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Automatic Stochastic Control of Impulses on a Three-Dimensional Crystallographic Lattice

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In Cybernetics there are numerous problems, requiring the modelling of the propagation of impulses through nets of centres, communication stations or through-flow reservoirs.

Conditions for the Automatic Control of the spread of impulses on a three-dimensional crystallographic lattice of the group I$\bar{4}$3 are investigated. Results are given of the mapping of the points of the Control Space onto the State Space of the Complex using the Chapman-Kolmogorov equation with a matrix of transition probabilities of the type (728,728) for $n = 1501$. On the bases of the information theory of control, founded upon the ε-entropy and the ε-capacity of a certain set, the choice of the appropriate variety of the states is done taking account of the prescribed accuracy of Control and of the given resolution of the analyzer.

1.0 There is a class of problems in Cybernetics connected with the oriented spread of impulses in nets with many nodes (neural nets, nets of retranslating communication centres), the spread of means or stuff (pharmaca, catalyzers, radioactive material) in through-flow models with interconnected perfectly-mixed reservoirs and in compartmental systems, leading to similar abstract models [6]. The spread of impulses or means can be considered as a communication problem. At the same time there is

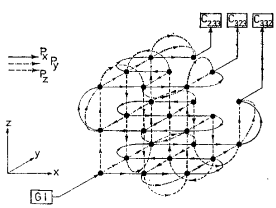

Fig. 1. A specimen of the lattice considered. GI — generator of impulses $C_{2,3,3}$, $C_{3,2,3}$, $C_{3,3,2}$ — counters of impulses.

Table 1

Case	Control vector			State vector				
	P_x	P_y	P_z	$x_{8,9,9}$	$x_{9,8,9}$	$x_{9,9,8}$	Σ	n
1	0·9	0·05	0·05	0·0983015660	0·3894614070	0·4893429526	0·9771059256	1501
2	0·65	0·05	0·3	0·2273123826	0·4233222367	0·3272666574	0·9779012767	1501
3	0·35	0·05	0·6	0·2831331187	0·4100617569	0·2864333693	0·9796282450	1501
4	0·05	0·05	0·9	0·3606922940	0·4476952846	0·1674391379	0·9758273631	1501
5	0·05	0·35	0·6	0·3131555342	0·3360939016	0·3269330495	0·9761824853	1501
6	0·05	0·65	0·3	0·3144887833	0·3046030539	0·3567378162	0·9758296535	1501
7	0·05	0·9	0·05	0·3394842043	0·1460477881	0·4914668099	0·9769988023	1501
8	0·3	0·65	0·05	0·2448484437	0·2160259248	0·5207299997	0·9816043682	1501
9	0·6	0·35	0·05	0·2083746801	0·2501747248	0·5233536200	0·9819030250	1501
10	0·8	0·1	0·1	0·1745246645	0·3677874253	0·4329309410	0·9752430307	1501
11	0·6	0·1	0·3	0·2454988790	0·3857993628	0·3486313556	0·9799295973	1501
12	0·3	0·1	0·6	0·2920742407	0·3804497702	0·3073246153	0·9798486262	1501
13	0·1	0·1	0·8	0·3280107313	0·3870483669	0·2591436329	0·9742027310	1501
14	0·1	0·3	0·6	0·3099253090	0·3396758565	0·3291506675	0·9787518330	1501
15	0·1	0·6	0·3	0·3068161244	0·3043671123	0·3678124148	0·9789956514	1501
16	0·1	0·8	0·1	0·3111496468	0·2265588742	0·4376217973	0·9753303183	1501
17	0·3	0·6	0·1	0·2662051725	0·2541375674	0·4608922029	0·9812349428	1501
18	0·6	0·3	0·1	0·2294277210	0·2920431926	0·4597609680	0·9812318815	1501
19	0·333	0·333	0·333	0·2792895092	0·3268782481	0·3771534691	0·9833212264	1501

often a requirement to control this spread, thus uniting Control and Communication and using the principle of feedback. Some of these problems have been already treated in two-dimensional euclidean space [2], [6].

2.0 It is the purpose of this paper to describe the theoretical premisses for automatic stochastic control of impulses on a three-dimensional crystallographic lattice. As in [2], the following convention is introduced: translations are indicated by arrows, which enables to pass directly to the graph of the Markov process on the lattice. We can thus associate with the lattice on Fig. 1 a three-dimensional graph of the pertinent Markov process of the spread of impulses.

