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Discrete Optimal Control Problems with Nonsmooth Costs

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In this paper three important discrete optimal control problems with nonsmooth cost functionals are formulated. Various possibilities of their solution are discussed and demonstrated. For a special class of controlled systems a separable-programming technique is recommended which can be applied to many other control problems as well.

1. INTRODUCTION

This paper can be divided into 4 main sections (§ 2, 3, 4, 5). In § 2, 3 and 4 three various discrete optimal control problems with nonsmooth cost functionals are formulated, namely so called minimum overshoot problem, two-sided minimum overshoot problem and minimum output error problem. Usual optimality conditions cannot be applied to these problems. However, all of them can be easily transcripted into the form of a linear or separable-programming problem. In the case, when the size of the resulting problem is too big or when equality constraints are absent, an effective subgradient optimization technique can be applied. ([2; 5; 7; 8]). The application of separable programming is not restricted to problems mentioned above. A broad class of optimal control problems can be solved for instance by the δ-method of separable programming. This is explained in § 5.

Numerical results are obtained by means of a linear-programming procedure included in the mathematical programming system MPS 360 Version 2 and all computations were executed on the IBM 370/135 computer.

The following notation is employed: E^n is the Euclidean n-space, $\langle \cdot, \cdot \rangle$ is the scalar product, x^j is the j-th coordinate of a vector x, A^T is the transpose of a matrix A, E is the unit matrix, ϑ is the zero matrix, $f'(x, y)$ is the directional derivative of the function f at x in the direction y, A^l is the l-th row of a matrix A.

Given a linear stationary dynamical system described by the difference equation

(1) $$x_{i+1} = Ax_i + Bu_i\,, \quad x_i \in E^n\,, \quad u_i \in E^m\,, \quad i = 0, 1, \ldots, k-1\,,$$

where x_i is the state of the system and u_i is the control applied to the system at time i, A resp. B is a $[n \times n]$ resp. $[n \times m]$ constant matrix and $k \geqq 1$ is a given integer. Let

(2) $$x_0 = \hat{x}_0$$

be the given initial state,

(3) $$x_k = x_T$$

be the given terminal state of the system and

(4) $$U = \{u \mid u^j \in [-1, 1]\,, \quad j = 1, 2, \ldots, m\}$$

be the set of admissible controls. Let further

(5) $$v_i = \langle c, x_i \rangle\,, \quad i = 0, 1, \ldots, k$$

be the scalar output of the system (1) at time i, where $c \in E^n$ is a constant vector.

It is to find a control sequence $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}$ and a corresponding trajectory $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_k$ determined by (1) and (2) which minimize the cost functional

(6) $$\varphi_1(u_0, u_1, \ldots, u_{k-1}) = \max_{i=1,2\ldots k-1} v_i$$

subject to the constraints (3), (4) and (5).

Here and in § 3 and 5 we shall assume that the controlled systems are controllable from the initial state $\hat{x}_0$ to the terminal state x_T by admissible controls.

Outputs v_i as functions of controls $u_0, u_1, \ldots, u_{i-1}$ are continuous and therefore the cost φ_1 is also continuous. Hence, this problem is a minimization of the continuous function on the nonempty compact set, the solution of which allways exists. Moreover, the linearity of the functions v_i implies the convexity of φ_1. Our problem is therefore that of convex programming.

In its solution we can choose between two basically different ways:

(i) Transcription into the linear-programming form by adding an artificial variable and some additional constraints.

(ii) Using any subgradient method, capable to handle equality and inequality constraints.

