Table (2.31) becomes

1	$-w^{-1}$	
$-w^{-1}$	1	$-w^{-1}$
$1 - w^{-2}$	0	

and since $\mathscr{V}(-w^{-1}) = \varrho^{-1} < 1$ (recall that $\varrho > 1$) we conclude that a is stable.

If $\mathfrak{F}$ is a subfield of $\mathfrak{C}$ valuated by (2.25), the crucial role plays the computation of m^*. The following algorithm is proposed in [47; 31]; there are many others [58].

Given a polynomial $m \in \mathfrak{F}[z^{-1}]$ of degree $n \geqq 0$, compute

$$m^{=}m = \bar{\gamma}_p z^p + \ldots + \bar{\gamma}_1 z + \gamma_0 + \gamma_1 z^{-1} + \ldots + \gamma_p z^{-p}\,, \quad \gamma_p \neq 0$$

and set

$$g = \gamma_0 z^{-p} + \gamma_1 z^{-(p+1)} + \ldots + \gamma_p z^{-2p}\,.$$

Perform the recurrent division

$$g = f_k q_k + r_k\,, \quad \partial r_k < \partial f_k\,, \quad k = 0, 1, \ldots$$

by f_k, where

$$f_0 = z^{-p}\,,$$

$$f_k = q_{k-1}^{\sim}\,, \quad k = 1, 2, \ldots$$

Then

$$\lim_{k \to \infty} q_k = q$$

and if $q \in \mathfrak{F}[z^{-1}]$, we have $m^* = q$ modulo a unit of $\mathfrak{F}[z^{-1}]$. If $\mathfrak{F}$ is not topologically complete it may happen that q does not belong to $\mathfrak{F}[z^{-1}]$ and, therefore, it cannot be equal to m^*.

Having computed q via the above iterative technique, we can use the definition of m^* to take

$$m^- = z^{-(n-p)}(m, q^{\sim})\,, \quad m^+ = \frac{m}{m^-}$$

and thus avoid the computation of roots of polynomials at any stage of the synthesis procedure.

Example 2.13. Consider

$$m = z^{-1} - 2z^{-2} \in \mathfrak{R}[z^{-1}]$$

and use the iterative technique to compute m^-, m^+, and m^*.

We have

$$m^{=}m = -2z + 5 - 2z^{-1}$$

and hence

$$g = 5z^{-1} - 2z^{-2}.$$

Initializing with

$$f_0 = z^{-1},$$

we obtain after scaling

$$q_0 = 1 - \tfrac{2}{5}z^{-1},$$

$$q_1 = 1 - \tfrac{10}{21}z^{-1},$$

$$q_2 = 1 - \tfrac{42}{85}z^{-1},$$

$$q_3 = 1 - \tfrac{170}{341}z^{-1},$$

etc. and, evidently

$$q = 1 - 0{\cdot}5z^{-1}.$$

It follows that

$$m^- = z^{-1}(z^{-1} - 0{\cdot}5), \quad m^+ = -2$$

modulo a unit of $\mathfrak{R}[z^{-1}]$ and hence

$$m^* = -2(1 - 0{\cdot}5z^{-1}) = z^{-1} - 2.$$

When only the m^* is required, we can compute

$$(1 - 0{\cdot}5z)^{-1}\, m^= m(1 - 0{\cdot}5z^{-1})^{-1} = 4 = (-2)(-2)$$

and obtain

$$m^* = -2(1 - 0{\cdot}5z^{-1}) = z^{-1} - 2.$$

Given a polynomial matrix over $\mathfrak{F}_{l,m}$, the matrix factorizations can be reduced to factorizations of invariant polynomials and the above procedure is still applicable. Nevertheless, the following original algorithm for direct computation of M_1^* and M_2^* is useful; there are many others [59; 60; 64].

Given a polynomial matrix $M \in \mathfrak{F}_{l,m}[z^{-1}]$ of degree $n \geqq 0$ and rank $M = m$, compute

$$M'^= M = \bar{\Gamma}'_p z^p + \ldots + \bar{\Gamma}'_1 z + \Gamma_0 + \Gamma_1 z^{-1} + \ldots + \Gamma_p z^{-p}, \quad \Gamma_p \neq 0$$

and set

$$G = \Gamma_0 z^{-p} + \Gamma_1 z^{-(p+1)} + \ldots + \Gamma_p z^{-2p}.$$

Perform the recurrent left division

(2.32) $$G = F_{1,k}Q_{1,k} + R_{1,k}, \quad \partial R_{1,k} < \partial F_{1,k}, \quad k = 0, 1, \ldots,$$

by $F_{1,k}$, where

$$F_{1,0} = I_m z^{-p},$$

$$F_{1,k} = Q'^{\sim}_{1,k-1}, \quad k = 1, 2, \ldots$$

Then

$$\lim_{k\to\infty} Q_{1,k} = Q_1$$

and if $Q_1 \in \mathfrak{F}_{m,m}[z^{-1}]$, we have $M_1^* = E_1 Q_1$, where E_1 is a unit of $\mathfrak{F}_{m,m}[z^{-1}]$.

Similarly, given a polynomial matrix $M \in \mathfrak{F}_{l,m}[z^{-1}]$ of degree $n \geqq 0$ and rank $M = l$, compute

$$MM'^{=} = \bar{A}_q' z^q + \ldots + \bar{A}_1' z + A_0 + A_1 z^{-1} + \ldots + A_q z^{-q}\,, \quad A_q \neq 0$$

and set

$$L = A_0 z^{-q} + A_1 z^{-(q+1)} + \ldots + A_q z^{-2q}\,.$$

Perform the recurrent right division

$$L = Q_{2,k} F_{2,k} + R_{2,k}\,, \quad \partial R_{2,k} < \partial F_{2,k}\,; \quad k = 0, 1, \ldots,$$

by $F_{2,k}$, where

$$F_{2,0} = I_l z^{-q}\,,$$

$$F_{2,k} = Q_{2,k-1}'^{\sim}\,, \quad k = 1, 2, \ldots$$

Then

$$\lim_{k\to\infty} Q_{2,k} = Q_2$$

and if $Q_2 \in \mathfrak{F}_{l,l}[z^{-1}]$, we have $M_2^* = Q_2 E_2$, where E_2 is a unit of $\mathfrak{F}_{l,l}[z^{-1}]$.

Unfortunately, no general proof of this algorithm is known at present. It is presented here just as a conjecture backed by computational experience.

Example 2.14. Given

$$M = \begin{bmatrix} z^{-1} & z^{-1}(1 - 2z^{-1}) \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix}$$

over $\mathfrak{R}[z^{-1}]$, use the iterative technique to compute M_1^* and M_2^*.

We have

$$M'^{=}M = \begin{bmatrix} 1 & 1 - 2z^{-1} \\ 1 - 2z & 2(1 - 2z)(1 - 2z^{-1}) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \\ -2 & -4 \end{bmatrix} z + \begin{bmatrix} 1 & 1 \\ 1 & 10 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 0 & -4 \end{bmatrix} z^{-1}$$

and hence

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 10 \end{bmatrix} z^{-1} + \begin{bmatrix} 0 & -2 \\ 0 & -4 \end{bmatrix} z^{-2}\,.$$

Initializing with

$$F_{1,0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^{-1}\,,$$

we obtain after scaling

$$Q_{1,0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{16}{9} \\ 0 & -\frac{2}{9} \end{bmatrix} z^{-1},$$

$$Q_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{64}{41} \\ 0 & -\frac{18}{41} \end{bmatrix} z^{-1},$$

$$Q_{1,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{256}{169} \\ 0 & -\frac{82}{169} \end{bmatrix} z^{-1},$$

$$Q_{1,3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1024}{681} \\ 0 & -\frac{338}{681} \end{bmatrix} z^{-1},$$

etc. and evidently,

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1{\cdot}5 \\ 0 & -0{\cdot}5 \end{bmatrix} z^{-1} = \begin{bmatrix} 1 & -1{\cdot}5z^{-1} \\ 0 & 1-0{\cdot}5z^{-1} \end{bmatrix}.$$

The matrix E_1 can be computed as follows. Since

$$\begin{bmatrix} 1 & 0 \\ -1{\cdot}5z & 1-0{\cdot}5z \end{bmatrix}^{-1} M'^{=}M \begin{bmatrix} 1 & -1{\cdot}5z^{-1} \\ 0 & 1-0{\cdot}5z^{-1} \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix},$$

we obtain

$$M_1^* = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1{\cdot}5z^{-1} \\ 0 & 1-0{\cdot}5z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 1-2z^{-1} \\ 0 & z^{-1}-2 \end{bmatrix}.$$

Similarly, we have

$$MM'^{=} = \begin{bmatrix} -2z^{-1}+6-2z & (1-2z^{-1})(1-2z) \\ (1-2z^{-1})(1-2z) & (1-2z^{-1})(1-2z) \end{bmatrix} =$$

$$= \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} z + \begin{bmatrix} 6 & 5 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} z^{-1}$$

and hence

$$L = \begin{bmatrix} 6 & 5 \\ 5 & 5 \end{bmatrix} z^{-1} + \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} z^{-2}.$$

Initializing with

$$F_{2,0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^{-1},$$

we obtain after scaling

$$Q_{2,0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{2}{5} \\ 0 & -\frac{2}{5} \end{bmatrix} z^{-1},$$

$$Q_{2,1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{10}{21} \\ 0 & -\frac{10}{21} \end{bmatrix} z^{-1},$$

$$Q_{2,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{42}{85} \\ 0 & -\frac{42}{85} \end{bmatrix} z^{-1},$$

$$Q_{2,3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{170}{341} \\ 0 & -\frac{170}{341} \end{bmatrix} z^{-1},$$

etc. and, evidently,

$$Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -0{\cdot}5 \\ 0 & -0{\cdot}5 \end{bmatrix} z^{-1} = \begin{bmatrix} 1 & -0{\cdot}5z^{-1} \\ 0 & 1-0{\cdot}5z^{-1} \end{bmatrix}.$$

The matrix E_2 can be computed as follows. Since

$$\begin{bmatrix} 1 & -0{\cdot}5z^{-1} \\ 0 & 1-0{\cdot}5z^{-1} \end{bmatrix}^{-1} MM'^{=} \begin{bmatrix} 1 & 0 \\ -0{\cdot}5z & 1-0{\cdot}5z \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix},$$

we obtain

$$M_2^* = \begin{bmatrix} 1 & -0{\cdot}5z^{-1} \\ 0 & 1-0{\cdot}5z^{-1} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & z^{-1}-2 \\ 0 & z^{-1}-2 \end{bmatrix}.$$

Example 2.15. Given

$$M = \begin{bmatrix} 1 & 1-z^{-2} \\ 1 & 1 \end{bmatrix} \in \mathfrak{Q}_{2,2}[z^{-1}],$$

use the iterative technique to compute M_1^*.

Since

$$M'^{=}M = \begin{bmatrix} 2 & 2-z^{-2} \\ 2-z^2 & -z^2+3-z^{-2} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} z^2 + \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} z^{-2},$$

we get

$$G = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} z^{-2} + \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} z^{-4}.$$

Initializing with

$$F_{1,0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^{-2},$$

we obtain after scaling

$$Q_{1,0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} z^{-2},$$

$$Q_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} z^{-2}$$

etc., that is,

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} z^{-2} = \begin{bmatrix} 1 & 1 - \frac{1}{2} z^{-2} \\ 0 & 1 \end{bmatrix},$$

an element of $\mathfrak{Q}_{2,2}[z^{-1}]$.

Observe that $Q_{1,k}$ converges to Q_1 in a finite number of steps. This is due to the fact that mp $M'^{=}M = 1$, a unit of $\mathfrak{Q}[z^{-1}]$.

Since

$$\begin{bmatrix} 1 & 0 \\ 1 - \frac{1}{2}z^2 & 1 \end{bmatrix}^{-1} M'^{=}M \begin{bmatrix} 1 & 1 - \frac{1}{2}z^{-2} \\ 0 & 1 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix},$$

we obtain

$$M_1^* = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 - \frac{1}{2}z^{-2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} - \frac{1}{2}z^{-2} \\ 1 & \frac{1}{2} - \frac{1}{2}z^{-2} \end{bmatrix}.$$

To evaluate the performance of a least squares control for systems defined over a subfield of $\mathfrak{C}$ valuated by (2.25), we have to compute the quadratic norm $\|E\|^2$ of the error sequence.

There is an algorithm [2; 3; 46] to compute the quadratic norm

$$\|e\|^2 = \langle e'^{=}e \rangle = \left\langle \left(\frac{b}{a}\right)^{=} \left(\frac{b}{a}\right) \right\rangle$$

of a single real error. We state here without proof its generalization to errors defined over an arbitrary subfield $\mathfrak{F}$ of $\mathfrak{C}$ valuated by (2.25).

Given polynomials $a, b \in \mathfrak{F}[z]$, $\partial b \leqq \partial a = n$, $n \geqq 0$, a being stable. Introduce polynomials

$$a_k = \alpha_{k,0} + \alpha_{k,1}z + \ldots + \alpha_{k,n-k}z^{n-k},$$

$$b_k = \beta_{k,0} + \beta_{k,1}z + \ldots + \beta_{k,n-k}z^{n-k}$$

which are defined recursively by

$$(2.33) \qquad za_{k+1} = a_k - \frac{\alpha_{k,0}}{\bar{\alpha}_{k,n-k}} a_k^{\sim}, \quad k = 0, 1, \ldots, n-1,$$

$$a_0 = a$$

Therefore, the quadratic norm of a multivariable system error is the sum of the quadratic norms of its single error components.

Example 2.16. Given the error sequence

$$E = \frac{\begin{bmatrix} z^{-1} - 2 \\ -2z^{-1} - 2 \end{bmatrix}}{z^{-1} - 2} = \frac{\begin{bmatrix} 1 - 2z \\ -2 - 2z \end{bmatrix}}{1 - 2z}$$

over $\mathfrak{R}$, compute $\|E\|^2$.

We denote

$$e_1 = \frac{1 - 2z}{1 - 2z}, \quad e_2 = \frac{-2 - 2z}{1 - 2z}.$$

Table (2.31) becomes

$$\begin{array}{ccc} -2 & 1 & \\ 1 & -2 & -0{\cdot}5 \\ -1{\cdot}5 & 0 & \end{array}$$

Then table (2.35) for e_1 yields

$$\begin{array}{ccc} -2 & 1 & \\ 1 & -2 & -0{\cdot}5 \\ -1{\cdot}5 & 0 & \\ -1{\cdot}5 & 0 & \end{array}$$

and

$$\|e_1\|^2 = \frac{1}{-2}\left(\frac{1^2}{-2} + \frac{(-1{\cdot}5)^2}{-1{\cdot}5}\right) = 1 .$$

Table (2.35) for e_2 yields

$$\begin{array}{ccc} -2 & -2 & \\ 1 & -2 & 1 \\ -3 & 0 & \\ -1{\cdot}5 & 0 & \end{array}$$

and

$$\|e_2\|^2 = \frac{1}{-2}\left(\frac{(-2)^2}{-2} + \frac{(-3)^2}{-1{\cdot}5}\right) = 4 .$$

Therefore

$$\|E\|^2 = 1 + 4 = 5 .$$

Example 2.17. Given the error

$$E = \frac{z - 0{\cdot}5\,\mathrm{i}}{z + 0{\cdot}5\,\mathrm{i}}$$

over $\mathfrak{A}$, the field of algebraic numbers, compute $\|E\|^2$.

Find a stable control sequence $U \in \mathfrak{F}^+_{m,1}\{z^{-1}\}$ such that the error sequence E vanishes in a minimum time $k_{\min}$ and thereafter.

(3.2) *Finite time optimal control problem.*

Given a system S which is a (not necessarily minimal) realization of

$$S = \frac{B}{a} \in \mathfrak{F}_{l,m}\{z^{-1}\}, \quad B \neq 0$$

and a reference sequence

$$W = \frac{Q}{p} \in \mathfrak{F}_{l,1}\{z^{-1}\}, \quad Q \neq 0 .$$

Find a finite control equence $U \in \mathfrak{F}_{m,1}[z^{-1}]$ such that the error sequence E vanishes in a minimum time $k_{\min}$ and thereafter.

(3.3) *Least squares control problem.*

Given a system S which is a (not necessarily minimal) realization of

$$S = \frac{B}{a} \in \mathfrak{F}_{l,m}\{z^{-1}\}, \quad B \neq 0$$

and a reference sequence

$$W = \frac{Q}{p} \in \mathfrak{F}_{l,1}\{z^{-1}\}, \quad Q \neq 0 .$$

Find a stable control sequence $U \in \mathfrak{F}^+_{m,1}\{z^{-1}\}$ such that the quadratic norm $\|E\|^2$ of the error sequence E is minimized.

It is to be noted that the control sequence is required to be stable in all control problems. This is rather a strict assumption motivated by physical realizability of the optimal control. However, an optimal control which is bounded instead of stable may be well acceptable in the engineering practice. This is to be born in mind when applying the synthesis procedures.

Even if these problems can be considered classical the author is not aware of any solution of the open-loop optimal control problems in the literature. The only exception is [60], where a restricted version of problem (3·3) is considered. The open-loop optimal control problems for single-variable systems have been systematically formulated and solved for the first time in [30; 31; 32; 33; 34; 35].