The investigated lattice on Fig. 1 has been constructed artifically by using strata of two-dimensional crystallographic lattices of the **pgg** group (in the notation of Hermann and Mauguin), by orienting and connecting them as shown in principle on Fig. 1. The reason for choosing planar elements of the **pgg** group were the good theoretical results obtained from the point of view of control with a similar two-dimensional lattice of the group **pgg** [2].

Ct. Novák [3] has identified the three-dimensional group as $I\bar{4}3$ m, No. 217, according to Int. Tab. I, [4], p. 326, belonging to the Cubic System. At the same time he has shown the relation of this group to the group I222, No. 23, according to Int. Tab. I [4], p. 109 (of the Orthorhombic System) and to group $I\bar{4}2$ m, No. 121, according to Int. Tab. I [4], p. 211 (of the Tetragonal System).

The 2 later groups would visibly result e.g. from making the length of the oriented edges of the lattice on Fig. 1 proportional to the control probabilities (all 3 edges from a node with different lengths, or 2 with same length and the third edge with a different length).

Fig. 1 shows only a specimen of the lattice considered, with $(3 \times 3 \times 3) - 1$ nodes, the node which has been cut away being (3, 3, 3). Impulses are repeatedly applied to node (1, 1, 1) from a generator of impulses GI. Their propagation through the lattice is governed by 3 probabilities p_x, p_y, p_z in the directions indicated by the arrows. Notice the external connections which complete the conditions on some peripheral nodes.

Three counters of arriving impulses are supposed at the nodes (3, 2, 3), (2, 3,3) and (3, 3, 2) and they function as absorbing elements.

Actually the lattice used for theoretical investigation had $(9 \times 9 \times 9) - 1 = = 728$ nodes, the node cut away being (9, 9, 9). The place of application of the impulses is node (1, 1, 1). The control of their spread is effected by the vector $\mathbf{U} = (p_x, p_y, p_z)$ applied to the Complex i.e. to all nodes, excepting the 3 connected with the counters. Counters of impulses are supposed at the nodes (9, 8, 9), (8, 9, 9) and (9, 9, 8) and their indications serve to determine the relative frequencies $x^*_{9,8,9}$, $x^*_{8,9,9}$, $x^*_{9,9,8}$, converging, when the number of applied impulses increases to ∞, towards the probabilities $x_{9,8,9}$, $x_{8,9,9}, {}_x x_{9,9,8}$ which are considered as the state variables of the Complex.

3.0 The aim of the Control is to obtain a State $\mathbf{X}$ of the Complex prescribed by the vector of the Command variables $\mathbf{R}$, with a given accuracy $\varepsilon_S = \mathbf{R} - \mathbf{X}$.

The feedback loop of the Automatic Control is as follows: the three-dimensional lattice, the Analyser of the measured state variables of the Complex, the Control Element acting upon the Complex through the appropriate Control vector $\mathbf{U}$. The Control Element gets information about the desired state of the Complex through the vector $\mathbf{R}$ of the Command variables from the Command Element.

The spread of the impulses on the lattice is modelled by a Markov Chain. The Chapman-Kolmogorov equation in matrix form

$$(1) \qquad \mathbf{S}(n) = \mathbf{S}(0) \,.\, [p_{jk}]^n$$

is used to determine $\mathbf{S}(n)$ – the vector of the probabilities of the different states of the Complex at discrete time n.

$\mathbf{S}(n)$ is a row vector with 728 coordinates,

$\mathbf{S}(0)$ is the vector of the initial state probabilities.

Supposing, that at time 0 an impulse is applied to node (1, 1, 1), the coordinates of this row vector are: 1 followed by 727 zeros;

$[p_{jk}]$ is the matrix of transition probabilities of type (728, 728), which has 529 984 elements;

n is the number of steps (discrete time). Computations of eq. (1) have been carried out till $n = 1501$.