With respect to the efficiency of modern simplex algorithms, we suppose that the first way is more suitable in this case. This transcription can be carried out for instance as follows:

Let ξ be an unconstrained artificial variable. Then our problem can be written down in the form

$$\xi \to \min$$

subj. to

$$\begin{aligned} &\langle c, x_i \rangle \leqq \xi , \quad i = 1, 2, \ldots, k - 1 , \\ (7) \qquad &x_{i+1} = Ax_i + Bu_i , \\ &x_0 \quad = \hat{x}_0 , \\ &x_k \quad = x_T , \\ &u_i \in U , \quad i = 0, 1, \ldots, k - 1 . \end{aligned}$$

Now we set

$$(8) \qquad z^T = (u_0^1, u_0^2, \ldots, u_0^m, u_1^1, \ldots, u_1^m, \ldots, u_{k-1}^m, \xi) \in E^{km+1}$$

and transcript the problem (7) into the standart form of linear programming

$$\langle \eta_1, z \rangle \to \min$$

subj. to

$$\begin{aligned} (9) \qquad & A_1 z \leqq b_1 , \\ & A_2 z = b_2 , \\ & z^i \in [-1, 1] , \quad i = 1, 2, \ldots, km , \end{aligned}$$

where

$$\eta_1^T = (0, 0, \ldots, 0, 1) \in E^{km+1}$$

(10)

$$A_1 = \begin{bmatrix} c^TB & 0 & \ldots & \ldots & 0 & -1 \\ c^TAB & c^TB & 0 & \ldots & 0 & -1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c^TA^{k-2}B & c^TA^{k-3}B & \ldots & c^TB & \underbrace{0 0 \ldots 0}_{m \times} & -1 \end{bmatrix} \text{ is a } [(k-1) \times (km+1)] \text{ matrix,}$$

$$A_2 = [A^{k-1}B \;\; A^{k-2}B \;\; \ldots \;\; AB \;\; B \;\; 0] \text{ is a } [n \times (km+1)] \text{ matrix,}$$

$$b_1 = -\begin{bmatrix} c^TA\hat{x}_0 \\ c^TA^2\hat{x}_0 \\ \vdots \\ c^TA^{k-1}\hat{x}_0 \end{bmatrix} \in E^{k-1} \quad \text{and} \quad b_2 = x_T - A^k\hat{x}_0 .$$

The linear programming problem (9) has $km + 1$ variables (km lower and upper bounded), $k - 1$ inequality and n equality constraints. Due to the special form of the matrix A_1 the bounded variable revised simplex algorithm with a product form of the inverse will be very effective.

We shall now clarify results of this method in the following example:
Let the controlled system be given by the equation

$$x_{i+1} = \begin{bmatrix} 0{\cdot}905 & 0{\cdot}092 & 0 & 0 \\ 0 & 0{\cdot}932 & 0 & 0 \\ 0 & 0 & 0{\cdot}97 & 0{\cdot}095 \\ 0 & 0 & 0 & 0{\cdot}932 \end{bmatrix} x_i + \begin{bmatrix} 0{\cdot}095 & 0{\cdot}005 \\ 0 & 0{\cdot}097 \\ 0{\cdot}005 & 0{\cdot}099 \\ 0{\cdot}097 & 0 \end{bmatrix} u_i ,$$

$$\hat{x}_0 = \begin{bmatrix} 3 \\ -2 \\ -5 \\ 4 \end{bmatrix}, \quad x_T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad k = 28 , \quad v_i = x_i^3 .$$

First we have solved this control problem irrespective of the overshoot (value of the cost (6)) as a two point boundary value problem. Afterwords we have minimized the value of φ_1 according to the method described above. Both sequences of outputs

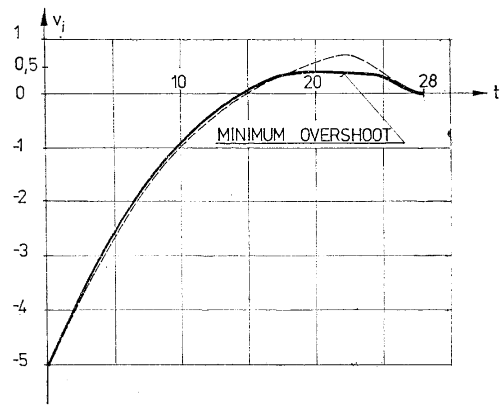

Fig. 1.

v_i are depicted in Fig. 1. (To visualize the outputs, evolution curves are fitted to the respective points v_i).

The optimal control sequence of this minimum overshoot problem is presented in Table 1.