3.2. Stable time optimal control problem

Let $\mathfrak{F}$ be an arbitrary field with valuation $\mathscr{V}$ and write

$$S = \frac{B}{a} = B_1 A_2^{-1}, \quad \text{rank} \quad B_1 = r,$$

$$B_1 = B_1^- B_1^+ . \tag{3.4}$$

By the definition of B_1^-, see (2.19) and (2.30), we get

$$B_1^- = [B_{11}^- \ 0],$$

where $B_{11}^- \in \mathfrak{F}_{l,r}[z^{-1}]$, $0 \in \mathfrak{F}_{l,m-r}[z^{-1}]$ and rank $B_{11}^- = r$.

Then we have the following result.

Theorem 3.1. *Problem* (3.1) *has a solution if and only if the linear Diophantine equation*

$$B_{11}^- X + Yp = Q \tag{3.5}$$

has a solution X°, Y° *such that* $\partial Y^\circ = \min$ *subject to*

$$U = A_2 (B_1^+)^{-1} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{3.6}$$

belongs to $\mathfrak{F}_{m,1}^+\{z^{-1}\}$, *where*

$$U_1 = \frac{X^\circ}{p},$$

$$U_2 \in \mathfrak{F}_{m-r,1}\{z^{-1}\}.$$

The optimal control is not unigue, in general, and all optimal controls are given by (3.6). *Moreover,*

$$E = Y^\circ$$

and

$$k_{\min} = 0, \qquad Y^\circ = 0,$$
$$= 1 + \partial Y^\circ, \quad Y^\circ \neq 0.$$

Proof. Write

$$E = W - SU = \frac{Q}{p} - B_1 A_2^{-1} U = \frac{Q}{p} - [B_{11}^- \ 0]\, B_1^+ A_2^{-1} U = \frac{Q}{p} - B_{11}^- U_1,$$

where

$$B_1^+ A_2^{-1} U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

and

$$U_1 \in \mathfrak{F}_{r,1}\{z^{-1}\}\,, \quad U_2 \in \mathfrak{F}_{m-r,1}\{z^{-1}\}\,.$$

Since the error is to vanish in a finite time and thereafter, $\boldsymbol{E}$ must be a polynomial matrix in $\mathfrak{F}_{l,1}[z^{-1}]$, say Y. Therefore

$$Y = \frac{Q}{p} - B_{11}^{-} U_1 = \frac{Q - pB_{11}^{-} U_1}{p} \tag{3.7}$$

and since $(p, Q) = 1$ up to a unit in $\mathfrak{F}[z^{-1}]$, we must take

$$U_1 = \frac{X}{p}\,, \tag{3.8}$$

where $X \in \mathfrak{F}_{r,1}[z^{-1}]$ is a polynomial matrix to be specified.

In fact, the X and Y satisfy equation (3.5) by virtue of (3.7) and (3.8). Among all solutions of equation (3.5) we have to take only those which make the $\boldsymbol{U}$ stable and within this class further those which minimize the degree of $\boldsymbol{E}$. Therefore

$$\boldsymbol{U} = A_2 (B_1^{+})^{-1} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix},$$

where

$$U_1 = \frac{X^\circ}{p}\,,$$

$U_2 \in \mathfrak{F}_{m-r,1}\{z^{-1}\}$ arbitrary but such that $A_2(B_1^{+})^{-1}\begin{bmatrix} 0 \\ U_2 \end{bmatrix} \in F_{m,1}^{+}\{z^{-1}\}$

and

$$\boldsymbol{E} = Y^\circ\,,$$

the X°, Y° being a solution of equation (3.5) such that $\partial Y^\circ = \min$ among all solutions yielding a stable $\boldsymbol{U}$. Then

$$\begin{aligned} k_{\min} &= 0 \qquad \text{if} \quad Y^\circ = 0\,, \\ &= 1 + \partial Y^\circ \quad \text{otherwise.} \end{aligned}$$

The stability of $\boldsymbol{U}$ cannot be inferred until the general solution of equation (5) is found. □

Example 3.1. Consider the system over the field $\mathfrak{R}$ valuated by (2.25) which is a realization of

$$\boldsymbol{S} = \frac{\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1})^2 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1},$$

the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1 - z^{-1}}.$$

and solve problem (3.1).

We carry out factorization (3.4)

$$B_1 = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and since

$$B_{11}^{-} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix},$$

equation (3.5) becomes

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.9}$$

It is to be noted that the matrix

$$\begin{bmatrix} B_{11}^{-} & 0 \\ 0 & p \end{bmatrix} = \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & z^{-1}(1 - z^{-1}) & 0 \\ 0 & 0 & 1 - z^{-1} \end{bmatrix}$$

has the invariant polynomials 1, $z^{-1}(1 - z^{-1})$, $z^{-1}(1 - z^{-1})$ while the matrix

$$\begin{bmatrix} B_{11}^{-} & Q \\ 0 & p \end{bmatrix} = \begin{bmatrix} z^{-1} & 0 & 1 \\ 0 & z^{-1}(1 - z^{-1}) & 1 \\ 0 & 0 & 1 - z^{-1} \end{bmatrix}$$

has the invariant polynomials 1, z^{-1}, $z^{-1}(1 - z^{-1})^2$. Since they are not equal, the above matrices are not associates and equation (3.9) has no solution by Theorem 1.1. Hence our problem has no solution.

Example 3.2. Consider the system over the field $\mathfrak{R}$ valuated by (2.25) which is a realization of

$$S = \frac{\begin{bmatrix} z^{-1} & 0 \\ z^{-2} & z^{-1}(1 - z^{-1}) \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} & 0 \\ z^{-2} & z^{-1} \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1},$$

the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ 2(1 - z^{-1}) \end{bmatrix}}{1 - z^{-1}}$$

and solve problem (3.1).

Equation (3.5) now reads

$$(3.10)\qquad \begin{bmatrix} z^{-1} & 0 \\ z^{-2} & z^{-1} \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} 1 \\ 2(1 - z^{-1}) \end{bmatrix}$$

and it has a solution. We find

$$\begin{bmatrix} z^{-1} & 0 \\ z^{-2} & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix}$$

and, by Theorem 1.1, equation (3.10) is converted into the set of polynomial equations

$$z^{-1}\bar{x}_1 + \bar{y}_1(1 - z^{-1}) = 1\,,$$
$$z^{-1}\bar{x}_2 + \bar{y}_2(1 - z^{-1}) = 2 - 3z^{-1}$$

and

$$X = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}.$$

We obtain

$$\bar{x}_1 = 1 + (1 - z^{-1})\, t_1\,, \quad \bar{y}_1 = 1 - z^{-1} t_1\,,$$
$$\bar{x}_2 = -1 + (1 - z^{-1})\, t_2\,, \quad \bar{y}_2 = 2 - z^{-1} t_2\,,$$

for arbitrary $t_1, t_2 \in \Re[z^{-1}]$ and

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1 - z^{-1})\,,$$
$$Y = \begin{bmatrix} 1 \\ 2 + z^{-1} \end{bmatrix} - \begin{bmatrix} z^{-1} & 0 \\ z^{-2} & z^{-1} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

by (1.19).

All tentative controls have the form

$$\boldsymbol{U}_1 = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{1 - z^{-1}} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix},$$

that is,

$$\boldsymbol{U} = \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{1 - z^{-1}} + \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 \\ -\dfrac{1}{1 - z^{-1}} \end{bmatrix} + \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

and no one is stable. We conclude that problem (3.1) has no solution in the sense of our definition. However, the solution may be well acceptable in the enginnering practice because it is bounded.

Example 3.3. Solve problem (3.1) for a realization of

$$S = \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} = \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} [1]^{-1}$$

over the field $\mathfrak{R}$ valuated by (2.25) and for the reference sequence

$$W = \frac{\begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ -1 \end{bmatrix}}{z^{-1} - 2}.$$

We are to solve the equation

$$\begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} X + Y(z^{-1} - 2) \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ -1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \sqrt{2}\,(1 - z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix},$$

equation (3.11) reduces to the set of polynomial equations

$$z^{-1}\bar{x}_1 + \bar{y}_1(z^{-1} - 2) = \frac{1}{\sqrt{2}}$$

$$\bar{y}_2(z^{-1} - 2) = z^{-1} - 2$$

and

$$X = [x_1], \quad Y = \begin{bmatrix} 1 & 0 \\ \sqrt{2}\,(1 - z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}.$$

We obtain

$$\bar{x}_1 = \frac{1}{2\sqrt{2}} + (z^{-1} - 2)\, t_1, \quad \bar{y}_1 = -\frac{1}{2\sqrt{2}} - z^{-1} t_1,$$

$$\bar{y}_2 = 1$$

for arbitrary $t_1, t_2 \in \mathfrak{R}[z^{-1}]$ and

$$X = \frac{1}{2\sqrt{2}} + t_1(z^{-1} - 2),$$

$$Y = \begin{bmatrix} -\dfrac{1}{2\sqrt{2}} \\ \dfrac{1 + z^{-1}}{2} \end{bmatrix} - \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} [t_1].$$

The solution X°, Y° satisfying $\partial Y^\circ = \min$ reads

$$X^\circ = \frac{1}{2\sqrt{2}}, \quad Y^\circ = \begin{bmatrix} -\dfrac{1}{2\sqrt{2}} \\ \dfrac{1 + z^{-1}}{2} \end{bmatrix}$$

on setting $t_1 = 0$.

The control

$$U = \frac{1}{2\sqrt{2}} \frac{1}{z^{-1} - 2}$$

is optimal since it is stable. The associated error becomes

$$E = \begin{bmatrix} -\dfrac{1}{2\sqrt{2}} \\ \frac{1}{2} + \frac{1}{2}z^{-1} \end{bmatrix}$$

and $k_{\min} = 2$.

Example 3.4. Consider problem (3.1) for a realization of

$$S = \frac{[z^{-1}(z^{-1} - 2) \quad (1 - z^{-1})(z^{-1} - 2)]}{1 - z^{-1}} = [z^{-1} - 2 \quad 0] \begin{bmatrix} 1 - z^{-1} & 1 - z^{-1} \\ 1 - z^{-1} & -z^{-1} \end{bmatrix}^{-1}$$

over the field $\mathfrak{Q}$ valuated by (2.25) and the reference sequence

$$W = \frac{1}{1 - z^{-1}}.$$

We make decomposition (3.4)

$$B_1 = [1 \quad 0] \begin{bmatrix} z^{-1} - 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since rank $B_1 = 1$, we find

$$B_{11}^- = 1 .$$

Thus equation (3.5) becomes

$$X + Y(1 - z^{-1}) = 1$$

and its general solution obtains as

$$X = 1 + (1 - z^{-1})\,t ,$$
$$Y = 0 - t$$

where $t \in \mathfrak{Q}[z^{-1}]$ arbitrary. The particular solution X°, Y° satisfying $\partial Y^\circ = \min$ becomes

$$X^\circ = 1 , \quad Y^\circ = 0$$

and hence

$$U_1 = \frac{1}{1 - z^{-1}}.$$

All optimal controls are then

$$U = \begin{bmatrix} 1 - z^{-1} & 1 - z^{-1} \\ 1 - z^{-1} & -z^{-1} \end{bmatrix} \begin{bmatrix} z^{-1} - 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{1}{1 - z^{-1}} \\ U_2 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{z^{-1} - 2} + \begin{bmatrix} 1 - z^{-1} \\ -z^{-1} \end{bmatrix} U_2$$

for an arbitrary $U_2 \in \mathfrak{Q}^+\{z^{-1}\}$; the control is not unique. The resulting error is unique and

$$E = 0, \quad k_{\min} = 0.$$

This nonuniqueness of the optimal control is due to the fact that $l < m$.

Example 3.5. Consider a system given by

$$S = \frac{\begin{bmatrix} z^{-1} & z^{-1} \\ 0 & z^{-1}(1 - 2z^{-1} - z^{-2}) \end{bmatrix}}{1 - z^{-1}} =$$

$$= \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1} - z^{-2}) \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1}) \\ 0 & 1 - z^{-1} \end{bmatrix}^{-1},$$

the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1 - z^{-1}}$$

and solve problem (3.1).

We shall demonstrate the importance of the ground field $\mathfrak{F}$. First consider $\mathfrak{F} = \mathfrak{Q}$ with valuation (2.25); then factorization (3.4) yields

$$B_1 = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1} - z^{-2}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and equation (3.5) reads

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1} - z^{-2}) \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Evidently,

$$X^\circ = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \quad Y^\circ = \begin{bmatrix} 1 \\ 1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \end{bmatrix}$$

and the optimal control

$$U = \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1}) \\ 0 & 1 - z^{-1} \end{bmatrix} \frac{\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

yields the error

$$E = \begin{bmatrix} 1 \\ 1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \end{bmatrix}, \quad k_{\min} = 3 .$$

Now consider $\mathfrak{F} = \mathfrak{R}$, again with valuation (2.25); then factorization (3.4) becomes

$$B_1 = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - (1 + \sqrt{2})\, z^{-1}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 - (1 - \sqrt{2})\, z^{-1} \end{bmatrix}$$

and equation (3.5) reads

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - (1 + \sqrt{2})\, z^{-1}) \end{bmatrix} X + Y(1 - z^{-2}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Evidently,

$$X^{\circ} = \begin{bmatrix} 1 \\ -\dfrac{1}{\sqrt{2}} \end{bmatrix}, \quad Y^{\circ} = \begin{bmatrix} 1 \\ 1 + \dfrac{1 + \sqrt{2}}{\sqrt{2}} z^{-1} \end{bmatrix}$$

and the optimal control

$$U = \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1}) \\ 0 & 1 - z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 - (1 - \sqrt{2})\, z^{-1} \end{bmatrix}^{-1} \frac{\begin{bmatrix} 1 \\ -\dfrac{1}{\sqrt{2}} \end{bmatrix}}{1 - z^{-1}} =$$

$$= \frac{\begin{bmatrix} -\dfrac{1 - \sqrt{2}}{\sqrt{2}} - (1 - \sqrt{2})\, z^{-1} \\ -\dfrac{1}{\sqrt{2}} \end{bmatrix}}{1 - (1 - \sqrt{2})\, z^{-1}}$$

yields the error

$$E = \begin{bmatrix} 1 \\ 1 + \dfrac{1 + \sqrt{2}}{\sqrt{2}} z^{-1} \end{bmatrix}, \quad k_{\min} = 2 .$$

Therefore, a larger field gives more opportunity to improve the optimal control.

Example 3.6. This example illustrates that ∂Y° is to be minimal among all solutions of (3.5) yielding a *stable* U, not among all existing solutions.

Let the system over the field $\mathfrak{R}$ valuated by (2.25) be given by

$$S = \frac{\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1})^2 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1},$$

the reference sequence by

$$W = \frac{\begin{bmatrix} 1 \\ (1 - z^{-1})^2 \end{bmatrix}}{1 - z^{-1}}$$

and solve problem (3.1).

We have to solve the equation

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} 1 \\ (1 - z^{-1})^2 \end{bmatrix},$$

i.e. the set of polynomial equations

$$z^{-1}\bar{x}_1 + \bar{y}_1(1 - z^{-1}) = 1$$

$$z^{-1}(1 - z^{-1})\,\bar{x}_2 + \bar{y}_2(1 - z^{-1}) = (1 - z^{-1})^2,$$

where

$$X = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad Y = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}.$$

The general solution can be written as

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 - z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1 - z^{-1}),$$

$$Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 - z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

by (1.19). The solution X^0, Y^0 satisfying $\partial Y^0 = \min$ without any respect to U becomes

$$X^0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

on setting $t_1 = 0$, $t_2 = 0$ but the control

$$U = \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1 - z^{-1}} = \frac{\begin{bmatrix} 1 - z^{-1} \\ 1 \end{bmatrix}}{1 - z^{-1}}$$

is not stable.

However, all possible controls are given as

$$U = \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} 1 + (1 - z^{-1})\, t_1 \\ -1 + t_2 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} 1 + (1 - z^{-1})\, t_1 \\ \dfrac{-1 + t_2}{1 - z^{-1}} \end{bmatrix}$$

and they will be stable if and only if $t_2 = 1 - (1 - z^{-1})\, t$ for any $t \in \Re[z^{-1}]$. Therefore, the solution X^0, Y^0 such that $\partial Y^0 = \min$ subject to U stable becomes

$$X^0 = \begin{bmatrix} (1 + \tau_0) - \tau_0 z^{-1} \\ 0 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} 1 - \tau_0 z^{-1} \\ 1 - z^{-1} \end{bmatrix}$$

on setting $t_1 = \tau_0$, $t_2 = 1$ where $\tau_0 \in \Re$ arbitraty. Then the optimal control is

$$U = \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} (1 + \tau_0) - \tau_0 z^{-1} \\ 0 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} (1 + \tau_0) - \tau_0 z^{-1} \\ 0 \end{bmatrix}$$

and the resulting error

$$E = \begin{bmatrix} 1 - \tau_0 z^{-1} \\ 1 - z^{-1} \end{bmatrix}, \quad k_{\min} = 2\,.$$

Example 3.7. Given a system

$$S = \frac{\begin{bmatrix} z^{-1} & z^{-1} \\ 0 & z^{-1}(1 - 2z^{-1})(z^{-1} - 2) \end{bmatrix}}{1 - z^{-1}} =$$

$$= \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1})(z^{-1} - 2) \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1}) \\ 0 & 1 - z^{-1} \end{bmatrix}^{-1}$$

over the field $\Re$ valuated by (2.25) and the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix}}{1 - z^{-1}},$$

solve problem (3.1).