The choice of the Control vector it theoretically based on the mapping of the end-points of the Control vectors from the state-space of the Control Element into the state space of the Complex. The cases of Control considered are on Fig. 2. The

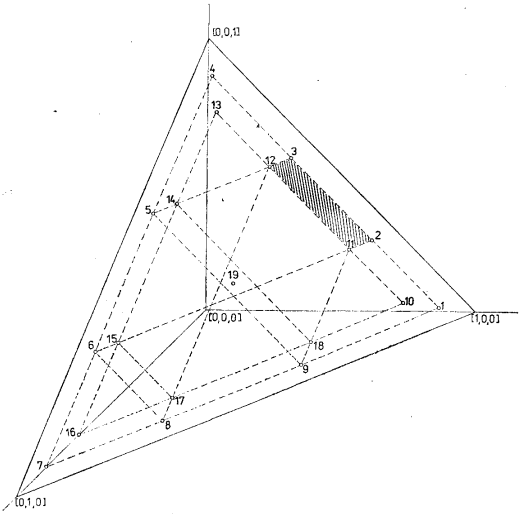

Fig. 2. The State Space of the Control element.

corresponding points in the State Space of the Complex are on Fig. 3, but they are shown already tilted around the axis ([1, 0, 0]; [0, 1, 0]) into the ground plane ([0, 0, 0]; [1, 0, 0]; [0, 1, 0]). The numerical values and results, obtained by applying eq. (1) and shown on Fig. 2 and Fig. 3 are given in Tab. 1, where n is the number of steps and $\sum$ is the sum $(x_{8,9,9} + x_{9,8,9} + x_{8,8,9})$ for $n = 1501$.

The evolution of this sum $\sum$ for cases 4, 12, 19 in time is given on Fig. 4, showing a good convergency towards the value 1.

4.0 An important problem in the design of the feedback loop for Automatic Control is the choice of the number of the states of the Complex. Use is done of the basic inequality, known from the work of B. N. Petrov et alii [1]

$$H_{\varepsilon S}(\mathscr{R}) < \log_2 J < L_{\varepsilon_{ut}}(\mathscr{R}) \tag{2}$$

where $\mathscr{R}$ is the set of the end-points of the vector of the Command variables,

H_{ε_S} is the minimum ε_S-entropy of the set $\mathscr{R}$,
ε_S is the admissible control deviation,
$L_{\varepsilon_{ut}}$ is the ε_{ut}-capacity of the set $\mathscr{R}$,
ε_{ut} is the utilitary threshold of resolution in the state space of the measured variables of the Complex,

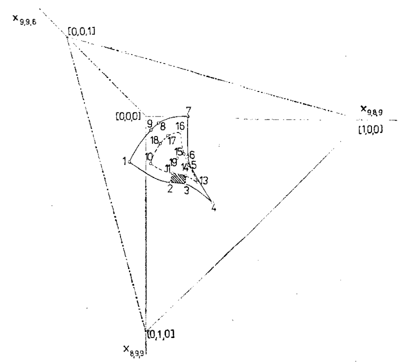

Fig. 3. The State Space of the Complex.

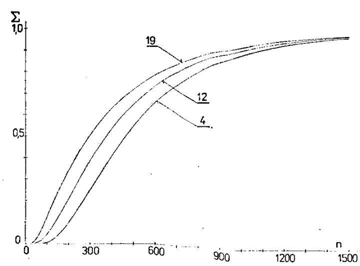

Fig. 4. The sum $\sum = (x_{8,9,9} + x_{9,8,9} + x_{9,9,8})$ as function of the number of steps n.

J is the number of states of the Complex, $\log_2 J$ is the variety of the number of states.

Let us consider the mapping of the quadrilateral 2–3–12–11 of Fig. 2 on the state space of the Complex as shown on Fig. 3, where the quadrilateral 2–3–12–11 is given in its true form (after rotation around the axis ([1, 0, 0]; [0, 1, 0]) into the ground plane ([0, 0, 0]; [1, 0, 0]; [0, 1, 0]). The sides of this quadrilateral on Fig. 3 are $\delta_{2,3} = 0{\cdot}07042$; $\delta_{11,2} = 0{\cdot}04689$; $\delta_{11,12} = 0{\cdot}06149$; $\delta_{12,3} = 0{\cdot}03731$. Its surface can be approximated as $0{\cdot}042 \times 0{\cdot}066$.