Tab. 1.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
u_i^1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
u_i^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
i	14	15	16	17	18	19	20	21	22	23	24	25	26	27
u_i^1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0·86	0·5	1	1	1
u_i^2	1	1	1	0·15	0·1	0·2	0·24	0·42	0·91	0·04	−0·49	−1	−1	−1

3. TWO-SIDED MINIMUM OVERSHOOT PROBLEM

Given a controlled system (1) with the output (5), the initial state (2), the terminal state (3), and the set of admissible controls (4). Let N and k be given integers, $1 \leqq N < k$. It is to find a control sequence $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}$ and a corresponding trajectory $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_k$ determined by (1) and (2), which minimize the cost functional

$$\varphi_2(u_0, u_1, \ldots, u_{k-1}) = \max_{i=N,N+1,\ldots,k-1} |v_i - v_k| \tag{11}$$

subject to the constraints (3), (4) and (5).

This problem is that of convex programming again for the same reasons as in the one-side case and its solution also exists.

In [1], [6] a method is described for treating of a certain class of discrete optimal control problems with the absolute value in their costs. In the case of problem beeing discussed we come to similar results, if we use the well-known δ-method of separable programming. Moreover, this method can be applied for more general discrete systems as well, what will be shown in § 5.

We introduce first the vector variable

$$z^T = (u_0^1, \ldots, u_0^m, u_1^1, \ldots, u_1^m, \ldots, u_{k-1}^m, v_N - v_k, v_{N+1} - v_k, \ldots, v_{k-1} - v_k) \in \in E^{km+k-N} \tag{12}$$

and denote by ϱ a maximal admissible value of $|v_i - v_k|$ for $i = N, N+1, \ldots, k-1$. (Evidently, if ϱ is chosen too small, the solution need not exist). Now we write

$$v_i - v_k = -\varrho + s_i + t_i \tag{13}$$

and

$$|v_i - v_k| = \varrho - s_i + t_i\,, \tag{14}$$

where s_i, t_i are new nonnegative bounded variables,

$$0 \leqq s_i,\ t_i \leqq \varrho\,, \quad i = N,\ N+1, \ldots, k-1\,. \tag{15}$$

They must satisfy the following regularity condition: The variable t_i can have a nonzero value, if

$$s_i = \varrho\,, \quad i = N,\ N+1, \ldots, k-1\,. \tag{16}$$

We shall call them "separating". Now we set

(17)
$$z'^T = \left(u_0^1, \ldots, u_0^m,\ u_1^1, \ldots, u_1^m, \ldots, u_{k-1}^m,\ s_N,\ t_N,\ s_{N+1},\ t_{N+1}, \ldots, s_{k-1},\ t_{k-1}\right) \in E^{km+2(k-N)}$$

and transcript our problem into the following form:

$$\max_{i=km+1,km+3,\ldots,km+2(k-N)-1} \left(-\langle e_i, z'\rangle + \langle e_{i+1}, z'\rangle\right) \to \min$$

subj. to

$$A_3 z' = b_3\,, \tag{18}$$

$$-1 \leqq z'^i \leqq 1 \quad \text{for} \quad i = 1, 2, \ldots, km\,,$$
$$0 \leqq z'^i \leqq \varrho \quad \text{for} \quad i = km+1\,, \quad km+2, \ldots, km+2(k-N)$$

and satisfy the regularity condition (16). In the equality constraints the $[(n+k-N) \times (km+2(k-N))]$ matrix