We find factorization (3.4)

$$B_1 = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} - 2 \end{bmatrix}.$$

Then we are to solve the equation

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix},$$

the solution being

$$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1 - z^{-1}),$$

$$Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

for any $t_1, t_2 \in \Re[z^{-1}]$. The solution X^0, Y^0 becomes

$$X^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

on setting $t_1 = 0, t_2 = 0$.

The control

$$U = \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1}) \\ 0 & 1 - z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} - 2 \end{bmatrix}^{-1} \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is optimal since it is stable, and it yields the error

$$E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k_{\min} = 1 .$$

Note that the control sequence is finite, not only stable, even though B_1^+ is not a unit.

Example 3.8. Given a realization of

$$S = \frac{z^{-1} + z^{-2}}{1 + z^{-1} + z^{-2}}$$

over the field $\mathfrak{Z}_2$ (with valuation (2.24), of course), solve problem (3.1) for the reference sequence

$$W = 1 + z^{-2} .$$

As no polynomial of $\mathfrak{Z}_2[z^{-1}]$ is stable save the units in $\mathfrak{Z}_2[z^{-1}]$, we have

$$B_{11}^- = z^{-1} + z^{-2} .$$

Equation (3.5) then becomes

$$(z^{-1} + z^{-2}) X + Y = 1 + z^{-2}$$

and its general solution is

$$X = 0 + t ,$$

$$Y = 1 + z^{-2} - (z^{-1} + z^{-2}) t$$

for any $t \in \mathfrak{Z}_2[z^{-1}]$. Remember that all calculations have to be carried out in the modulo 2 arithmetics.

The solution X^0, Y^0 satisfying $\partial Y^0 = \min$ is, evidently,

$$X^0 = 1\,, \quad Y^0 = 1 + z^{-1}$$

on setting $t = 1$. Therefore,

$$\boldsymbol{U} = 1 + z^{-1} + z^{-2}$$

is the optimal control and

$$\boldsymbol{E} = 1 + z^{-1}\,, \quad k_{\min} = 2$$

is the resulting error.

3.3. Finite time optimal control problem

Let $\mathfrak{F}$ be an arbitrary field with valuation $\mathscr{V}$ and write

$$\boldsymbol{S} = \frac{B}{a} = B_1 A_2^{-1}\,, \quad \text{rank} \quad B_1 = r\,.$$

By the definition of B_1 in (2.19) we get

$$B_1 = [B_{11}\ 0]$$

where $B_{11} \in \mathfrak{F}_{l,r}[z^{-1}]$, $0 \in \mathfrak{F}_{l,m-r}[z^{-1}]$ and rank $B_{11} = r$.

Then we have the following result.

Theorem 3.2. *Problem* (3.2) *has a solution if and only if the linear Diophantine equation*

(3.12) $$B_{11}X + Yp = Q$$

has a solution X^0, Y^0 *such that* $\partial Y^0 = \min$ *subject to*

(3.13) $$\boldsymbol{U} = A_2 \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{bmatrix}$$

belongs to $\mathfrak{F}_{m,1}[z^{-1}]$, *where*

$$\boldsymbol{U}_1 = \frac{X^0}{p}\,,$$

$$\boldsymbol{U}_2 \in \mathfrak{F}_{m-r,1}\{z^{-1}\}\,.$$

The optimal control is not unique, in general, and all optimal controls are given by (3.13). *Moreover,*

$$E = Y^0$$

and

$$k_{\min} = 0\,, \qquad Y^0 = 0\,,$$
$$= 1 + \partial Y^0\,, \quad Y^0 \neq 0\,.$$

Proof. Write

$$E = W - SU = \frac{Q}{p} - B_1 A_2^{-1} U = \frac{Q}{p} - [B_{11}\ \ 0]\, A_2^{-1} U = \frac{Q}{p} - B_{11} U_1\,,$$

where

$$A_2^{-1} U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

and

$$U_1 \in \mathfrak{F}_{r,1}\{z^{-1}\}\,, \quad U_2 \in \mathfrak{F}_{m-r,1}\{z^{-1}\}\,.$$

Since the error is to vanish in a finite time and thereafter, E must be a polynomial matrix in $\mathfrak{F}_{l,1}[z^{-1}]$, say Y. Therefore

$$Y = \frac{Q}{p} - B_{11} U_1 = \frac{Q - pB_{11} U_1}{p} \tag{3.14}$$

and since $(p, Q) = 1$ up to a unit in $\mathfrak{F}[z^{-1}]$, we must take

$$U_1 = \frac{X}{p}\,, \tag{3.15}$$

where $X \in \mathfrak{F}_{r,1}[z^{-1}]$ is an unspecified polynomial matrix as yet.

In fact, the X and Y satisfy equation (3.12) by virtue of (3.14) and (3.15). Among all solutions of equation (3.12) we have to take only those which make the U polynomial and within the class only those which minimize the degree of E. Therefore

$$U = A_2 \begin{bmatrix} U_1 \\ U_2 \end{bmatrix},$$

where

$$U_1 = \frac{X^0}{p}\,,$$

$$U_2 \in \mathfrak{F}_{m-r,1}\{z^{-1}\} \quad \text{arbitrary but such that} \quad A_2 \begin{bmatrix} 0 \\ U_2 \end{bmatrix} \in \mathfrak{F}_{m,1}[z^{-1}]$$

and

$$E = Y^0\,,$$

where X^0, Y^0 is a solution of equation (3.12) such that $\partial Y^0 = \min$ among all solutions yielding a polynomial $\boldsymbol{U}$. Then

$$\begin{aligned} k_{\min} &= 0 && \text{if} \quad Y^0 = 0\,, \\ &= 1 + \partial Y^0 && \text{otherwise.} \end{aligned}$$

The finiteness of $\boldsymbol{U}$ cannot be inferred until the general solution of equation (3.12) is found. □

Example 3.9. Consider the system which is a realization of

$$\boldsymbol{S} = \frac{\begin{bmatrix} z^{-1} & z^{-1} - z^{-3} \\ z^{-3} & z^{-3} \end{bmatrix}}{1 - z^{-1}} =$$

$$= \begin{bmatrix} z^{-1} & 0 \\ z^{-3} & z^{-5} \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1})(1 - z^{-2}) \\ 0 & 1 - z^{-1} \end{bmatrix}^{-1}$$

over the field $\mathfrak{R}$ valuated by (2.25), the reference sequence

$$\boldsymbol{W} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1 - z^{-2}}$$

and solve problem (3.2).

Equation (12) reads

$$\begin{bmatrix} z^{-1} & 0 \\ z^{-3} & z^{-5} \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.16}$$

Write

$$\begin{bmatrix} z^{-1} & 0 \\ z^{-3} & z^{-5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z^{-2} & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-5} \end{bmatrix};$$

then equation (3.16) reduces to the set of polynomial equations

$$z^{-1}\bar{x}_1 + \bar{y}_1(1 - z^{-1}) = 1\,,$$

$$z^{-5}\bar{x}_2 + \bar{y}_2(1 - z^{-1}) = -z^{-2}$$

and

$$X = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 \\ z^{-2} & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}.$$

The general solution obtains as

$$\bar{x}_1 = 1 + (1 - z^{-1})\, t_1\,, \quad \bar{y}_1 = 1 - z^{-1} t_1\,,$$

$$\bar{x}_2 = -1 + (1 - z^{-1})\, t_2\,, \quad \bar{y}_2 = -z^{-2} - z^{-3} - z^{-4} - z^{-5} t_2$$

where $t_1, t_2 \in \mathfrak{R}[z^{-1}]$ arbitrary; hence

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}(1 - z^{-1}),$$

$$Y = \begin{bmatrix} 1 \\ -z^{-3} - z^{-4} \end{bmatrix} - \begin{bmatrix} z^{-1} & 0 \\ z^{-3} & z^{-5} \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}.$$

The particular solution X^0, Y^0 satisfying $\partial Y^0 = \min$ is given as

$$X^0 = \begin{bmatrix} z^{-2} \\ -1 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} 1 + z^{-1} + z^{-2} \\ 0 \end{bmatrix}$$

on setting $t_1 = -1 - z^{-1}$, $t_2 = 0$. Then

$$U_1 = \frac{\begin{bmatrix} z^{-2} \\ -1 \end{bmatrix}}{1 - z^{-1}}$$

and

$$U = \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1})(1 - z^{-2}) \\ 0 & 1 - z^{-1} \end{bmatrix} \frac{\begin{bmatrix} z^{-2} \\ -1 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is the unique optimal control, and it yields the error

$$E = \begin{bmatrix} 1 + z^{-1} + z^{-2} \\ 0 \end{bmatrix}, \quad k_{\min} = 3\,.$$

Example 3.10. Consider the Galois field $\mathfrak{F} = \mathfrak{Z}_2[z]_{z^2+z+1}$, an algebraic extension of $\mathfrak{Z}_2$ consisting of the elements $\{0, 1, \varepsilon, \varepsilon^2\}$, where $\varepsilon^3 = 1$. The addition and multiplication tables are given below.

+	0	1	ε	ε^2
0	0	1	ε	ε^2
1	1	0	ε^2	ε
ε	ε	ε^2	0	1
ε^2	ε^2	ε	1	0

.	0	1	ε	ε^2
0	0	0	0	0
1	0	1	ε	ε^2
ε	0	ε	ε^2	1
ε^2	0	ε^2	1	ε

The only valuation is the trivial one, see (2.24).

Given the system

$$S = \frac{\begin{bmatrix} z^{-1} & 0 \\ \varepsilon z^{-1} & \varepsilon^2 \end{bmatrix}}{1 + z^{-1}} = \begin{bmatrix} 0 & z^{-1} \\ \varepsilon^2 & \varepsilon z^{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 + z^{-1} \\ 1 + z^{-1} & 0 \end{bmatrix}^{-1}$$

and the reference sequence

$$W = \frac{\begin{bmatrix}\varepsilon \\ \varepsilon^2\end{bmatrix}}{1 + z^{-1}}$$

over the above defined field $\mathfrak{F}$, solve problem (3.2).

Equation (3.12) becomes

$$\begin{bmatrix}0 & z^{-1} \\ \varepsilon^2 & \varepsilon z^{-1}\end{bmatrix} X + Y(1 + z^{-1}) = \begin{bmatrix}\varepsilon \\ \varepsilon^2\end{bmatrix}. \tag{3.17}$$

We write

$$\begin{bmatrix}0 & z^{-1} \\ \varepsilon^2 & \varepsilon z^{-1}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ \varepsilon^2 & \varepsilon\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & z^{-1}\end{bmatrix};$$

then equation (3.17) reduces to the set of polynomial equations

$$\bar{\bar{x}}_1 + \bar{\bar{y}}_1(1 + z^{-1}) = 0,$$

$$z^{-1}\bar{\bar{x}}_2 + \bar{\bar{y}}_2(1 + z^{-1}) = \varepsilon$$

and

$$X = \begin{bmatrix}\bar{\bar{x}}_1 \\ \bar{\bar{x}}_2\end{bmatrix}, \quad Y = \begin{bmatrix}0 & 1 \\ \varepsilon^2 & \varepsilon\end{bmatrix}\begin{bmatrix}\bar{\bar{y}}_1 \\ \bar{\bar{y}}_2\end{bmatrix}.$$

The general solution can be written as

$$\bar{\bar{x}}_1 = 0 + (1 - z^{-1})\, t_1, \quad \bar{\bar{y}}_1 = 0 - t_1,$$

$$\bar{\bar{x}}_2 = \varepsilon + (1 - z^{-1})\, t_2, \quad \bar{\bar{y}}_2 = \varepsilon - z^{-1}t_2$$

for arbitrary $t_1, t_2 \in \mathfrak{F}[z^{-1}]$ and

$$X = \begin{bmatrix}0 \\ \varepsilon\end{bmatrix} + \begin{bmatrix}t_1 \\ t_2\end{bmatrix}(1 + z^{-1}),$$

$$Y = \begin{bmatrix}\varepsilon \\ \varepsilon^2\end{bmatrix} - \begin{bmatrix}0 & z^{-1} \\ \varepsilon^2 & \varepsilon z^{-1}\end{bmatrix}\begin{bmatrix}t_1 \\ t_2\end{bmatrix}.$$

The solution X^0, Y^0 satisfying $\partial Y^0 = \min$ is obtained on setting $t_1 = \tau_0$, $t_2 = 0$, where $\tau_0 \in \mathfrak{F}$ arbitrary, and

$$X^0 = \begin{bmatrix}\tau_0 + \tau_0 z^{-1} \\ \varepsilon\end{bmatrix}, \quad Y^0 = \begin{bmatrix}\varepsilon \\ \varepsilon^2(1 - \tau_0)\end{bmatrix}.$$

The optimal control is not unique,

$$U = \begin{bmatrix}0 & 1 + z^{-1} \\ 1 + z^{-1} & 0\end{bmatrix}\frac{\begin{bmatrix}\tau_0 + \tau_0 z^{-1} \\ \varepsilon\end{bmatrix}}{1 + z^{-1}} = \begin{bmatrix}\varepsilon \\ \tau_0 + \tau_1 z^{-1}\end{bmatrix},$$

and it yields the error

$$E = \begin{bmatrix} \varepsilon \\ \varepsilon^2(1 - \tau_0) \end{bmatrix}, \quad k_{\min} = 1 .$$

Example 3.11. Consider once more the system given by

$$S = \frac{\begin{bmatrix} z^{-1} & 0 \\ z^{-2} & z^{-1}(1 - z^{-1}) \end{bmatrix}}{1 - z^{-1}}$$

and the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ 2(1 - z^{-1}) \end{bmatrix}}{1 - z^{-1}}$$

of Example 3.2 and solve problem (3.2).

All tentative controls have the form

$$U = \begin{bmatrix} 1 \\ -\dfrac{1}{1 - z^{-1}} \end{bmatrix} + \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

and it is easy to see that U is not a polynomial matrix regardless of t_1 and t_2. Therefore, problem (3.2) has no solution at all.

Example 3.12. The method of the paper is general enough to effectively treat systems whose transfer function matrix is singular. For example, let a system over the field $\mathfrak{R}$ valuated by (2.24) be given by

$$S = \frac{\begin{bmatrix} z^{-1} & z^{-1} \\ z^{-1} & z^{-1} \end{bmatrix}}{(1 - z^{-1})(z^{-1} - 2)} = \begin{bmatrix} z^{-1} & 0 \\ z^{-1} & 0 \end{bmatrix} \begin{bmatrix} (1 - z^{-1})(z^{-1} - 2) & -1 \\ 0 & 1 \end{bmatrix}^{-1} \tag{3.18}$$

and solve problem (3.2) for the reference sequence

$$W = \frac{\begin{bmatrix} 2 \\ z^{-1} \end{bmatrix}}{z^{-1} - 2} .$$

Since

$$B_{11} = \begin{bmatrix} z^{-1} \\ z^{-1} \end{bmatrix},$$

we are to solve the equation

$$\begin{bmatrix} z^{-1} \\ z^{-1} \end{bmatrix} X + Y(z^{-1} - 2) = \begin{bmatrix} 2 \\ z^{-1} \end{bmatrix} .$$

We can write

$$\begin{bmatrix} z^{-1} \\ z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix}$$

and, hence, we obtain

$$z^{-1}\bar{\bar{x}}_1 + \bar{\bar{y}}_1(z^{-1} - 2) = 2 ,$$

$$\bar{\bar{y}}_2(z^{-1} - 2) = z^{-1} - 2 ,$$

where

$$X = \begin{bmatrix} \bar{\bar{x}}_1 \\ \bar{\bar{x}}_2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \bar{\bar{y}}_1 \\ \bar{\bar{y}}_2 \end{bmatrix}.$$

The general solution is

$$\bar{\bar{x}}_1 = 1 + (z^{-1} - 2)\,t , \quad \bar{\bar{y}}_1 = -1 - z^{-1}t$$

$$\bar{\bar{y}}_2 = 1$$

and

$$X = 1 + t(z^{-1} - 2) ,$$

$$Y = \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \begin{bmatrix} z^{-1} \\ z^{-1} \end{bmatrix} t$$

for arbitrary $t \in \Re[z^{-1}]$. Evidently,

$$X^0 = 1 , \quad Y^0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

when one sets $t = 0$. Then

$$\boldsymbol{U}_1 = \frac{1}{z^{-1} - 2}$$

and

$$\boldsymbol{U} = \begin{bmatrix} (1 - z^{-1})(z^{-1} - 2) & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{z^{-1} - 2} \\ \boldsymbol{U}_2 \end{bmatrix} = \begin{bmatrix} 1 - z^{-1} - \boldsymbol{U}_2 \\ \boldsymbol{U}_2 \end{bmatrix}$$

is the optimal control for any $\boldsymbol{U}_2 \in \Re[z^{-1}]$. The resulting error becomes

$$\boldsymbol{E} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

and

$$k_{\min} = 1 .$$

Because of the singularity of B_1, the admissible reference sequences W for which the problem has a solution given the system (3.18) are quite restricted. It can be shown using Theorem 1.1 that if

$$\boldsymbol{W} = \frac{\begin{bmatrix} a \\ b \end{bmatrix}}{p} ,$$

where $a, b, p \in \mathfrak{R}[z^{-1}]$, the relation

$$p \mid b - a$$

must hold.

