ is $\mathcal{I}$-stable.

Proof. Let us suppose that $X = (x^i)_{i \in I} \in \mathbf{X}_I$, $p \in P$, $\mathcal{I} \subset \mathcal{M} \subset \mathcal{K}$, and that for all $i \in I$

$$u_i(x^i) \geqq \max \{u_i(y^i) : Y = (y^i)_{i \in I} \in \mathbf{A}_{\{i\}}(p)\} . \tag{24}$$

Let us suppose, further, that there exists $k \in I$ such that $X \notin \mathbf{A}_{\{k\}}(p)$; i.e.

$$px^k < pa^k . \tag{25}$$

Because of $X \in \mathbf{X}_I$, the inequality

$$\sum_{i \in I} px^i \leqq \sum_{i \in I} pa^i$$

follows from (3) immediately, and (25) implies that there exists $l \in I$ such that $px^l < pa^l$. Let us construct $Y = (y^i)_{i \in I} \in \mathbf{X}$ such that

$$y_0^l = x_0^l + (pa^l - px^l)/p_0 ,$$
$$y_j^l = x_j^l , \quad j = 1, \ldots, m .$$

This Y belongs to $\mathbf{A}_{\{l\}}(p)$, as $py^l = pa^l$, and

$$u_l(y^l) = u_l(x^l) + \frac{1}{p_0} (pa^l - px^l) < u(x^l) ,$$

which is a contradiction with (24). Consequently, (25) can not be true, and $X \in \mathbf{A}_{\{i\}}(p)$ for all $i \in I$, if (24) is true for all $i \in I$. Let us suppose, now, that ξ is $\mathcal{I}$-stable. Then

$$u_i(x^i) = \xi^i \geqq v(\{i\}) = \max \{u_i(y^i) : Y \in \mathbf{A}_{\{i\}}(p)\} \tag{26}$$

for all $i \in I$. It means that condition (24) is fulfilled and, as $X \in \mathbf{X}_I$ by assumption, also $X \in \mathbf{A}_{\{i\}}(p)$ for all $i \in I$. Consequently, (X, p) is an $\mathcal{I}$-equilibrium and, by Theorem 2, it is also an $\mathcal{M}$-equilibrium.

Corollary 5. Under the assumptions of Theorem 4, (X, p) is an $\mathcal{M}$-equilibrium if and only if ξ is $\mathcal{M}$-stable.

Corollary 6. Under the assumptions of Theorem 4, if ξ is $\mathcal{I}$-stable then (X,p) is an equilibrium in the sense of [4], as follows from Theorem 4 and Corollary 3 if we put $\mathcal{M} = \mathcal{I}$.

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RNDr Milan Mareš, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 180 76 Praha 8. Czechoslovakia.