(19)
$$A_3 =$$
$$= \begin{bmatrix} A^{k-1}B & A^{k-2}B & \ldots & A^{k-N}B & A^{k-N-1}B & \ldots & AB & B & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ c^T A^{N-1}B & c^T A^{N-2}B & \ldots & c^T B & \vartheta & \ldots & & \vartheta & -1 & -1 & 0 & 0 & \ldots & 0 \\ c^T A^N B & c^T A^{N-1}B & \ldots & c^T AB & c^T B & \ldots & \vartheta & \vartheta & 0 & 0 & -1 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c^T A^{k-2}B & c^T A^{k-3}B & \ldots & c^T A^{k-N-1}B & c^T A^{k-N-2}B & \ldots & c^T B & \vartheta & 0 & 0 & \ldots & \ldots & -1 & -1 \end{bmatrix}$$

and the vector

$$b_3 = \begin{bmatrix} x_T - A^k \hat{x}_0 \\ \langle c, x_T - A^N \hat{x}_0 \rangle - \varrho \\ \langle c, x_T - A^{N+1} \hat{x}_0 \rangle - \varrho \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \langle c, x_T - A^{k-1} \hat{x}_0 \rangle - \varrho \end{bmatrix} \in E^{n+k-N}\,. \tag{20}$$

In the cost functional the vectors e_i are rows of the unit matrix E of order $km + 2(k-N)$.

Using the same way as in § 2, we set now

$$z''^T = \left(z'^T, \xi\right) \in E^{km+2(k-N)+1}\,,$$

 where ξ is a new nonnegative scalar variable. The problem (18) can be now written down in the standart linear-programming form

$$\langle \eta_2, z'' \rangle \to \min$$

subj. to

(22)
$$A_4 z'' = b_4 ,$$
$$A_5 z'' \leqq b_5 ,$$
$$-1 \leqq z''^i \leqq 1 \quad \text{for} \quad i = 1, 2, \ldots, km ,$$
$$0 \leqq z''^i \leqq \varrho \quad \text{for} \quad i = km + 1 , \quad km + 2, \ldots, km + 2(k - N) ,$$
$$z''^{km + 2(k-N)+1} \geqq 0$$

with the restriction, that z''^i corresponding to separating variables must satisfy the regularity condition (16). In this programm

(23)
$$A_4 = \left[\begin{array}{c|c} A_3 & 0 \\ & 0 \\ & \vdots \\ & 0 \end{array}\right]$$

is a $[(n + k - N) \times (km + 2(k - N) + 1]$ matrix,

$$A_5 = \left[\begin{array}{c|c} \vartheta & \begin{matrix} -1 & 1 & 0 & 0 & \ldots & \ldots & 0 & -1 \\ 0 & 0 & -1 & 1 & \ldots & \ldots & 0 & -1 \\ \ldots & & & & & & & \ldots \\ 0 & 0 & \ldots & & \ldots & -1 & 1 & -1 \end{matrix} \end{array}\right]$$
$$\underbrace{}_{km}$$

is a $[(k - N) \times (km + 2(k - N) + 1)]$ matrix,

$$b_4 = b_3 \quad \text{and} \quad b_5^T = (-\varrho, -\varrho, \ldots, -\varrho) \in E^{k-N} .$$

Due to the regularity condition, one could expect that the usual simplex algorithm cannot be applied for solving this problem. However, this is not true because of the validity of the following theorem:

Theorem 1. Let $\hat{z}''$ be a solution of (22), obtained by any method of linear programming, omitting condition (16). Then the first km components of $\hat{z}''$ form the optimal solution of the two-sided minimum overshoot problem and the last component equals the optimal value of φ_2 minus ϱ.

Previous Theorem is an easy corollary of the general preposition, proved in [4]. Thus, we have obtained an usual linear-programming problem with $km + 2(k - N) + 1$ variables ($km + 2(k - N)$ lower and upper bounded), $k - N$ inequality and $n + k - N$ equality constraints.

Remark. According to the requirements put on the solved control problem, we can combine costs of both minimum overshoot problems as follows

$$(24)\qquad \varphi_3(u_0, u_1, \ldots, u_{k-1}) = \max_{\substack{i=1,2,\ldots,N-1 \\ j=N,N+1,\ldots,k-1}} \{v_i, |v_j - v_k|\} .$$

4. MINIMUM OUTPUT ERROR PROBLEM

In a number of practical control situations, we may be required to get the output of a dynamical system to agree "as closely as possible" with some desired value at a prespecified time i. It we use the c_0 norm of the sequence of errors, we can formulate this problem in the following way:

Given a controlled system (1) with the output (5), the initial state (2), the set of admissible controls (4) and the prescribed sequence of outputs $v_1^*, v_2^*, v_3^*, \ldots, v_k^*$, k is a given integer. It is to find a control sequence $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{k-1}$ and a corresponding trajectory $\hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k$, determined by (1) and (2), which minimize the cost functional

$$(25)\qquad \varphi_4(u_0, u_1, \ldots, u_{k-1}) = \max_{i=1,2,\ldots,k} |v_i - v_i^*|$$

subject to the constraints (4) and (5).