Example 3.13. There is another sort of nonuniqueness of the optimal controls due to the internal structure of the system.

Let the system over the field $\mathfrak{R}$ valuated by (2.25) be given by

$$S = \frac{\begin{bmatrix} z^{-1}(z^{-1}-2) & 0 \\ -z^{-1}(1-z^{-1}) & z^{-2} \end{bmatrix}}{(1-z^{-1})(z^{-1}-2)} =$$

$$= \begin{bmatrix} z^{-1}(z^{-1}-2) & z^{-2} \\ -z^{-1}(1-z^{-1}) & z^{-2} \end{bmatrix} \begin{bmatrix} (1-z^{-1})(z^{-1}-2) & z^{-1}(1-z^{-1}) \\ 0 & -(1-z^{-1}) \end{bmatrix}^{-1}$$

and the reference sequence by

$$W = \frac{\begin{bmatrix} 1 \\ 1{\cdot}5 \end{bmatrix}}{1-z^{-1}};$$

find a solution to problem (3.2).

Equation (3.12) becomes

$$\begin{bmatrix} z^{-1}(z^{-1}-2) & z^{-2} \\ -z^{-1}(1-z^{-1}) & z^{-2} \end{bmatrix} X + Y(1-z^{-1}) = \begin{bmatrix} 1 \\ 1{\cdot}5 \end{bmatrix}.$$

Writing

$$\begin{bmatrix} z^{-1}(z^{-1}-2) & z^{-2} \\ -z^{-1}(1-z^{-1}) & z^{-2} \end{bmatrix} = \begin{bmatrix} z^{-1}-2 & 1 \\ -(1-z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-2} \end{bmatrix},$$

the above equation reduces to the set of polynomial equations

$$z^{-1}\bar{x}_1 + \bar{y}_1(1-z^{-1}) = 0{\cdot}5\,,$$

$$z^{-2}\bar{x}_2 + \bar{y}_2(1-z^{-1}) = 2 - 0{\cdot}5z^{-1}$$

and

$$X = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad Y = \begin{bmatrix} z^{-1}-2 & 1 \\ -(1-z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}.$$

The general solution reads

$$\bar{x}_1 = 0{\cdot}5 + (1-z^{-1})\,t_1\,, \quad \bar{y}_1 = 0{\cdot}5 - z^{-1}t_1\,,$$

$$\bar{x}_2 = 1{\cdot}5 + (1-z^{-1})\,t_2\,, \quad \bar{y}_2 = 2 + 1{\cdot}5z^{-1} - z^{-2}t_2$$

for $t_1, t_2 \in \mathfrak{R}[z^{-1}]$ arbitrary and

$$X = \begin{bmatrix} 0{\cdot}5 \\ 1{\cdot}5 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1 - z^{-1}),$$

$$Y = \begin{bmatrix} 1 \quad + 2z^{-1} \\ 1{\cdot}5 + 2z^{-1} \end{bmatrix} - \begin{bmatrix} z^{-1}(z^{-1} - 2) & z^{-2} \\ -z^{-1}(1 - z^{-1}) & z^{-2} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}.$$

It can be seen that the solution X^0, Y^0 satisfying $\partial Y^0 = \min$ is obtained by setting $t_1 = -t_2 = \tau_0$, $\tau_0 \in \mathfrak{R}$ arbitrary, and

$$X^0 = \begin{bmatrix} (0{\cdot}5 + \tau_0) - \tau_0 z^{-1} \\ (1{\cdot}5 - \tau_0) + \tau_0 z^{-1} \end{bmatrix},$$

$$Y^0 = \begin{bmatrix} 1 + (2 + 2\tau_0) z^{-1} \\ 1{\cdot}5 + (2 + \tau_0) z^{-1} \end{bmatrix}.$$

Then neither the optimal control

$$\boldsymbol{U} = \begin{bmatrix} (1 - z^{-1})(z^{-1} - 2) & z^{-1}(1 - z^{-1}) \\ 0 & -(1 - z^{-1}) \end{bmatrix} \frac{\begin{bmatrix} (0{\cdot}5 + \tau_0) - \tau_0 z^{-1} \\ (1{\cdot}5 - \tau_0) + \tau_0 z^{-1} \end{bmatrix}}{1 - z^{-1}} =$$

$$= \begin{bmatrix} -(1 + 2\tau_0) + (2 + 2\tau_0) z^{-1} \\ -(1{\cdot}5 - \tau_0) - \tau_0 z^{-1} \end{bmatrix}$$

nor the error

$$\boldsymbol{E} = \begin{bmatrix} 1 + (2 + 2\tau_0) z^{-1} \\ 1{\cdot}5 + (2 + \tau_0) z^{-1} \end{bmatrix}$$

is unique. All the errors give $k_{\min} = 2$, however.

There are two typical solutions:

$$\tau_0 = -1 \quad \text{gives} \quad \boldsymbol{E} = \begin{bmatrix} 1 \\ 1{\cdot}5 + z^{-1} \end{bmatrix}$$

and

$$\tau_0 = -2 \quad \text{gives} \quad \boldsymbol{E} = \begin{bmatrix} 1 - 2z^{-1} \\ 1{\cdot}5 \end{bmatrix}.$$

3.4. Least squares control problem

Let $\mathfrak{F}$ be a subfield of the field $\mathfrak{C}$ of complex numbers valuated by (2.25) and write

$$\boldsymbol{S} = \frac{B}{a} = B_1 A_2^{-1}, \quad \text{rank} \quad B_1 = r,$$

$$B_1 = B_1^- B_1^+ . \tag{3.19}$$

By the definition of B_1^-, see (2.19) and (2.30), we have

$$B_1^- = [B_{11}^- \ 0],$$

where $B_{11}^- \in \mathfrak{F}_{l,r}[z^{-1}]$, $0 \in \mathfrak{F}_{l,m-r}[z^{-1}]$ and rank $B_{11}^- = r$.

Further let

$$B_{11}^{-=\prime} B_{11}^- = (B_{11}^-)^{*=\prime} (B_{11}^-)^* \tag{3.20}$$

and denote

$$d = \partial B_{11}^- - \partial(B_{11}^-)^* . \tag{3.21}$$

For convenience, we shall use the notation

$$(B_{11}^-)^* =_{\text{def}} H .$$

Then we have the following result.

Theorem 3.3. *Let $\mathfrak{F}$ be a subfield of $\mathfrak{C}$ valuated by* (2.25)*. Then problem* (3.3) *has a solution if and only if the linear Diophantine equation*

$$z^{-d} H^{\sim\prime} X + Y p = B_{11}^{-\sim\prime} Q \tag{3.22}$$

has a solution X^0, Y^0 such that $\partial Y^0 < \partial z^{-d} H^{\sim\prime}$ and

$$U = A_2 (B_1^+)^{-1} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \tag{3.23}$$

$$E = W - B_{11}^- U_1 \tag{3.24}$$

belong to $\mathfrak{F}_{m,1}^+\{z^{-1}\}$ and $\mathfrak{F}_{l,1}^+\{z^{-1}\}$ respectively, where

$$U_1 = \frac{H^{-1} X^0}{p},$$

$$U_2 \in \mathfrak{F}_{m-r,1}\{z^{-1}\} .$$

The optimal control is not unique, in general, and all optimal controls are given by (3.23)*. Moreover, E is given by* (3.24) *and satisfies*

$$B_{11}^{-\sim\prime} E = Y^0 ; \tag{3.25}$$

also

$$\|E\|_{\min}^2 = \langle ((H^{\sim\prime})^{-1} Y^0)^{=\prime} ((H^{\sim\prime})^{-1} Y^0) \rangle + \langle W^{=\prime}(I_l - B_{11}^- H^{-1} (H^{=\prime})^{-1} B_{11}^{-=\prime}) W \rangle .$$

Proof. In order to minimize $\|E\|^2$ we shall assume that E is stable whereby

$$\|E\|^2 = \langle E^{=\prime} E \rangle .$$

Then we will manipulate the expression $\langle \boldsymbol{E}^{=\prime}\boldsymbol{E}\rangle$ so as to make the minimizing choice of $\boldsymbol{U}$ obvious.

Write

$$\boldsymbol{E} = \boldsymbol{W} - \boldsymbol{S}\boldsymbol{U} = \boldsymbol{W} - [B_{11}^{-}\ 0]\, B_1^{+} A_2^{-1} \boldsymbol{U} = \boldsymbol{W} - B_{11}^{-}\boldsymbol{U}_1 ,$$

where

$$B_1^{+} A_2^{-1} \boldsymbol{U} = \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{bmatrix}$$

and

$$\boldsymbol{U}_1 \in \mathfrak{F}_{r,1}\{z^{-1}\}, \quad \boldsymbol{U}_2 \in \mathfrak{F}_{m-r,1}\{z^{-1}\}.$$

Then

$$\begin{aligned} (3.26) \qquad \boldsymbol{E}^{=\prime}\boldsymbol{E} &= \boldsymbol{W}^{=\prime}\boldsymbol{W} - \boldsymbol{W}^{=\prime} B_{11}^{-}\boldsymbol{U}_1 - \boldsymbol{U}_1^{=\prime} B_{11}^{-=\prime}\, \boldsymbol{W} + \boldsymbol{U}_1^{=\prime} B_{11}^{-=\prime} B_{11}^{-}\boldsymbol{U}_1 = \\ &= ((H^{=\prime})^{-1} B_{11}^{-=\prime}\boldsymbol{W} - H\boldsymbol{U}_1)^{=\prime} ((H^{=\prime})^{-1} B_{11}^{-=\prime}\boldsymbol{W} - H\boldsymbol{U}_1) + \\ &\quad + \boldsymbol{W}^{=\prime}\boldsymbol{W} - \boldsymbol{W}^{=\prime} B_{11} H^{-1} (H^{=\prime})^{-1} B_{11}^{-=\prime}\boldsymbol{W}. \end{aligned}$$

Since the last two terms in (3.26) are independent of $\boldsymbol{U}_1$ (and hence $\boldsymbol{U}$), the expression $\langle \boldsymbol{E}^{=\prime}\boldsymbol{E}\rangle$ attains its minimum for the same control sequence $\boldsymbol{U}$ as the expression $\langle \boldsymbol{E}_1^{=\prime}\boldsymbol{E}_1\rangle$ does, where

$$\boldsymbol{E}_1 = (H^{=\prime})^{-1} B_{11}^{-=\prime}\boldsymbol{W} - H\boldsymbol{U}_1 .$$

Using (2.28) and (3.21) we have

$$(3.27) \qquad (H^{=\prime})^{-1} B_{11}^{-=\prime} = \frac{(H^{\sim\prime})^{-1} B_{11}^{-\sim\prime}}{z^{-d}}$$

and, therefore,

$$(3.28) \qquad \boldsymbol{E}_1 = \frac{(H^{\sim\prime})^{-1} B_{11}^{-\sim\prime} Q}{z^{-d} p} - H\boldsymbol{U}_1 .$$

Now take the partial fraction expansion

$$\frac{(H^{\sim\prime})^{-1} B_{11}^{-\sim\prime} Q}{z^{-d} p} = \frac{X}{p} + \frac{(H^{\sim\prime})^{-1} Y}{z^{-d}}$$

of the first term on the right-hand side of (3.28). It follows that the X and Y are coupled by equation (3.22).

Collecting the terms gives us

$$(3.29) \qquad \boldsymbol{E}_1 = \frac{(H^{\sim\prime})^{-1} Y}{z^{-d}} + \boldsymbol{A},$$

where

$$(3.30) \qquad \boldsymbol{A} = \frac{X}{p} - H\boldsymbol{U}_1 .$$

Hence, by virtue of (3.29)

$$(3.30)\qquad \langle E_1^{=\prime} E_1\rangle = \left\langle \left(\frac{(H^{\sim\prime})^{-1}Y}{z^{-d}}\right)^{=\prime}\left(\frac{(H^{\sim\prime})^{-1}Y}{z^{-d}}\right)\right\rangle + $$

$$+ \left\langle \left(\frac{(H^{\sim\prime})^{-1}Y}{z^{-d}}\right)^{=\prime} A\right\rangle + \left\langle A^{=\prime}\left(\frac{(H^{\sim\prime})^{-1}Y}{z^{-d}}\right)\right\rangle + \langle A^{=\prime}A\rangle .$$

Any solution of equation (3.22) can be written in the form

$$(3.32)\qquad X = X^0 + D^{-1}Tp ,$$

$$(3.33)\qquad Y = Y^0 - z^{-d}H^{\sim\prime}D^{-1}T$$

by (1.19), where $T \in \mathfrak{F}_{r,1}[z^{-1}]$ is arbitrary and $D \in \mathfrak{F}_{r,r}[z^{-1}]$ is defined in (1.20), and where

$$(3.34)\qquad \partial Y^0 < \partial z^{-d}H^{\sim\prime} .$$

Substituting (3.33) into (3.31) we obtain

$$\langle E_1^{=\prime} E_1\rangle = \left\langle \left(\frac{(H^{\sim\prime})^{-1}Y^0}{z^{-d}}\right)^{=\prime}\left(\frac{(H^{\sim\prime})^{-1}Y^0}{z^{-d}}\right)\right\rangle - \left\langle \left(\frac{(H^{\sim\prime})^{-1}Y^0}{z^{-d}}\right)^{=\prime} D^{-1}T\right\rangle -$$

$$- \left\langle (D^{-1}T)^{=\prime}\left(\frac{(H^{\sim\prime})^{-1}Y^0}{z^{-d}}\right)\right\rangle + \langle (D^{-1}T)^{=\prime}(D^{-1}T)\rangle + \left\langle \left(\frac{(H^{\sim\prime})^{-1}Y^0}{z^{-d}}\right)^{=\prime} A\right\rangle -$$

$$- \langle (D^{-1}T)^{=\prime}A\rangle + \left\langle A^{=\prime}\left(\frac{(H^{\sim\prime})^{-1}Y^0}{z^{-d}}\right)\right\rangle - \langle A^{=\prime}D^{-1}T\rangle + \langle A^{=\prime}A\rangle .$$

The key observation is that

$$\left(\frac{(H^{\sim\prime})^{-1}Y^0}{z^{-d}}\right)^{=\prime} = z^{-(\partial z^{-d}H^{\sim\prime} - \partial Y^0)}H^{-1}Y^{0\sim\prime}$$

is divisible by z^{-1} due to (3.34) and hence

$$\left\langle \left(\frac{(H^{\sim\prime})^{-1}Y^0}{z^{-d}}\right)^{=\prime} D^{-1}T\right\rangle = 0$$

and

$$\left\langle \left(\frac{(H^{\sim\prime})^{-1}Y^0}{z^{-d}}\right)^{=\prime} A\right\rangle = 0 .$$

Therefore,

$$\langle E_1^{=\prime} E_1\rangle = \langle ((H^{\sim\prime})^{-1}Y^0)^{=\prime}((H^{\sim\prime})^{-1}Y^0)\rangle + \langle (A - D^{-1}T)^{=\prime}(A - D^{-1}T)\rangle .$$

The first term on the right-hand side above cannot be affected by any choice of $\boldsymbol{U}_1$. The best we can do to minimize $\langle \boldsymbol{E}_1^{=}{}'\boldsymbol{E}_1 \rangle$ is to set $A - D^{-1}T = 0$. By virtue of (3.30) we obtain

$$\frac{X}{p} - H\boldsymbol{U}_1 - D^{-1}T = 0\,,$$

i.e.

$$X - D^{-1}Tp = pH\boldsymbol{U}_1\,.$$

But

(3.32) $$X - D^{-1}Tp = X^0$$

by (3.32) and hence the $\langle \boldsymbol{E}_1^{=}{}'\boldsymbol{E}_1 \rangle$ is minimized by setting

(3.35) $$\boldsymbol{U}_1 = \frac{H^{-1}X^0}{p}\,.$$

It means that $\|\boldsymbol{E}\|^2 = \langle \boldsymbol{E}^{=}{}'\boldsymbol{E} \rangle$ is minimized by the same $\boldsymbol{U}_1$ provided the $\boldsymbol{E}$ is stable.

Thus

$$\boldsymbol{U} = A_2(B_1^+)^{-1} \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{bmatrix}$$

is the optimal control provided it is stable. It follows that $\boldsymbol{U}_2$ can be taken as an arbitrary element of $\mathfrak{F}_{m-r,1}\{z^{-1}\}$ but such that

$$A_2(B_1^+)^{-1} \begin{bmatrix} 0 \\ \boldsymbol{U}_2 \end{bmatrix} \in \mathfrak{F}_{m,1}^{+}\{z^{-1}\}\,.$$

We also have

$$\|\boldsymbol{E}\|^2_{\min} = \langle ((H^{\sim}{}')^{-1}\,Y^0)^{=}{}'\,((H^{\sim}{}')^{-1}\,Y^0)\rangle + \langle \boldsymbol{W}^{=}{}'\,(I_l - B_{11}^{-}H^{-1}(H^{=}{}')^{-1}\,B_{11}^{-=}{}')\,\boldsymbol{W}\rangle$$

by taking (3.26) into account.