4.1 For a prescribed admissible deviation $\varepsilon_S = 0{\cdot}02$, the minimum number of sets in the ε_S-covering of the set $\mathscr{R}$ is estimated as

$$N_{\varepsilon S}^{\min}(\mathscr{R}) = \left(\frac{0{\cdot}042}{2\varepsilon_S} + 1\right)\left(\frac{0{\cdot}066}{2\varepsilon_S} + 1\right)$$

with the condition that both factors of the product are integers. We take $N_{\varepsilon S}^{\min}(\mathscr{R}) = 2 \times 2 = 4$.

4.2 The ε_{ut}-entropy of the set $\mathscr{R}$ is then

$$H_{\varepsilon S}(\mathscr{R}) = \log_2 N_{\varepsilon S}^{\min}(\mathscr{R}) = 2\,.$$

This is shown on Fig. 5.

4.3 The ε_{ut}-capacity of the set $\mathscr{R}$ is computed from the estimate of the number $N_{\varepsilon_{ut}}^{\max}(\mathscr{R})$ of the maximum number of points in the ε_{ut}-discernible subset of the set $\mathscr{R}$ as

$$L_{\varepsilon_{ut}}(\mathscr{R}) = \log_2 N_{\varepsilon_{ut}}^{\max}(\mathscr{R})\,.$$

For a selected $\varepsilon_{ut} = 0{\cdot}015$ the number $N_{\varepsilon_{ut}}^{\max}(\mathscr{R})$ is estimated as follows:

$$N_{\varepsilon_{ut}}^{\max}(\mathscr{R}) = \left(\frac{0{\cdot}042}{\varepsilon_{ut}} + 1\right)\left(\frac{0{\cdot}066}{\varepsilon_{ut}} + 1\right)$$

with the condition that both factors of the product are integers. We take $N_{\varepsilon_{ut}}^{\max}(\mathscr{R}) = 3 \times 5 = 15$.

The ε_{ut}-capacity is then

$$L_{\varepsilon_{ut}}(\mathscr{R}) = \log_2 15 = 3{\cdot}9068906\,.$$

This is shown on Fig. 5.

Similarly for $\varepsilon_{ut} = 0{\cdot}005$ the ε_{ut}-capacity is 6·9772, for $\varepsilon_{ut} = 0{\cdot}010$ it is 5·1292, for $\varepsilon_{ut} = 0{\cdot}025$ it is 2·5849, and for $\varepsilon_{ut} = 0{\cdot}030$ it is 2·3219. The dashed curve on Fig. 5, connecting these isolated points indicates merely the general (rough) shape of the upper boundary in the inequality (2).

4.4 Taking into account also the lower boundary given by $H_{\varepsilon s}(\mathscr{R})$ = const. = 2 on Fig. 5, and considering for technical reasons the same subdivision of the scale of the counters placed at nodes (9, 8, 9), (8, 9, 9) and (9, 9, 8) we choose the number of states of the Complex for purposes of Control in the quadrilateral zone 2 – 3 – 12 – – 11 of the State Space of the Complex as $J = 2 \times 2 \times 2 = 8$, which gives a variety

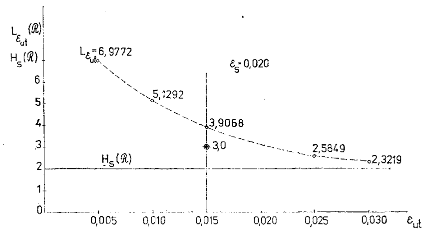

Fig. 5. The choice of the variety of States of the controlled Complex.

of states equal to 3, shown by a doubly circled point on Fig. 5, This point is well far away from both the boundaries of the admissible variety of States given by inequality (2).

This gives an important indication for the design of the feedback control loop for the prescribed $\varepsilon_S = 0{\cdot}020$ and the chosen $\varepsilon_{ut} = 0{\cdot}015$.

5.0 The computations according to equation (1) with stochastic matrices with 529 984 elements raised to exponent values reaching $n = 1501$ have been carried out by J. Grim on the IBM 370/135 computer of the Institute of Information Theory and Automation of the Czechoslovak Academy of Sciences. The stochastic character of the matrices has been constantly checked during the computation.

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It is hoped that the results obtained will serve as indications and possible incentive for further work on the oriented probabilistic propagation of impulses on other three-dimensional crystallographic group lattices in different interdisciplinary areas of Cybernetics.

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