This problem and that in the previous section differ only slightly and therefore the method described in § 3 can be applied to the minimum output error problem as well. However, the last one seems to us to be a little bit easier because of absence of the equality constraints. This fact leads us to the proposal to use alternatively some subgradient method for solving this problem. Moreover, the theory of subgradient optimization enables us to state the necessary and sufficient optimality conditions in a very compact form.

First let us denote

$$(26)\qquad \begin{aligned} r_i &= \langle c, A^i \hat{x}_0 \rangle - v_i^* , \\ W_i &= [A^{i-1}B \; A^{i-2}B \; \ldots \; AB \; B \; \underbrace{0 \; 0 \; \ldots \; 0}_{(k-i)m\times}] , \\ s_i &= W_i^T c , \quad i = 1, 2, \ldots, k , \\ z^T &= (u_0^1, u_0^2, \ldots, u_0^m, u_1^1, \ldots, u_1^m, \ldots, u_{k-1}^m) \in E^{km} , \\ \Omega &= \underbrace{U \times U \times \ldots \times U}_{k\times} \end{aligned}$$

Then

$$(27)\qquad v_i - v_i^* = r_i + \langle s_i, z \rangle , \quad i = 1, 2, \ldots, k$$

and the minimum output error problem can be written down in the following form

(28) $$\varphi_5(z) = \max_{i=1,2,\ldots,k} \{r_i + \langle s_i, z\rangle, -r_i - \langle s_i, z\rangle\} \to \min$$

subj. to

$$z \in \Omega .$$

The vector $w \in E^n$ is a subgradient at $x_1 \in E^n$ of the convex function f on E^n if

(29) $$f(x_2) - f(x_1) \geqq \langle w, x_2 - x_1\rangle \quad \text{for all} \quad x_2 \in E^n .$$

The set of all subgradients at x is the compact convex set $\partial f(x)$ called the subdifferential. The function φ_5 has the directional derivative at z in the direction d for any z, $d \in E^{km}$ and

(30) $$\varphi_5'(z, d) = \max_{w \in \partial\varphi_5(z)} \langle d, w\rangle .$$

In our case

(31) $$\partial\varphi_5(z) = \operatorname{co} V(z) ,$$

where

(32) $$\begin{aligned} V(z) &= V_1(z) \cup V_2(z) , \\ V_1(z) &= \{s_i \mid r_i + \langle s_i, z\rangle = \varphi_5(z) , \quad i = 1, 2, \ldots, k\} \\ V_2(z) &= \{-s_i \mid -r_i - \langle s_i, z\rangle = \varphi_5(z) , \quad i = 1, 2, \ldots, k\} . \end{aligned}$$

Theorem stated below is an applications of the optimality conditions proved in [2] for our case.

Theorem 2. The point $\hat{z} \in E^{km}$ is the solution of the minimum output error problem in the sense of (26), if and only if

(33) $$\min_{z \in \Omega} \max_{w \in \operatorname{co} V(\hat{z})} \langle w, z - \hat{z}\rangle = 0 .$$

In [7] and [8] a very general method was proposed for the minimization of a convex (possibility nonsmooth) functional on a specified set of a Hilbert space. In [5] the application of this method to problems with piecewise linear costs is recommended. The method consists in the construction of the sequence

(34) $$z_{\nu+1} = \mathrm{P}_\Omega(z_\nu - \alpha_\nu w_\nu) , \quad \nu = 0, 1, \ldots,$$

starting in some feasible point $z_0 \in \Omega$. P_Ω is the projector from E^{km} onto Ω i.e.