Further

$$\boldsymbol{E} = \boldsymbol{W} - \boldsymbol{S}\boldsymbol{U} = \frac{Q}{p} - B_{11}^{-}\boldsymbol{U}_1$$

and the error sequence $\boldsymbol{E}$ satisfies the relation

$$B_{11}^{-\sim}{}'\boldsymbol{E} = \frac{B_{11}^{-\sim}{}'Q}{p} - B_{11}^{-\sim}{}'B_{11}^{-}\boldsymbol{U}_1 =$$

$$= \frac{B_{11}^{-\sim}{}'Q}{p} - B_{11}^{-\sim}{}'B_{11}^{-}\frac{H^{-1}X^0}{p} = \frac{B_{11}^{-\sim}{}'Q - z^{-d}H^{\sim}{}'X^0}{p} = \frac{Y^0 p}{p} = Y^0$$

on using (3.35), (3.20), and (3.22).

Since decomposition (3.20) is unique modulo a unitary element in $\mathfrak{F}_{r,r}$, see [55; 64], we have to show that the optimal control is independent of a particular choice of this element. Indeed, let

$$H_\omega = \Omega H$$

also satisfies (3.20), where $\Omega \in \mathfrak{F}_{r,r}$ satisfies $\bar{\Omega}'\Omega = \Omega^{\sim\prime}\Omega = I_r$. Then $H_\omega^{\sim\prime} = H^{\sim\prime}\Omega^{\sim\prime}$ and we are to solve the equation

$$z^{-d}H_\omega^{\sim\prime}X_\omega + Y_\omega p = B_{11}^{-\sim\prime}Q$$

instead of (3.22). Since Ω is a unit in $\mathfrak{F}_{r,r}[z^{-1}]$, we get

$$X_\omega = (\Omega^{\sim\prime})^{-1}X\,, \quad Y_\omega = Y$$

where X, Y is a solution of (3.22). Therefore $Y_\omega^0 = Y^0$ and

$$\boldsymbol{U}_{1\omega} = \frac{H_\omega^{-1}X_\omega^0}{p} = \frac{H^{-1}\Omega^{-1}(\Omega^{\sim\prime})^{-1}X^0}{p} = \boldsymbol{U}_1\,. \qquad \square$$

Example 3.14. Given the system which is a realization of

$$\boldsymbol{S} = \frac{\begin{bmatrix} z^{-1} & 0 \\ z^{-1}(1-2z^{-1}) & z^{-1}(1-2z^{-1}) \end{bmatrix}}{1-z^{-1}} =$$

$$= \begin{bmatrix} z^{-1} & 0 \\ z^{-1}(1-2z^{-1}) & z^{-1}(1-2z^{-1}) \end{bmatrix} \begin{bmatrix} 1-z^{-1} & 0 \\ 0 & 1-z^{-1} \end{bmatrix}^{-1}$$

over the field $\mathfrak{R}$, solve problem (3.3) for the reference sequence

$$\boldsymbol{W} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1-z^{-1}}\,.$$

We first find the decomposition (3.19)

$$B_1 = \begin{bmatrix} z^{-1} & 0 \\ z^{-1}(1-2z^{-1}) & z^{-1}(1-2z^{-1}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$B_{11}^{-} = \begin{bmatrix} z^{-1} & 0 \\ z^{-1}(1-2z^{-1}) & z^{-1}(1-2z^{-1}) \end{bmatrix}, \quad B_{11}^{-\sim\prime} = \begin{bmatrix} z^{-1} & z^{-1}-2 \\ 0 & z^{-1}-2 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ z^{-1}-2 & z^{-1}-2 \end{bmatrix}, \quad H^{\sim\prime} = \begin{bmatrix} z^{-1} & 1-2z^{-1} \\ 0 & 1-2z^{-1} \end{bmatrix}, \quad d = 1\,.$$

Then equation (3.22) reads

$$(3.36)\qquad \begin{bmatrix} z^{-2} & z^{-1}(1-2z^{-1}) \\ 0 & z^{-1}(1-2z^{-1}) \end{bmatrix} X + Y(1-z^{-1}) = \begin{bmatrix} 2z^{-1}-2 \\ z^{-1}-2 \end{bmatrix}.$$

We write

$$\begin{bmatrix} z^{-2} & z^{-1}(1-2z^{-1}) \\ 0 & z^{-1}(1-2z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1-2z^{-1} & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-2}(1-2z^{-1}) \end{bmatrix} \begin{bmatrix} z^{-1} & 1-2z^{-1} \\ -1 & 2 \end{bmatrix}$$

and hence equation (3.36) is equivalent to the set of polynomial equations

$$z^{-1}\bar{x}_1 + \bar{y}_1(1-z^{-1}) = 2z^{-1} - 2\,,$$

$$z^{-2}(1-2z^{-1})\,\bar{x}_2 + \bar{y}_2(1-z^{-1}) = -5z^{-1} + 4z^{-2}$$

by Theorem 1.1 and

$$X = \begin{bmatrix} z^{-1} & 1-2z^{-1} \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 \\ 1-2z^{-1} & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}.$$

The general solution can be written as

$$\bar{x}_1 = 0 + (1-z^{-1})\,t_1\,, \quad \bar{y}_1 = -2 - z^{-1}t_1\,,$$

$$\bar{x}_2 = 1 + (1-z^{-1})\,t_2\,, \quad \bar{y}_2 = -5z^{-1} - 2z^{-2} - z^{-2}(1-2z^{-1})\,t_2$$

and

$$X = \begin{bmatrix} -1+2z^{-1} \\ z^{-1} \end{bmatrix} + \begin{bmatrix} 2 & -1+2z^{-1} \\ 1 & z^{-1} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1-z^{-1})\,,$$

$$Y = \begin{bmatrix} -2 \\ -2-z^{-1}-2z^{-2} \end{bmatrix} - \begin{bmatrix} z^{-1} & 0 \\ z^{-1}(1-2z^{-1}) & z^{-2}(1-2z^{-1}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

by (1.19). The particular solution X^0, Y^0 for which $\partial Y^0 < 2$ is evidently obtained as

$$X^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} -2 - z^{-1} \\ -2 - 2z^{-1} \end{bmatrix}$$

on setting $t_1 = 1$, $t_2 = 0$.

Now we compute

$$U_1 = \begin{bmatrix} 1 & 0 \\ z^{-1}-2 & z^{-1}-2 \end{bmatrix}^{-1} \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1-z^{-1}}\,;$$

Hence

$$U = \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} U_1 = \frac{\begin{bmatrix} z^{-1} - 2 \\ 3 - z^{-1} \end{bmatrix}}{z^{-1} - 2}$$

and, by (3.25),

$$E = \begin{bmatrix} z^{-1} & z^{-1} - 2 \\ 0 & z^{-1} - 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 - z^{-1} \\ -2 - 2z^{-1} \end{bmatrix} = \frac{\begin{bmatrix} z^{-1} - 2 \\ -2z^{-1} - 2 \end{bmatrix}}{z^{-1} - 2}.$$

Since both U and E are stable, the U qualifies as the optimal control and

$$\|E\|^2_{\min} = 1 + 4 = 5 .$$

For effective computation of $\|E\|^2_{\min}$ see Example 2.16.

Example 3.15. Consider again the system

$$S = \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} = \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} [1]^{-1}$$

over $\mathfrak{R}$, the reference sequence

$$W = \frac{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -1 \end{bmatrix}}{z^{-1} - 2},$$

and solve problem (3.3).

We compute factorization (3.19)

$$B_1 = \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} [1]$$

and

$$B_{11}^{-} = \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix}, \quad B_{11}^{-\sim\prime} = [z^{-1} \quad \sqrt{2}\,(z^{-1} - 1)] ,$$

$$H = z^{-1} - 2 , \quad H^{\sim\prime} = 1 - 2z^{-1} , \quad d = 1 .$$

Then we are to solve the equation

$$z^{-1}(1 - 2z^{-1}) X + Y(z^{-1} - 2) = \sqrt{2} - \frac{1}{\sqrt{2}} z^{-1} ,$$

obtaining

$$X = 0 + (z^{-1} - 2)\, t ,$$
$$Y = -\frac{1}{\sqrt{2}} - z^{-1}(1 - 2z^{-1})\, t$$

for any $t \in \mathfrak{R}[z^{-1}]$. The particular solution X^0, Y^0 satisfying $\partial Y^0 < 2$ is obtained for $t = 0$ as

$$X^0 = 0\,, \quad Y^0 = -\frac{1}{\sqrt{2}}\,.$$

Then the optimal control

$$U = 0$$

yields the error

$$E = \frac{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -1 \end{bmatrix}}{z^{-1} - 2}$$

and

$$\|E\|^2_{\min} = \tfrac{1}{6} + \tfrac{1}{3} = \tfrac{1}{2}\,.$$

It is to be noted that problem (3.1) and problem (3.3) may have different solutions, even if the system enjoys the "minimum-phase" property. Compare the above result with Example 3.3.

Example 3.16. Given a realization of

$$S = \frac{[1 - z^{-1} \quad 1 - z^{-1}]}{z^{-1} - 2} = [1 - z^{-1} \quad 0] \begin{bmatrix} z^{-1} - 2 & -1 \\ 0 & 1 \end{bmatrix}^{-1}$$

over the field $\mathfrak{R}$, solve problem (3.3) for the reference sequence

$$W = \frac{1}{z^{-1} - 2}\,.$$

We compute decomposition (3.19)

$$B_1 = [1 - z^{-1} \quad 0] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and hence

$$B_{11}^- = 1 - z^{-1}\,, \quad B_{11}^{-\sim\prime} = z^{-1} - 1\,,$$

$$H = 1 - z^{-1}\,, \quad H^{\sim\prime} = z^{-1} - 1\,, \quad d = 0\,.$$

The equation

$$(z^{-1} - 1)\,X + Y(z^{-1} - 2) = z^{-1} - 1$$

has the general solution

$$X = 1 + (z^{-1} - 2)\,t\,,$$

$$Y = 0 - (z^{-1} - 1)\,t$$

for arbitrary $t \in \Re[z^{-1}]$ and the solution X^0, Y^0 satisfying $\partial Y^0 < 1$ becomes

$$X^0 = 1\,, \quad Y^0 = 0$$

on setting $t = 0$.

Thus

$$U_1 = \frac{1}{(1 - z^{-1})(z^{-1} - 2)}$$

and

$$U = \begin{bmatrix} z^{-1} - 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1 - z^{-1}} - U_2 \\ U_2 \end{bmatrix},$$

where $U_2 \in \Re^+\{z^{-1}\}$ arbitrary, is the only candidate for optimal control. It yields the best possible error

$$E = 0\,, \quad \|E\|^2_{\min} = 0\,,$$

but it is not stable. Therefore, the problem has no solution in the sense of our definition.

Example 3.17. Consider a realization of the transfer function

$$S = \frac{\begin{bmatrix} \sqrt{2}\, z^{-1}(1 - z^{-1}) \\ z^{-1} \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} \sqrt{2}\, z^{-1}(1 - z^{-1}) \\ z^{-1} \end{bmatrix} [1 - z^{-1}]^{-1}$$

over $\Re$ and solve problem (3.3) for the reference sequence

$$W = \frac{\begin{bmatrix} \sqrt{2}\, z^{-1} \\ 1 \end{bmatrix}}{1 - z^{-1}}\,.$$

Since

$$B_1 = \begin{bmatrix} \sqrt{2}\, z^{-1}(1 - z^{-1}) \\ z^{-1} \end{bmatrix} [1]\,,$$

we have

$$B_{11}^- = \begin{bmatrix} \sqrt{2}\, z^{-1}(1 - z^{-1}) \\ z^{-1} \end{bmatrix}, \quad B_{11}^{-\sim\prime} = [\sqrt{2}\,(z^{-1} - 1) \quad z^{-1}]\,,$$

$$H = z^{-1} - 2\,, \quad H^{\sim\prime} = 1 - 2z^{-1}\,, \quad d = 1$$

and the equation

$$z^{-1}(1 - 2z^{-1})X + Y(1 - z^{-1}) = -z^{-1}(1 - 2z^{-1})$$

is to be solved.

Its general solution reads

$$X = -1 + (1 - z^{-1})\,t\,,$$
$$Y = \quad 0 - z^{-1}(1 - 2z^{-1})\,t$$

for any $t \in \Re[z^{-1}]$ and the solution X^0, Y^0 with $\partial Y^0 < 2$ becomes

$$X^0 = -1\,, \quad Y^0 = 0$$

when setting $t = 0$.

Then

$$U = (1 - z^{-1})\frac{-1}{(z^{-1} - 2)(1 - z^{-1})} = -\frac{1}{z^{-1} - 2}\,.$$

Even though the U is stable, it does not represent the optimal control because the resulting error

$$E = W - B_{11}^{-} U_1 = \frac{\begin{bmatrix} -\sqrt{2}\, z^{-1} \\ -2(1 - z^{-1}) \end{bmatrix}}{(1 - z^{-1})(z^{-1} - 2)}$$

is not stable. Hence, there is no solution.

This example has illustrated that it is not rigorous to end up when computing U. We have to check the error, too. If the resulting error is not stable, its quadratic norm will not be finite contradicting our hypothesis.

Example 3.18. Given again a realization of

$$S = \frac{\begin{bmatrix} z^{-1} & z^{-1} - z^{-3} \\ z^{-3} & z^{-3} \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} & 0 \\ z^{-3} & z^{-5} \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1})(1 - z^{-2}) \\ 0 & 1 - z^{-1} \end{bmatrix}^{-1}$$

over $\Re$ and the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1 - z^{-1}}\,,$$

solve problem (3.3).

We first compute

$$B_{11}^{-} = \begin{bmatrix} z^{-1} & 0 \\ z^{-3} & z^{-5} \end{bmatrix}, \qquad B_{11}^{-\sim\prime} = \begin{bmatrix} z^{-4} & z^{-2} \\ 0 & 1 \end{bmatrix},$$

$$H = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} z^{-2} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad H^{\sim\prime} = \begin{bmatrix} \sqrt{2}\, z^{-2} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} z^{-2} \end{bmatrix}, \quad d = 3\,.$$

Then equation (3.22) can be written as

$$(3.37)\qquad \begin{bmatrix} \sqrt{2}\, z^{-5} & 0 \\ \dfrac{1}{\sqrt{2}} z^{-3} & \dfrac{1}{\sqrt{2}} z^{-5} \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} z^{-4} \\ 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} \sqrt{2}\, z^{-5} & 0 \\ \dfrac{1}{\sqrt{2}} z^{-3} & \dfrac{1}{\sqrt{2}} z^{-5} \end{bmatrix} = \begin{bmatrix} 2z^{-2} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{1}{\sqrt{2}} z^{-3} & 0 \\ 0 & -\sqrt{2}\, z^{-7} \end{bmatrix} \begin{bmatrix} 1 & z^{-2} \\ 0 & 1 \end{bmatrix},$$

equation (3.37) reduces to the set of polynomial equations

$$\frac{1}{\sqrt{2}} z^{-3} \bar{x}_1 + \bar{y}_1(1 - z^{-1}) = 0,$$

$$-\sqrt{2}\, z^{-7} \bar{x}_2 + \bar{y}_2(1 - z^{-1}) = z^{-4}$$

and

$$X = \begin{bmatrix} 1 & z^{-2} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad Y = \begin{bmatrix} 2z^{-2} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}.$$

The general solution is

$$\bar{x}_1 = 0 + (1 - z^{-1})\, t_1, \qquad \bar{y}_1 = 0 - \frac{1}{\sqrt{2}} z^{-3} t_1,$$

$$\bar{x}_2 = -\frac{1}{\sqrt{2}} + (1 - z^{-1})\, t_2, \quad \bar{y}_2 = z^{-4} + z^{-5} + z^{-6} - \sqrt{2}\, z^{-7} t_2$$

and, by (1.19),

$$X = \begin{bmatrix} \dfrac{1}{\sqrt{2}} z^{-2} \\ -\dfrac{1}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} 1 & -z^{-2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1 - z^{-1}),$$

$$Y = \begin{bmatrix} z^{-4} + z^{-5} + z^{-6} \\ 0 \end{bmatrix} - \begin{bmatrix} \sqrt{2}\, z^{-5} & 0 \\ \dfrac{1}{\sqrt{2}} z^{-3} & \dfrac{1}{\sqrt{2}} z^{-5} \end{bmatrix} \begin{bmatrix} 1 & -z^{-2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}.$$

The solution X^0, Y^0 satisfying $\partial Y^0 < 5$ is

$$X^0 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} \end{bmatrix}, \quad Y^0 = \begin{bmatrix} z^{-4} \\ -\frac{1}{2} z^{-3} - \frac{1}{2} z^{-4} \end{bmatrix}$$

when setting

$$t_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-1}, \quad t_2 = 0.$$

Then

$$U = \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1})(1 - z^{-2}) \\ 0 & 1 - z^{-1} \end{bmatrix} \frac{\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} z^{-2} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}}{1 - z^{-1}} =$$

$$= \begin{bmatrix} 1{\cdot}5 - 0{\cdot}5z^{-2} \\ -1 \end{bmatrix}$$

is the optimal control and it yields the error

$$E = \begin{bmatrix} z^{-4} & z^{-2} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} z^{-4} \\ -\frac{1}{2}z^{-3} - \frac{1}{2}z^{-4} \end{bmatrix} = \begin{bmatrix} 1 + 0{\cdot}5z^{-1} + 0{\cdot}5z^{-2} \\ -0{\cdot}5z^{-3} - 0{\cdot}5z^{-4} \end{bmatrix}.$$

Apparently,

$$\|E\|^2_{\min} = 1{\cdot}5 + 0{\cdot}5 = 2.$$

Example 3.19. Consider the system

$$S = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix}$$

over $\mathfrak{R}$, the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ 1 - 0{\cdot}5z^{-1} \end{bmatrix}}{1 - 0{\cdot}5z^{-1}}$$

and find a solution to problem (3.3).