(35) $$\mathrm{P}_\Omega(z) \in \Omega , \quad \|z - \mathrm{P}_\Omega(z)\| = \inf_{z_1 \in \Omega} \|z - z_1\| ,$$

(36) $$w_\nu \in \operatorname{co} V(z_\nu)$$

and α_ν is an appropriate stepsize. In [7] the convergence of this method is proved under merely the conditions

(37) $$\alpha_\nu = \frac{\lambda_\nu}{\|w_\nu\|}, \quad \lambda_\nu \to 0, \quad \sum_{\nu=0}^{\infty} \lambda_\nu = \infty .$$

An effective choice of α_ν, based upon the estimate of the optimal value of φ_5, is suggested in [8]. Another subgradient methods, capable to treat even nonconvex problems, are collected in [2].

5. SEPARABLE PROGRAMMING IN DISCRETE OPTIMAL CONTROL PROBLEMS

A broad class of discrete optimal control problems can be solved by methods of separable programming. If we use so called δ-method, the problem must be first recast into the form of a linear-programming one, where some or all variables have to satisfy a regularity condition of the type (16). This problem we call "approximating" and for its solution a simplex algorithm with properly restricted basis entry can be applied. Therefore we find an optimal or locally optimal solution in a finite number of simplex transformations. The detailed description of the δ-method can be found for instance in [4]. Here we shall only specify classes of discrete optimal control problems, to which this method can be applied and show, how the approximating problem can be obtained.

Let the state space variables be constrained by inequalities

(38) $$\alpha_i^j \leqq x_i^j \leqq \beta_i^j, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, n,$$

and the set of admissible controls be given by (4). Let further dynamical properties of the controlled system be described by the difference equation

(39) $$x_{i+1}^l = \sum_{j=1}^{n} {}^l q_i^j(x_i^j) + \sum_{j=1}^{m} {}^l p_i^j(u_i^j), \quad i = 0, 1, \ldots, k-1, \quad l = 1, 2, \ldots, n,$$

where ${}^l q_i^j$, ${}^l p_i^j[E^1 \to E^1]$ are continuous functions, with the initial condition

(40) $$\sum_{j=1}^{n} {}^l r^j(x_0^j) = 0, \quad l = 1, 2. \ldots, \varrho, \quad \varrho \leqq n,$$

and the terminal condition

(41) $$\sum_{j=1}^{n} {}^l s^j(x_k^j) = 0, \quad l = 1, 2, \ldots, \sigma, \sigma \leqq n,$$

where ${}^l r^j$, ${}^l s^j[E^1 \to E^1]$ are continuous functions, k is a given integer. If we are to transfer the system (39) from an initial state satisfying (40) to a terminal state satisfying (41) in such a way that the cost functional

$$J_1(u_0, u_1, \dots, u_{k-1}) = \sum_{i=0}^{k-1} \sum_{j=1}^{m} \Phi_i^j(u_i^j) + \sum_{i=0}^{k} \sum_{j=1}^{n} \Psi_i^j(x_i^j), \tag{42}$$

where Φ_i^j, $\Psi_i^j[E^1 \to E^1]$ are continuous functions, is minimized, then the δ-method can be easily applied.

In the case the state space variables are unbounded, we have to restrict ourselves to systems, described by the equation

$$x_{i+1}^l = A_i^l x_i + \sum_{j=1}^{m} {}^l p_i^j(u_i^j), \quad i = 0, 1, \dots, k-1, \quad l = 1, 2, \dots, n, \tag{43}$$

where A_i are constant $[n \times n]$ matrices, and to cost functionals of the type

$$J_2(u_0, u_1, \dots, u_{k-1}) = \sum_{i=0}^{k-1} \sum_{j=1}^{m} \Phi_i^j(u_i^j) + \sum_{i=0}^{k} \langle c_i, x_i \rangle, \tag{44}$$

where $c_i \in E^n$ are constant vectors.

The construction of the approximating problem we shall now demonstrate on the minimum overshoot problem, where the linear controlled system (1) is replaced by the nonlinear system, described by the equation

$$x_{i+1}^l = A^l x_i + \sum_{j=1}^{m} {}^l p^j(u_i^j), \quad i = 0, 1, \dots, k-1, \quad l = 1, 2, \dots, n. \tag{45}$$

This problem is much more complicated than that, discussed in § 2, because of its nonconvexity. Therefore, finding of mere local minima can be guaranteed.

First we find sufficiently precise polygonal approximations of the functions ${}^l p^j$. For this purpose we express all variables u_i^j, the corresponding functions ${}^l p^j$ of them are nonlinear at least for one l, in the form

$$u_i^j = \sum_{\nu=1}^{h} u_{i\nu}^j, \tag{46}$$

where

$$0 \leqq u_{i\nu}^j \leqq \delta_\nu \quad \text{and} \quad \sum_{\nu=1}^{h} \delta_\nu = 2. \tag{47}$$

The variables $u_{i\nu}^j$ are separating variables and the regularity condition implies, that $u_{i\nu}^j > 0$ only if

$$u_{i\mu}^j = \delta_\mu \quad \text{for all} \quad \mu = 1, 2, \dots, \nu - 1, \quad i = 0, 1, \dots, k-1, \quad j = 1, 2, \dots, m. \tag{48}$$

The functions ${}^l p^j$ we polygonally approximate by piecewise linear functions ${}^l \bar{p}^j$

$${}^l \bar{p}^j = {}^l p^j(-1) + \sum_{\nu=1}^{h} \frac{{}^l p^j(-1 + \sum_{\mu=1}^{\nu} \delta_\mu) - {}^l p^j(-1 + \sum_{\mu=1}^{\nu-1} \delta_\mu)}{\delta_\nu} u_{i\nu}^j = {}^l \bar{P}_0^j + \sum_{\nu=1}^{h} {}^l \bar{P}_\nu^j u_{i\nu}^j. \tag{49}$$

Now we replace in our minimum overshoot problem the system described by (45) by another "approximating" system, where all nonlinear functions ${}^{l}p^{j}$ are replaced by linear approximating functions ${}^{l}\bar{p}^{j}$ of separating variables $u^{j}_{i\nu}$. Thus, if all functions ${}^{l}p^{j}$ are nonlinear, we obtain an approximating problem

$$\xi \to \min$$

subj. to

$$\langle c, x_i \rangle \leqq \xi\,, \quad i = 1, 2, \ldots, k - 1\,,$$

(50)
$$x^{l}_{i+1} = A^{l}x_i + \sum_{j=1}^{m} {}^{l}\bar{P}^{j}_{0} + \sum_{j=1}^{m} \sum_{\nu=1}^{h} {}^{l}\bar{P}^{j}_{\nu} u^{j}_{i\nu}\,, \quad l = 1, 2, \ldots, n\,,$$

$$x_0 = \hat{x}_0\,,$$

$$x_k = x_T$$

$$0 \leqq u^{j}_{i\nu} \leqq \delta_\nu\,, \quad \sum_{\nu=1}^{h} \delta_\nu = 2\,, \quad i = 0, 1, \ldots, k - 1\,,$$

and the variables $u^{j}_{i\nu}$ have to satisfy the condition (48). Now we set

(51)
$$z^T = (\lambda^{1T}_0, \lambda^{2T}_0, \ldots, \lambda^{mT}_0, \lambda^{1T}_1, \ldots, \lambda^{mT}_1, \ldots, \lambda^{mT}_{k-1}, \xi) \in E^{kmh+1}\,,$$

where

(52)
$$\lambda^{jT}_i = (u^{j}_{i1}, u^{j}_{i2}, \ldots, u^{j}_{ih}) \in E^h\,, \quad i = 0, 1, \ldots, k - 1\,, \quad j = 1, 2, \ldots, m\,.$$

The system equation can be now written in the form

(53)
$$x_{i+1} = Ax_i + C + D\lambda_i\,,$$

where

$$\lambda^T_i = (\lambda^{1T}_i, \lambda^{2T}_i, \ldots, \lambda^{mT}_i)\,, \quad i = 0, 1, \ldots, k - 1\,,$$