It is easy to see that

$$B_{11}^- = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix}, \quad B_{11}^{-\sim\prime} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} - 2 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} - 2 \end{bmatrix}, \quad H^{\sim\prime} = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 - 2z^{-1} \end{bmatrix}, \quad d = 1$$

and hence the equation

$$\begin{bmatrix} z^{-2} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} X + Y(1 - 0{\cdot}5z^{-1}) = \begin{bmatrix} z^{-1} \\ (z^{-1} - 2)(1 - 0{\cdot}5z^{-1}) \end{bmatrix}$$

yields the general solution

$$X = \begin{bmatrix} 0{\cdot}5 \\ 0 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1 - 0{\cdot}5z^{-1}),$$

$$Y = \begin{bmatrix} z^{-1} \\ z^{-1} - 2 \end{bmatrix} - \begin{bmatrix} z^{-2} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

for any $t_1, t_2 \in \Re[z^{-1}]$. The solution X^0, Y^0 with $\partial Y^0 < 2$ is

$$X^0 = \begin{bmatrix} 0{\cdot}5 \\ 0 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} z^{-1} \\ z^{-1} - 2 \end{bmatrix}$$

on setting $t_1 = 0$, $t_2 = 0$.

The optimal control

$$U = \frac{\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} - 2^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 0{\cdot}5 \\ 0 \end{bmatrix}}{1 - 0{\cdot}5\, z^{-1}} = \frac{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}{z^{-1} - 2}$$

generates the error

$$E = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} - 2 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} \\ z^{-1} - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\|E\|^2_{\min} = 1 + 1 = 2\,.$$

Note that this optimal control is also the optimal control for problem (3.1), even though the system does not have the "minimum-phase" property.

Example 3.20. This example illustrates that the condition $\partial Y^0 < \partial z^{-d} H^{\sim\prime}$ may not yield a unique solution to (3.22) in which case the stability considerations for U are important.

Given a realization of

$$S = \frac{\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1})^2 \end{bmatrix}}{1 - 2z^{-1}} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} \begin{bmatrix} 1 - 2z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

over the filed $\Re$, solve problem (3.3) for the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ (1 - 2z^{-1})^2 \end{bmatrix}}{1 - 2z^{-1}}.$$

We compute

$$B_{11}^{-} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix}, \quad B_{11}^{-\sim\prime} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} - 2 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} - 2 \end{bmatrix}, \quad H^{\sim\prime} = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 - 2z^{-1} \end{bmatrix}, \quad d = 1$$

and solve the equation

$$\begin{bmatrix} z^{-2} & 0 \\ 0 & z^{-1}(1-2z^{-1}) \end{bmatrix} X + Y(1-2z^{-1}) = \begin{bmatrix} z^{-1} \\ (z^{-1}-2)(1-2z^{-1})^2 \end{bmatrix}.$$

Evidently, the general solution becomes

$$X = \begin{bmatrix} 2 \\ 5-2z^{-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1-2z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1-2z^{-1}),$$

$$Y = \begin{bmatrix} z^{-1} \\ -2 \end{bmatrix} - \begin{bmatrix} z^{-2} & 0 \\ 0 & z^{-1}(1-2z^{-1}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-2z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

for arbitrary $t_1, t_2 \in \Re[z^{-1}]$.

Now the particular solution X^0, Y^0 such that $\partial Y^0 < 2$ obtains as

$$X^0 = \begin{bmatrix} 2 \\ (5+\tau_0)-2z^{-1} \end{bmatrix}, \quad Y^0 = \begin{bmatrix} z^{-1} \\ -2-\tau_0 z^{-1} \end{bmatrix}$$

on setting $t_1 = 0$, $t_2 = \tau_0 \in \Re$ arbitrary. Computing

$$U = \begin{bmatrix} 1-2z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} 1 & 0 \\ 0 & z^{-1}-2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ (5+\tau_0)-2z^{-1} \end{bmatrix}}{1-2z^{-1}} =$$

$$= \begin{bmatrix} 2 \\ \dfrac{(5+\tau_0)-2z^{-1}}{(z^{-1}-2)(1-2z^{-1})} \end{bmatrix},$$

it is seen that the U will be stable if and only if $\tau_0 = -4$. Then

$$U = \frac{\begin{bmatrix} 2(z^{-1}-2) \\ 1 \end{bmatrix}}{z^{-1}-2}$$

is the optimal control and

$$E = \begin{bmatrix} 1 \\ -2\dfrac{1-2z^{-1}}{z^{-1}-2} \end{bmatrix}$$

is the corresponding error. It follows that

$$\|E\|^2_{\min} = 1 + 4 = 5.$$

Example 3.21. This example illustrates the importance of the ground field $\mathfrak{F}$. Consider the system given by

$$S = \frac{\begin{bmatrix} z^{-1} & z^{-1} \\ 0 & z^{-1}(1 - 2z^{-1} - z^{-2}) \end{bmatrix}}{1 - z^{-1}} =$$

$$= \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1} - z^{-2}) \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1}) \\ 0 & 1 - z^{-1} \end{bmatrix}^{-1},$$

the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1 - z^{-1}}$$

and solve problem (3.3).

If the system is viewed over the field $\mathfrak{Q}$, we compute

$$B_1^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_{11}^- = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1} - z^{-2}) \end{bmatrix}, \quad B_{11}^{-\sim\prime} = \begin{bmatrix} z^{-2} & 0 \\ 0 & -1 - 2z^{-1} + z^{-2} \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 - 2z^{-1} + z^{-2} \end{bmatrix}, \quad H^{\sim\prime} = \begin{bmatrix} z^{-2} & 0 \\ 0 & 1 - 2z^{-1} - z^{-2} \end{bmatrix}, \quad d = 1$$

and equation (3.22) reads

$$\begin{bmatrix} z^{-3} & 0 \\ 0 & z^{-1}(1 - 2z^{-1} - z^{-2}) \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} z^{-2} \\ -1 - 2z^{-1} + z^{-2} \end{bmatrix}.$$

Evidently,

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1 - z^{-1}),$$

$$Y = \begin{bmatrix} z^{-2} \\ -1 - 4z^{-1} - z^{-2} \end{bmatrix} - \begin{bmatrix} z^{-3} & 0 \\ 0 & z^{-1}(1 - 2z^{-1} + 3z^{-2}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

and

$$X^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} z^{-2} \\ -1 - 4z^{-1} - z^{-2} \end{bmatrix}.$$

The only candidate for optimal control is

$$U = \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1}) \\ 0 & 1 - z^{-1} \end{bmatrix} \frac{\begin{bmatrix} 1 & 0 \\ 0 & -1 - 2z^{-1} + z^{-2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1 - z^{-1}} =$$

$$= \frac{\begin{bmatrix} -2 - 2z^{-1} + z^{-2} \\ 1 \end{bmatrix}}{-1 - 2z^{-1} + z^{-2}}$$

and it is not stable. Hence problem (3.3) has no solution.

Now view the system over the field $\mathfrak{R}$. Then

$$B_1^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 - (1 - \sqrt{2})\, z^{-1} \end{bmatrix}, \quad B_{11}^- = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - (1 + \sqrt{2})\, z^{-1}) \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 - (1 - \sqrt{2})\, z^{-1} \end{bmatrix}, \quad B_{11}^{-\sim\prime} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} - (1 + \sqrt{2}) \end{bmatrix}$$

and equation (3.22) reads

$$\begin{bmatrix} z^{-2} & 0 \\ 0 & z^{-1}(1 - (1 + \sqrt{2})\, z^{-1}) \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} z^{-1} \\ z^{-1} - (1 + \sqrt{2}) \end{bmatrix}.$$

Evidently,

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1 - z^{-1}),$$

$$Y = \begin{bmatrix} z^{-1} \\ -(1 + \sqrt{2}) - (1 + \sqrt{2})\, z^{-1} \end{bmatrix} - \begin{bmatrix} z^{-2} & 0 \\ 0 & z^{-1}(1 - (1 + \sqrt{2})\, z^{-1} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

and the solution

$$X^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} z^{-1} \\ -(1 + \sqrt{2}) - (1 + \sqrt{2})\, z^{-1} \end{bmatrix}$$

satisfies $\partial Y^0 < 2$.

Then the optimal control

$$U = \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1}) \\ 0 & 1 - z^{-1} \end{bmatrix} \cdot \tag{3.38}$$

$$\cdot \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 - (1 - \sqrt{2})\, z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} - (1 + \sqrt{2}) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1 - z^{-1}} =$$

$$= \frac{\begin{bmatrix} -(2+\sqrt{2}) - (1-\sqrt{2})\, z^{-2} \\ 1 \end{bmatrix}}{(1-(1-\sqrt{2})\, z^{-1})(z^{-1} - (1+\sqrt{2}))}$$

yields the error

$$E = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} - (1+\sqrt{2}) \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} \\ -(1+\sqrt{2}) - (1+\sqrt{2})\, z^{-1} \end{bmatrix} =$$

$$= \frac{\begin{bmatrix} z^{-1} - (1+\sqrt{2}) \\ -(1+\sqrt{2}) - (1+\sqrt{2})\, z^{-1} \end{bmatrix}}{z^{-1} - (1+\sqrt{2})} .$$

and

(3.39) $$\|E\|^2_{\min} = 1 + \frac{3+\sqrt{2}}{(1+\sqrt{2})^2\,(2+\sqrt{2})} = \frac{13 + 8\sqrt{2}}{10 + 7\sqrt{2}} .$$

Therefore, a larger field may guarantee the existence of the optimal control. Since the reals are the topological closure of the rationals, optimal control (3.38) is the limit of all rational approximations and norm (3.39) is the infimum of the corresponding rational norms.

To illustrate advantages of the present approach over the classical method of Wiener, we shall demonstrate that the latter does not work for unstable systems. Recall [60] the classical formula for the optimal control

$$U = (S^*)^{-1}\, [(S^{*=\prime})^{-1}\, S^{=\prime} W]_+ ,$$

where S^* is the minimum-phase spectral factor of the system transfer function matrix S, i.e. $S^{*=\prime} S^* = S^{=\prime} S$, and $[(S^{*=\prime})^{-1}\, S^{=\prime} W]_+$ represents the partial fraction expansion of the $(S^{*=\prime})^{-1}\, S^{=\prime} W$ with unstable fractions deleted.

Example 3.22. Consider

$$S = \begin{bmatrix} \dfrac{z^{-1}}{1 - 2z^{-1}} & 0 \\ 0 & z^{-1} \end{bmatrix}, \quad W = \begin{bmatrix} \dfrac{1}{z^{-1} - 2} \\ \dfrac{1}{z^{-1} - 2} \end{bmatrix}$$

over the field $\Re$. Then

$$S^* = \begin{bmatrix} \dfrac{1}{z^{-1} - 2} & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[(S^{*=\prime})^{-1}S^{=\prime}W]_+ = \begin{bmatrix} \dfrac{(z-2)z}{(1-2z)(z^{-1}-2)} \\ \dfrac{z}{z^{-1}-2} \end{bmatrix}_+ = \begin{bmatrix} \dfrac{1-2z^{-1}}{z^{-1}(z^{-1}-2)^2} \\ \dfrac{1}{z^{-1}(z^{-1}-2)} \end{bmatrix}_+ =$$

$$= \begin{bmatrix} -\dfrac{1+0{\cdot}25z^{-1}}{(z^{-1}-2)^2} + \dfrac{0{\cdot}25}{z^{-1}} \\ \dfrac{0{\cdot}5}{z^{-1}-2} - \dfrac{0{\cdot}5}{z^{-1}} \end{bmatrix}_+ = \begin{bmatrix} -\dfrac{1+0{\cdot}25z^{-1}}{(z^{-1}-2)^2} \\ \dfrac{0{\cdot}5}{z^{-1}-2} \end{bmatrix}.$$

Therefore,

$$U = \begin{bmatrix} \dfrac{1}{z^{-1}-2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\dfrac{1+0{\cdot}25z^{-1}}{(z^{-1}-2)^2} \\ \dfrac{0{\cdot}5}{z^{-1}-2} \end{bmatrix} = \begin{bmatrix} -\dfrac{1+0{\cdot}25z^{-1}}{z^{-1}-2} \\ \dfrac{0{\cdot}5}{z^{-1}-2} \end{bmatrix}$$

and

$$E = \begin{bmatrix} \dfrac{1-z^{-1}+0{\cdot}25z^{-2}}{(z^{-1}-2)(1-2z^{-1})} \\ -0{\cdot}5 \end{bmatrix}, \quad \|E\|^2 \to \infty$$

but this is *not* the optimal control.

The method presented in this paper gives us

$$S = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1-2z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1}, \quad W = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{z^{-1}-2}$$

and the equation

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} X + Y(z^{-1}-2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

has the solution

$$X^0 = \begin{bmatrix} 0{\cdot}5 \\ 0{\cdot}5 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} -0{\cdot}5 \\ -0{\cdot}5 \end{bmatrix}.$$

Thus the optimal control is

$$U = \begin{bmatrix} 1-2z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} 0{\cdot}5 \\ 0{\cdot}5 \end{bmatrix}}{z^{-1}-2} = \frac{\begin{bmatrix} 0{\cdot}5 - z^{-1} \\ 0{\cdot}5 \end{bmatrix}}{z^{-1}-2}$$

and the resulting error

$$E = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \quad \|E\|^2_{\min} = 0.5 .$$

Note that the burdensome computations associated with the partial fractioning are elegantly avoided by solving a Diophantine equation.

Up to now we have confined ourselves to systems defined over a field $\mathfrak{F}$ which is a subfield of $\mathfrak{C}$ valuated by (2.25). If the system is defined over another field, the quadratic norm of E cannot be written as $\|E\|^2 = \langle E^{=\prime}E \rangle$ and Theorem 3.3 does not apply. It is necessary to develop a special procedure depending upon the valuation in $\mathfrak{F}$.

For example, let $\mathfrak{F}$ be an arbitrary field with the trivial valuation (2.24). Then the quadratic norm of an error sequence

$$E = \begin{bmatrix} \varepsilon_{10} + \varepsilon_{11}z^{-1} + \dots \\ \varepsilon_{20} + \varepsilon_{21}z^{-1} + \dots \\ \dots\dots\dots\dots\dots\dots \\ \varepsilon_{l0} + \varepsilon_{l1}z^{-1} + \dots \end{bmatrix} \in \mathfrak{F}_{l,1}\{z^{-1}\}$$

is defined as

$$\|E\|^2 = \sum_{i=1}^{l} \sum_{k=0}^{\infty} \mathscr{V}^2(\varepsilon_{ik}),$$

see (2.27), and it can be interpreted as the number of nonzero elements ε_{ik} in the error sequence.

A careful examination shows that no polynomial in $\mathfrak{F}[z^{-1}]$ is stable with respect to (2.14) save the units of $\mathfrak{F}[z^{-1}]$. Thus a sequence in $\mathfrak{F}_{l,1}\{z^{-1}\}$ is stable if and only if it is finite. Therefore, the least squares control problem (3.3) reduces to solving equation (3.12), whose general solution X, Y determines all finite control sequences that yield a finite error sequence, and then finding a solution X^1, Y^1 minimizing the number of nonzero elements in the error sequence.

Example 3.23. Consider a simple system over the field $\mathfrak{Z}_2$ (with valuation (2.24), of course) given by

$$S = \frac{1 + z^{-1} + z^{-2}}{1 + z^{-1}}$$

and solve problem (3.3) for the reference sequence

$$W = z^{-2} .$$

Equation (3.12) becomes

$$(1 + z^{-1} + z^{-2})X + Y = z^{-2}$$

and it has the general solution

$$(3.40) \qquad X = 1 + t\,, \quad Y = 1 + z^{-1} + (1 + z^{-1} + z^{-2})\, t$$

for arbitrary $t \in \mathfrak{Z}_2[z^{-1}]$.

Thus the controls

$$U = (1 + z^{-1})(1 + t)$$

yield the errors

$$E = 1 + z^{-1} + (1 + z^{-1} + z^{-2})\, t\,.$$

Setting

$$t = \tau_0 + \tau_1 z^{-1} + \ldots + \tau_n z^{-n}$$

for some n, we obtain

$$E = (1 + \tau_0) + (1 + \tau_0 + \tau_1)\, z^{-1} + (\tau_0 + \tau_1 + \tau_2)\, z^{-2} + \ldots$$
$$\ldots + (\tau_{n-2} + \tau_{n-1} + \tau_n)\, z^{-n} + (\tau_{n-1} + \tau_n)\, z^{-(n+1)} + \tau_n z^{-(n+2)}$$

and it can be easily verified that the choice

$$\tau_0 = 1\,, \quad \tau_1 = \tau_2 = \ldots = \tau_n = 0\,,$$

i.e. $t = 1$, minimizes the number of nonzero elements in E.

Hence

$$X^1 = 0\,, \quad Y^1 = z^{-2}$$

and the optimal control

$$U = 0$$

gives the error

$$E = z^{-2}\,, \quad \|E\|^2_{\min} = 1\,.$$

It is to be noted that problem (3.2) and problem (3.3) yield, in general, different optimal controls. True, the same equation is solved, but it is solved for different solutions. In this example the finite time optimal control is obtained as

$$U = 1 + z^{-1}\,, \quad E = 1 + z^{-1}$$

on setting $t = 0$ in (3.40).

4. CLOSED-LOOP STABILITY

4.1. The closed-loop system

In this part we shall consider the closed-loop system shown in Fig. 5, which consists of a system $\mathscr{S}$ to be controlled and a controller $\mathscr{R}$. It is to be noted that this is not the most general feedback configuration, but it is reasonably general and widely used in practice and, therefore, it will be taken here to solve various control problems.

Throughout the chapter, the most important concept will be that of minimal realization. An interesting point is that the closed-loop system may not be a minimal realization of its impulse response even if the original components $\mathcal{S}$ and $\mathcal{R}$ are. As a matter of fact, we shall see later that the optimum system synthesis calls for certain procedures, called the „zero-pole" cancellations, which produce a nonminimally realized closed-loop system. As a result, we cannot infer dynamical properties and, in particular, stability of such a closed-loop system from its impulse response description.

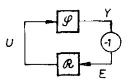

Fig. 5. The closed-loop systém.

We shall show that, besides stability, the impulse response of the closed-loop system must satisfy certain additional conditions to yield a stable closed-loop system. This fundamental result will be used in synthesizing optimal closed-loop control systems.

4.2. The characteristic and invariant polynomials

Consider the closed-loop system shown in Fig. 5, where $\mathcal{S}$ is a system defined over an arbitrary field $\mathfrak{F}$ valuated by $\mathcal{V}$ that is described by the equations

(4.1)
$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k ,$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k$$

and $\mathcal{R}$ is a system over $\mathfrak{F}$ defined by the equations

(4.2)
$$\mathbf{z}_{k+1} = \mathbf{F}\mathbf{z}_k + \mathbf{G}\mathbf{e}_k ,$$
$$\mathbf{u}_k = \mathbf{H}\mathbf{z}_k + \mathbf{J}\mathbf{e}_k .$$

Further, let

$$\mathbf{x} \in \mathfrak{F}^n , \quad \mathbf{z} \in \mathfrak{F}^p ,$$

and

$$\mathbf{u} \in \mathfrak{F}^m , \quad \mathbf{y} \in \mathfrak{F}^l , \quad \mathbf{e} \in \mathfrak{F}^l .$$

Since the closed-loop system must contain a delay of at least one time unit to be physically realizable, we shall agree on including the delay into the system $\mathscr{S}$ to be controlled. Therefore, any of the following equivalent conditions

$$\mathbf{D} = \mathbf{0}, \quad \partial \hat{B} < \partial \hat{a}, \quad z^{-1} \mid B \tag{4.3}$$

is assumed to hold for any system $\mathscr{S}$ considered henceforth.

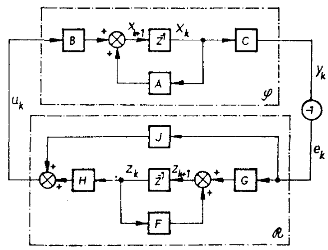

Fig. 6. A detail representation of the closed-loop.

A detail representation of the closed-loop system is given in Fig. 6. The state equation of the system shown therein becomes

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix},$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} - \mathbf{BJC} & \mathbf{BH} \\ -\mathbf{GC} & \mathbf{F} \end{bmatrix} \in \mathfrak{F}_{n+p,n+p}. \tag{4.4}$$

The characteristic polynomial of the closed-loop system is defined as

$$\hat{c} = \det (z\mathbf{I}_{n+p} - \mathbf{K}) \in \mathfrak{F}[z]$$

and it has the degree

$$\partial \hat{c} = n + p. \tag{4.5}$$

The invariant polynomials $\hat{c}_i$ of the closed-loop system are defined as the monic invariant polynomials of the matrix

$$z\mathbf{I}_{n+p} - \mathbf{K} \in \mathfrak{F}_{n+p,n+p}[z].$$

It is interesting that the invariant polynomials $\hat{c}_i$ can be obtained from the transfer function matrices of $\mathscr{S}$ and $\mathscr{R}$. To do so, we have to assume that $\mathscr{S}$ is a minimal realization of the impulse response matrix

$$S = \mathbf{C}(z\mathbf{I}_n - \mathbf{A})^{-1}\,\mathbf{B} \in \mathfrak{F}_{l,m}\{z^{-1}\} \tag{4.6}$$

and that $\mathscr{R}$ is a minimal realization of the impulse response matrix

$$R = \mathbf{H}(z\mathbf{I}_p - \mathbf{F})^{-1}\,\mathbf{G} + \mathbf{J} + \mathfrak{F}_{m,l}\{z^{-1}\}\,. \tag{4.7}$$

Using (2.4) we shall make the decompositions

$$S = \hat{B}_1\hat{A}_2^{-1} = \hat{A}_1^{-1}\hat{B}_2\,, \tag{4.8}$$

where $\hat{A}_1$ and $\hat{B}_2$ are left coprime while $\hat{B}_1$ and $\hat{A}_2$ are right coprime and

$$\det(z\mathbf{I}_n - \mathbf{A}) = \det \hat{A}_1 = \det \hat{A}_2 \tag{4.9}$$

modulo units of $\mathfrak{F}[z]$; also

$$R = \hat{S}_1\hat{R}_2^{-1} = \hat{R}_1^{-1}\hat{S}_2\,, \tag{4.10}$$

where $\hat{R}_1$ and $\hat{S}_2$ are left coprime while $\hat{S}_1$ and $\hat{R}_2$ are right coprime and

$$\det(z\mathbf{I}_p - \mathbf{F}) = \det \hat{R}_1 = \det \hat{R}_2 \tag{4.11}$$

modulo units of $\mathfrak{F}[z]$.

Then we have the following result.

Theorem 4.1. *Consider the closed-loop system shown in Fig. 5, where $\mathscr{S}$ and $\mathscr{R}$ are minimal realizations of*

$$S = \hat{B}_1\hat{A}_2^{-1} = \hat{A}_1^{-1}\hat{B}_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}$$

and

$$R = \hat{S}_1\hat{R}_2^{-1} = \hat{R}_1^{-1}\hat{S}_2 \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

respectively. Further denote

$$\hat{C}_1 = \hat{R}_1\hat{A}_2 + \hat{S}_2\hat{B}_1 \in \mathfrak{F}_{m,m}[z]\,, \tag{4.12}$$

$$\hat{C}_2 = \hat{A}_1\hat{R}_2 + \hat{B}_2\hat{S}_1 \in \mathfrak{F}_{l,l}[z]\,.$$

Then the characteristic polynomial $\hat{c}$ of the closed-loop system is given as

$$\hat{c} = \det \hat{C}_1 = \det \hat{C}_2$$

modulo a unit of $\mathfrak{F}[z]$.

Proof. We apply the well-known [12] formula

$$(4.13)\qquad \det\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det \mathbf{D}\ \det(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}) = \\ = \det \mathbf{A}\ \det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}),$$

where the indicated inverses are assumed to exist, to compute the characteristic polynomial

$$\hat{c} = \det(z\mathbf{I}_{n+p} - \mathbf{K}) = \\ = \det(z\mathbf{I}_p - \mathbf{F}) \det[z\mathbf{I}_n - \mathbf{A} + \mathbf{BJC} + \mathbf{BH}(z\mathbf{I}_p - \mathbf{F})^{-1}\,\mathbf{GC}] = \\ = \det(z\mathbf{I}_p - \mathbf{F}) \det(z\mathbf{I}_n - \mathbf{A} + \mathbf{B}\boldsymbol{R}\mathbf{C})$$

on using (4.13) and (4.7).

Now observe that

$$\det\begin{bmatrix} z\mathbf{I}_n - \mathbf{A} & \mathbf{B} \\ -\boldsymbol{R}\mathbf{C} & \boldsymbol{I}_m \end{bmatrix} = \det \boldsymbol{I}_m \det(z\mathbf{I}_n - \mathbf{A} + \mathbf{B}\boldsymbol{R}\mathbf{C}) = \\ = \det(z\mathbf{I}_n - \mathbf{A}) \det[\boldsymbol{I}_m + \boldsymbol{R}\mathbf{C}(z\mathbf{I}_n - \mathbf{A})^{-1}\,\mathbf{B}]$$

and

$$\det\begin{bmatrix} z\mathbf{I}_n - \mathbf{A} & -\mathbf{B}\boldsymbol{R} \\ \mathbf{C} & \boldsymbol{I}_l \end{bmatrix} = \det \boldsymbol{I}_l \det(z\mathbf{I}_n - \mathbf{A} + \mathbf{B}\boldsymbol{R}\mathbf{C}) = \\ = \det(z\mathbf{I}_n - \mathbf{A}) \det[\boldsymbol{I}_l + \mathbf{C}(z\mathbf{I}_n - \mathbf{A})^{-1}\,\mathbf{B}\boldsymbol{R}]$$

on using (4.13) and, hence,

$$\det(z\mathbf{I}_n - \mathbf{A} + \mathbf{B}\boldsymbol{R}\mathbf{C}) = \det(z\mathbf{I}_n - \mathbf{A}) \det(\boldsymbol{I}_l + \boldsymbol{SR}) = \\ = \det(z\mathbf{I}_n - \mathbf{A}) \det(\boldsymbol{I}_m + \boldsymbol{RS})$$

by virtue of (4.6).

Thus

$$(4.14)\qquad \hat{c} = \det(z\mathbf{I}_p - \mathbf{F}) \det(z\mathbf{I}_n - \mathbf{A}) \det(\boldsymbol{I}_l + \boldsymbol{SR}) = \\ = \det(z\mathbf{I}_p - \mathbf{F}) \det(z\mathbf{I}_n - \mathbf{A}) \det(\boldsymbol{I}_m + \boldsymbol{RS}).$$

Now

$$(4.15)\qquad \det(\boldsymbol{I}_l + \boldsymbol{SR}) = \det(I_l + \hat{A}_1^{-1}\hat{B}_2\hat{S}_1\hat{R}_2^{-1}) = \\ = \det[\hat{A}_1^{-1}(\hat{A}_1\hat{R}_2 + \hat{B}_2\hat{S}_1)\,\hat{R}_2^{-1}] = \\ = (\det \hat{A}_1)^{-1}\,(\det \hat{R}_2)^{-1} \det(\hat{A}_1\hat{R}_2 + \hat{B}_2\hat{S}_1)$$

and

$$(4.16)\qquad \det(\boldsymbol{I}_m + \boldsymbol{RS}) = \det(\boldsymbol{I}_m + \hat{R}_1^{-1}\hat{S}_2\hat{B}_1\hat{A}_2^{-1}) = \\ = \det[\hat{R}_1^{-1}(\hat{R}_1\hat{A}_2 + \hat{S}_2\hat{B}_1)\,\hat{A}_2^{-1}] = \\ = (\det \hat{R}_1)^{-1}(\det \hat{A}_2)^{-1}\det(\hat{R}_1\hat{A}_2 + \hat{S}_2\hat{B}_1)$$

by (4.8) and (4.10). Substituting (4.15) into (4.14) and taking (4.9) and (4.11) into account we obtain

$$\hat{c} = \det(\hat{A}_1\hat{R}_2 + \hat{B}_2\hat{S}_1)$$

modulo a unit of $\mathfrak{F}[z]$; substituting (4.16) into (4.14) and taking (4.9) and (4.11) into account we obtain

$$\hat{c} = \det(\hat{R}_1\hat{A}_2 + \hat{S}_2\hat{B}_1)$$

modulo a unit of $\mathfrak{F}[z]$. □

Note the importance of the assumption that both $\mathscr{S}$ and $\mathscr{R}$ be minimal realizations of $\boldsymbol{S}$ and $\boldsymbol{R}$, respectively. Otherwise (4.9) and/or (4.11) would not be valid and the final step in the proof above could not be taken.

We have created polynomial marices $\hat{C}_1$ and $\hat{C}_2$ whose determinants are essentially equal to the characteristic polynomial of the closed-loop system. In fact, much more is true. We shall prove below that the invariant polynomials of $\hat{C}_1$ and $\hat{C}_2$ are essentially equal to the invariant polynomials of the closed-loop system.

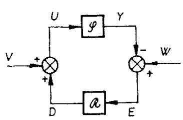

Fig. 7. The closed-loop system with external inputs.

To this effect we apply external signals $\boldsymbol{V}$ and $\boldsymbol{W}$ to the closed-loop system, see Fig. 7. Then all possible closed-loop impulse response matrices are listed below.

$$\begin{aligned} \boldsymbol{K}_{W/E} &= (\boldsymbol{I}_l + \boldsymbol{SR})^{-1}, & \boldsymbol{K}_{V/U} &= (\boldsymbol{I}_m + \boldsymbol{RS})^{-1}, \\ \boldsymbol{K}_{W/U} &= \boldsymbol{R}(\boldsymbol{I}_l + \boldsymbol{SR})^{-1}, & \boldsymbol{K}_{V/Y} &= \boldsymbol{S}(\boldsymbol{I}_m + \boldsymbol{RS})^{-1}, \\ \boldsymbol{K}_{W/Y} &= \boldsymbol{SR}(\boldsymbol{I}_l + \boldsymbol{SR})^{-1}, & \boldsymbol{K}_{V/D} &= -\boldsymbol{RS}(\boldsymbol{I}_m + \boldsymbol{RS})^{-1}. \end{aligned}$$

Note the identities

$$(4.17)\qquad \boldsymbol{R}(\boldsymbol{I}_l + \boldsymbol{SR})^{-1} = (\boldsymbol{I}_m + \boldsymbol{RS})^{-1}\boldsymbol{R},$$

$$(4.18)\qquad (\boldsymbol{I}_l + \boldsymbol{SR})^{-1}\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{I}_m + \boldsymbol{RS})^{-1},$$

which can be directly verified. Then using the decompositions (4.8) and (4.10) we can write

$$(4.19) \qquad K_{W/E} = (I_l + \hat{A}_1^{-1}\hat{B}_2\hat{S}_1\hat{R}_2^{-1})^{-1} =$$
$$= \hat{R}_2(\hat{A}_1\hat{R}_2 + \hat{B}_2\hat{S}_1)^{-1}\,\hat{A}_1 =$$
$$= \hat{R}_2\hat{C}_2^{-1}\hat{A}_1\;;$$
$$K_{W/U} = \hat{S}_1\hat{R}_2^{-1}(I_l + \hat{A}_1^{-1}\hat{B}_2 S_1\hat{R}_2^{-1})^{-1} =$$
$$= \hat{S}_1\hat{C}_2^{-1}\hat{A}_1$$

or by virtue of (4.17)

$$K_{W/U} = (I_m + \boldsymbol{RS})^{-1}\,\boldsymbol{R} =$$
$$= (I_{m'} + \boldsymbol{R}_1^{-1}\hat{S}_2\hat{B}_1\hat{A}_2^{-1})^{-1}\,\hat{R}_1^{-1}\hat{S}_2 =$$
$$= \hat{A}_2(\hat{R}_1\hat{A}_2 + \hat{S}_2\hat{B}_1)^{-1}\,\hat{S}_2 =$$
$$= \hat{A}_2\hat{C}_1^{-1}\hat{S}_2\;;$$
$$K_{W/Y} = \boldsymbol{S}(I_m + \boldsymbol{RS})^{-1}\,\boldsymbol{R} =$$
$$= \hat{B}_1\hat{A}_2^{-1}(I_m + \hat{R}_1^{-1}\hat{S}_2\hat{B}_1\hat{A}_2^{-1})^{-1}\,\hat{R}_1^{-1}\hat{S}_2 =$$
$$= \hat{B}_1(\hat{R}_1\hat{A}_2 + \hat{S}_2\hat{B}_1)^{-1}\,\hat{S}_2 =$$
$$= \hat{B}_1\hat{C}_1^{-1}\hat{S}_2\;;$$
$$K_{V/U} = (I_m + \hat{R}_1^{-1}\hat{S}_2\hat{B}_1\hat{A}_2^{-1})^{-1} =$$
$$= \hat{A}_2(\hat{R}_1\hat{A}_2 + \hat{S}_2\hat{B}_1)^{-1}\,\hat{R}_1 =$$
$$= \hat{A}_2\hat{C}_1^{-1}\hat{R}_1\;;$$
$$K_{V/Y} = \hat{B}_1\hat{A}_2^{-1}(I_m + \hat{R}_1^{-1}\hat{S}_2\hat{B}_1\hat{A}_2^{-1})^{-1} =$$
$$= \hat{B}_1\hat{C}_1^{-1}\hat{R}_1$$

or by virtue of (4.18)

$$K_{V/Y} = (I_l + \boldsymbol{S}\boldsymbol{R}^{-1})\,\boldsymbol{S} =$$
$$= (I_l + \hat{A}_1^{-1}\hat{B}_2\hat{S}_1\hat{R}_2^{-1})^{-1}\,\hat{A}_1^{-1}\hat{B}_2 =$$
$$= \hat{R}_2(\hat{A}_1\hat{R}_2 + \hat{B}_2\hat{S}_1)^{-1}\,\hat{B}_2 =$$
$$= \hat{R}_2\hat{C}_2^{-1}\hat{B}_2\;;$$
$$K_{V/D} = -\boldsymbol{R}(I_l + \boldsymbol{SR})^{-1}\,\boldsymbol{S} =$$
$$= -\hat{S}_1\hat{R}_2^{-1}(I_l + \hat{A}_1^{-1}\hat{B}_2\hat{S}_1\hat{R}_2^{-1})^{-1}\,\hat{A}_1^{-1}B_2 =$$
$$= -\hat{S}_1(\hat{A}_1\hat{R}_2 + \hat{B}_2\hat{S}_1)^{-1}\,\hat{B}_2 =$$
$$= -\hat{S}_1\hat{C}_2^{-1}\hat{B}_2\,.$$

Theorem 4.2. *Consider the closed-loop system shown in Fig. 5, where* $\boldsymbol{S}$ *and* $\boldsymbol{R}$ *are minimal realizations of*

$$\boldsymbol{S} = \hat{B}_1\hat{A}_2^{-1} = \hat{A}_1^{-1}\hat{B}_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}$$

and

$$\boldsymbol{R} = \hat{S}_1\hat{R}_2^{-1} = \hat{R}_1^{-1}\hat{S}_2 \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

respectively. Further denote

$$\hat{C}_1 = \hat{R}\hat{A}_2 + \hat{S}_2\hat{B}_1 \in \mathfrak{F}_{m,m}[z] \, .$$

$$\hat{C}_2 = \hat{A}_1\hat{R}_2 + \hat{B}_2\hat{S}_1 \in \mathfrak{F}_{l,l}[z] \, .$$

Then the nonunit invariant polynomials of $\hat{C}_1$ *are equal to the nonunit invariant polynomials of* $\hat{C}_2$ *up to units of* $\mathfrak{F}[z]$ *and both are equal to the nonunit invariant polynomials of the matrix* $z\mathbf{I}_{n+p} - \mathbf{K}$, *again up to units of* $\mathfrak{F}[z]$.