(54)
$$C = \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^n \end{bmatrix}, \quad C^l = \sum_{j=1}^{m} {}^{l}\bar{P}^{j}_{0}\,, \quad l = 1, 2, \ldots, n\,,$$

$$D = \begin{bmatrix} {}^{1}D^{1} & {}^{1}D^{2} & \ldots & {}^{1}D^{m} \\ {}^{2}D^{1} & {}^{2}D^{2} & \ldots & {}^{2}D^{m} \\ \ldots & \ldots & \ldots & \ldots \\ {}^{n}D^{1} & {}^{n}D^{2} & \ldots & {}^{n}D^{m} \end{bmatrix}, \quad {}^{l}D^{j} = ({}^{l}\bar{P}^{j}_{1}, {}^{l}\bar{P}^{j}_{2}, \ldots, {}^{l}\bar{P}^{j}_{h}) \in E^h\,.$$

Using relations (53), (54), we come to the final form of the solved problem, to which a simplex algorithm with restricted basis entry can be applied:

$$\langle \eta_3, z \rangle \to \min$$

subj. to

(55)
$$A_6 z \leqq b_6$$
$$A_7 z = b_7$$
$$0 \leqq z^i \leqq \delta_v\,, \quad i = ht + v\,, \quad t = 0, 1, \ldots, km - 1\,, \quad v = 1, 2, \ldots, h$$

and $z^i > 0$, if only $z^j = \delta_\mu\,, \quad j = ht + \mu\,, \quad t = 0, 1, \ldots, km - 1, \mu = 1, 2, \ldots, v - - 1$.

In (55)

$$\eta_3^T = (0, 0, 0, \ldots, 1) \in E^{kmh+1}$$

$$A_6 = \begin{bmatrix} c^T D & 0 & & \ldots & & 0 & -1 \\ c^T A D & c^T D & 0 & \ldots & & 0 & -1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c^T A^{k-2} D & c^T A^{k-3} D & \ldots & c^T D & \underbrace{0\ 0\ \ldots}_{mh\times} & 0 & -1 \end{bmatrix}$$

is a $[(k-1) \times (kmh + 1)]$ matrix ,

$$A_7 = [A^{k-1}D\ A^{k-2}D\ \ldots\ AD\ D\ 0] \quad \text{is a} \quad [n \times (kmh + 1)] \quad \text{matrix}\,,$$

(56)
$$b_6 = -\begin{bmatrix} c^T A \hat{x}_0 & + c^T C \\ c^T A^2 \hat{x}_0 & + c^T A C + c^T C \\ \ldots & \ldots \\ c^T A^{k-1} \hat{x}_0 + & \sum_{j=0}^{k-2} c^T A^{k-j-2} C \end{bmatrix} \in E^{k-1}$$

and

$$b_7 = x_T - A^k \hat{x}_0 - \sum_{j=0}^{k-1} A^{k-j-1} C\,.$$

The detailed description of the simplex algorithms for various approximating problems can be found in [3]. The choice of an integer h and upper bounds δ_v, $v = 1, 2, \ldots, h$ must respect the rate of nonlinearity of functions ${}^l p^j$ and the admissible error of the terminal condition.

If the discrete systems appearing in problems of § 3 and 4 are of the type (43) or (39) (with state space constraints), the δ-method of separable programming can be applied to them as well.

6. CONCLUSION

All optimization problems discussed in §§ 2, 3, 4 are of the minmax form. There is a great variety of such problems and some of them are currently investigated in the literature. Our problems were formulated for linear systems. However, they can be solved for nonlinear systems as well, but then they are no more convex. In § 5 an effective method for special class of nonlinear systems is suggested. In case of general nonlinear systems we would prefere subgradient methods rather then a transcription to the mathematical programming form.

Minmax criteria in control processes seems to us to be more suitable in many cases than various state space constraints. Therefore we hope, that numerical methods for solving of these problems will be further studied and developed.

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