Proof. First consider the following four impulse response matrices

$$\boldsymbol{K}_{W/Y} = \hat{B}_1\hat{C}_1^{-1}\hat{S}_2 \, ,$$

$$\boldsymbol{K}_{W/U} = \hat{A}_2\hat{C}_1^{-1}\hat{S}_2 \, ,$$

$$\boldsymbol{K}_{V/Y} = \hat{B}_1\hat{C}_1^{-1}\hat{R}_1 \, ,$$

$$\boldsymbol{K}_{V/U} = \hat{A}_2\hat{C}_1^{-1}\hat{R}_1 \, ,$$

and let $\hat{c}_{1i}$ denote the nonunit invariant polynomials of $\hat{C}_1$ and let p_i, q_i, s_i, t_i denote the nonunit invariant polynomials of $\boldsymbol{K}_{W/Y}$, $\boldsymbol{K}_{W/U}$, $\boldsymbol{K}_{V/Y}$, $\boldsymbol{K}_{V/U}$ respectively.

Then

$$p_i \mid \hat{c}_{1i}, \; q_i \mid \hat{c}_{1i}, \; s_i \mid \hat{c}_{1i}, \; t_i \mid \hat{c}_{1i}$$

and write

$$\hat{c}_{1i} = p_i p_{0i} \, ,$$

$$= q_i q_{0i} \, ,$$

$$= s_i s_{0i} \, ,$$

$$= t_i t_{0i} ,$$

where p_{0i}, q_{0i}, s_{0i}, t_{0i} are polynomials of $\mathfrak{F}[z]$ representing possible cancallations in the $\boldsymbol{K}_{W/Y}$, $\boldsymbol{K}_{W/U}$, $\boldsymbol{K}_{V/Y}$, $\boldsymbol{K}_V/_U$ respectively. Since, by definition, the matrices $\hat{R}_1$ and $\hat{S}_2$ are left coprime and the matrices $\hat{B}_1$ and $\hat{A}_2$ are right coprime, there can be no factor cancelled simultaneously in all four impulse response matrices, that is

$$(p_{0i}, q_{0i}, s_{0i}, t_{0i}) = 1 \, .$$

Otherwise speaking, the least common multiple of p_i, q_i, s_i and t_i is equal to $\hat{c}_{1i}$ up to a unit of $\mathfrak{F}[z]$.

Now have a look at Fig. 8, where a detailed representation of the system shown in Fig. 7 is given. It is seen that the system

(4.20)
$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + \begin{bmatrix} \mathbf{BJ} \\ \mathbf{G} \end{bmatrix} \mathbf{w}_k ,$$
$$\mathbf{y}_k = [\mathbf{C}\ 0] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + [0]\, \mathbf{w}_k ,$$

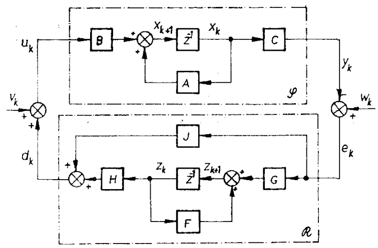

Fig. 8. A detail reprezentation of the closed-loop system with external inputs.

realizes $K_{W/Y}$; the system

(4.21)
$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + \begin{bmatrix} \mathbf{BJ} \\ \mathbf{G} \end{bmatrix} \mathbf{w}_k ,$$
$$\mathbf{u}_k = [-\mathbf{JC}\ \mathbf{H}] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + [\mathbf{J}]\, \mathbf{w}_k ,$$

realizes $K_{W/U}$; the system

(4.22)
$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathbf{v}_k ,$$
$$\mathbf{y}_k = [\mathbf{C}\ 0] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + [0]\, \mathbf{v}_k ,$$

realizes $K_{V/Y}$; and the system

(4.23)
$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathbf{v}_k ,$$
$$\mathbf{u}_k = [-\mathbf{JC}\ \mathbf{H}] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + [\mathbf{I}_m]\, \mathbf{v}_k ,$$

realizes $K_{V/U}$, where $\mathbf{K}$ is given in (4.4).

These realizations are not necessarily minimal but they all have the same state-transition matrix $\mathbf{K}$. Hence, denoting $\hat{k}_i$ the invariant polynomials of $z\mathbf{I}_{n+p} - \mathbf{K}$, we obtain

$$p_i \mid \hat{k}_i, \quad q_i \mid \hat{k}_i, \quad s_i \mid \hat{k}_i, \quad t_i \mid \hat{k}_i .$$

It follows that also $\hat{c}_{1i}$, the least common multiple of p_i, q_i, s_i and t_i divides $\hat{k}_i$. However, by Theorem 4.1,

$$\prod_i \hat{k}_i = \det(z\mathbf{I}_{n+p} - \mathbf{K}) = \det \hat{C}_1 = \prod_i \hat{c}_{1i}$$

up to a unit of $\mathfrak{F}[z]$ and hence $\hat{c}_{1i} = \hat{k}_i$ for all i up to a unit of $\mathfrak{F}[z]$.

Further consider the other impulse response matrices

$$\begin{aligned} K_{W/E} &= \hat{R}_2 \hat{C}_2^{-1} \hat{A}_1 , \\ K_{W/U} &= \hat{S}_1 \hat{C}_2^{-1} \hat{A}_1 , \\ K_{V/Y} &= \hat{R}_2 \hat{C}_2^{-1} \hat{B}_2 , \\ -K_{V/D} &= \hat{S}_1 \hat{C}_2^{-1} \hat{B}_2 \end{aligned}$$

and let $\hat{c}_{2i}$ denote the nonunit invariant polynomials of $\hat{C}_2$ and let r_i, q_i, s_i, u_i denote the nonunit invariant polynomials of $K_{W/E}$, $K_{W/U}$, $K_{V/Y}$, $K_{V/D}$ respectively.

Then

$$r_i \mid \hat{c}_{2i}, \; q_i \mid \hat{c}_{2i}, \; s_i \mid \hat{c}_{2i}, \; u_i \mid \hat{c}_{2i}$$

and since, by definition, the matrices $\hat{A}_1$ and $\hat{B}_2$ are left coprime and the matrices $\hat{S}_1$ and $\hat{R}_2$ are right coprime, and analogous reasoning gives us that the least common multiple of r_i, q_i, s_i and u_i is equal to $\hat{c}_{2i}$ up to a unit of $\mathfrak{F}[z]$.

From Fig. 8 it is seen that the system

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + \begin{bmatrix} \mathbf{BJ} \\ \mathbf{G} \end{bmatrix} \mathbf{w}_k , \tag{4.24}$$

$$\mathbf{e}_k = [-\mathbf{C} \;\; 0] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + [\mathbf{I}_l] \, \mathbf{w}_k ,$$

realizes $K_{W/E}$; the system

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + \begin{bmatrix} \mathbf{BJ} \\ \mathbf{G} \end{bmatrix} \mathbf{w}_k ,$$

$$\mathbf{u}_k = [-\mathbf{JC} \;\; \mathbf{H}] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + [\mathbf{J}] \, \mathbf{w}_k ,$$

realizes $K_{W/U}$; the system

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathbf{v}_k ,$$

$$\mathbf{y}_k = [\mathbf{C} \ 0] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + [0] \, \mathbf{v}_k ,$$

realizes $K_{Y/Y}$, and the system

(4.25)
$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathbf{v}_k$$

$$\mathbf{d}_k = [-\mathbf{JC} \ \mathbf{H}] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + [0] \, \mathbf{v}_k ,$$

realizes $K_{V/D}$, where $\mathbf{K}$ is given in (4.4).

These realizations are not necessarily minimal but they all have the same state-transition matrix $\mathbf{K}$. Hence

$$r_i \mid \hat{k}_i, \ q_i \mid \hat{k}_i, \ s_i \mid \hat{k}_i, \ u_i \mid \hat{k}_i .$$

It follows that also $\hat{c}_{2i}$, the least common multiple of r_i, q_i, s_i and u_i divides $\hat{k}_i$. However, by Theorem 4.1,

$$\prod_i \hat{k}_i = \det (z\mathbf{I}_{n+p} - \mathbf{K}) = \det \hat{C}_2 = \prod_i \hat{c}_{2i}$$

up to a unit of $\mathfrak{F}[z]$ and hence

$$\hat{c}_{2i} = \hat{k}_i$$

for all i up to a unit of $\mathfrak{F}[z]$. □

The pseudocharacteristic polynomial of the closed-loop system is defined as

$$c = \det (\mathbf{I}_{n+p} - z^{-1}\mathbf{K}) \in \mathfrak{F}[z^{-1}]$$

and it has a degree

$$\partial c \leqq \partial \hat{c} .$$

The pseudoinvariant polynomials of the closed-loop system are then defined as the invariant polynomials of the matrix $\mathbf{I}_{n+p} - z^{-1}\mathbf{K} \in \mathfrak{F}_{n+p,n+p}[z^{-1}]$.

To compute the pseudocharacteristic polynomial via the impulse response representations of $\boldsymbol{S}$ and $\boldsymbol{R}$, we have to take the decompositions

$$\boldsymbol{S} = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}$$

and

$$\boldsymbol{R} = S_1 R_2^{-1} = R_1^{-1} S_2 \in \mathfrak{F}_{m,l}\{z^{-1}\} .$$

Then arguments completely analogous to those in the proof of Theorem 4.1 yield

$$c = \det C_1 = \det C_2 \tag{4.26}$$

modulo units of $\mathfrak{F}[z^{-1}]$, where

$$\begin{aligned} C_1 &= R_1 A_2 + S_2 B_1 \in \mathfrak{F}_{m,m}[z^{-1}]\,, \\ C_2 &= A_1 R_2 + B_2 S_1 \in \mathfrak{F}_{l,l}[z^{-1}]\,. \end{aligned} \tag{4.27}$$

Of course,

$$c = \det(\mathbf{I}_{n+p} - z^{-1}\mathbf{K}) = z^{-(n+p)} \det(z\mathbf{I}_{n+p} - \mathbf{K}) = z^{-(n+p)}\hat{c}\,.$$

Similarly, the nonunit invariant polynomials of the matrix C_1 are equal to the nonunit invariant polynomials of C_2 up to units of $\mathfrak{F}[z^{-1}]$ and both are equal to the nonunit invariant polynomials c_i of the matrix $\mathbf{I}_{n+p} - z^{-1}\mathbf{K}$, again up to units of $\mathfrak{F}[z^{-1}]$. We also have

$$c_i = z^{-\partial \hat{c}_i} \hat{c}_i$$

and

$$\partial c_i \leqq \partial \hat{c}_i\,.$$

Example 4.1. Given a minimal realization of

$$S = \frac{\begin{bmatrix} z \\ z-1 \end{bmatrix}}{z(z-2)} = \begin{bmatrix} z \\ z-1 \end{bmatrix} [z(z-2)]^{-1} =$$
$$= \begin{bmatrix} z(z-2) & -z(z-2) \\ -(z-1) & z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and a minimal realization of

$$R = \frac{[z-1 \quad z]}{z} = [1 \quad 0] \begin{bmatrix} -z & z \\ z & -(z-1) \end{bmatrix}^{-1} = [z]^{-1} [z-1 \quad z]$$

over the field $\mathfrak{R}$, compute the invariant and pseudoinvariant polynomials of the closed-loop system.

We have

$$\hat{C}_1 = [z][z(z-2)] + [z-1 \quad z] \begin{bmatrix} z \\ z-1 \end{bmatrix} = z^3 - 2z\,,$$

$$\hat{C}_2 = \begin{bmatrix} z(z-2) & z(z-2) \\ -(z-1) & z \end{bmatrix} \begin{bmatrix} -z & z \\ z & -(z-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \quad 0]$$
$$= \begin{bmatrix} -2z^3 + 4z^2 + 1 & 2z^3 - 5z^2 + 2z \\ 2z^2 - z & -2z^2 + 2z \end{bmatrix},$$

and compute the canonical decompositions

$$\hat{C}_1 = [1]\,[z^3 - 2z]\,[1]\,,$$

$$\hat{C}_2 = \begin{bmatrix} -z^2 + 2z + 1 & -z + 2 \\ z & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & z^3 - 2z \end{bmatrix} \begin{bmatrix} -z^2 + 2z + 1 & z^2 - 2z \\ 1 & 1 \end{bmatrix}$$

and

$$\det \hat{C}_1 = z^3 - 2z\,.$$

$$\det \hat{C}_2 = -z^3 + 2z\,.$$

Thus the invariant polynomials of the closed-loop system, i.e. the monic invariant polynomials of the matrix $z\mathbf{I}_{n+p} - \mathbf{K}$, where $n + p = 2 + 1 = 3$, are

$$\hat{c}_1 = 1,\ \hat{c}_2 = 1,\ \hat{c}_3 = z^3 - 2z$$

and the characteristic polynomial is

$$\hat{c} = z^3 - 2z\,.$$

To compute the pseudoinvariant polynomials, we write

$$\boldsymbol{S} = \frac{\begin{bmatrix} z^{-1} \\ z^{-1}(1 - z^{-1}) \end{bmatrix}}{1 - 2z^{-1}} = \begin{bmatrix} z^{-1} \\ z^{-1}(1 - z^{-1}) \end{bmatrix} [1 - 2z^{-1}]^{-1} =$$

$$= \begin{bmatrix} 1 - 2z^{-1} & 0 \\ -(1 - z^{-1}) & 1 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix},$$

$$\boldsymbol{R} = [1 - z^{-1} \;\; 1] = [1 \;\; 0] \begin{bmatrix} 0 & 1 \\ 1 & -(1 - z^{-1}) \end{bmatrix}^{-1} = [1]^{-1}\,[1 - z^{1} \;\; 1]\,.$$

Then

$$C_1 = [1]\,[1 - 2z^{-1}] + [1 - z^{-1} \;\; 1] \begin{bmatrix} z^{-1} \\ z^{-1}(1 - z^{-1}) \end{bmatrix} = 1 - 2z^{-2}$$

$$C_2 = \begin{bmatrix} 1 - 2z^{-1} & 0 \\ -(1 - z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -(1 - z^{-1}) \end{bmatrix} + \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix} [1 \;\; 0] =$$

$$= \begin{bmatrix} z^{-1} & 1 - 2z^{-1} \\ 1 & -2 + 2z^{-1} \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 - 2z^{-2} \end{bmatrix} \begin{bmatrix} 1 & -2 + 2z^{-1} \\ 0 & 1 \end{bmatrix}$$

and

$$\det C_1 = 1 - 2z^{-2}\,,$$

$$\det C_2 = -1 + 2z^{-2}\,.$$

Therefore, the pseudoinvariant polynomials of the closed-loop system are

$$c_1 = 1,\ c_2 = 1,\ c_3 = 1 - 2z^{-2}$$

and the pseudocharacteristic polynomial is

$$c = 1 - 2z^{-2}$$

up to a unit of $\mathfrak{R}[z^{-1}]$.

Example 4.2. Consider the system $\mathscr{S} = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ over $\mathfrak{R}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$\boldsymbol{S} = \frac{[1\ 0]}{z-1} = [z-1]^{-1}[1\ 0] = [1\ 0]\begin{bmatrix} z-1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

and the controller $\mathscr{R} = \{\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}\}$ over $\mathfrak{R}$, where

$$\mathbf{F} = [-1], \quad \mathbf{G} = [1],$$

$$\mathbf{H} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\boldsymbol{R} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{z+1} = \begin{bmatrix} z+1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[z+1]^{-1}.$$

It is to be noted that $\mathscr{S}$ is not a minimal realization of $\boldsymbol{S}$.

Then

$$z\mathbf{I}_{n+p} - \mathbf{K} = \begin{bmatrix} z-1 & 0 & 1 \\ 0 & z-1 & 0 \\ -1 & 0 & z+1 \end{bmatrix}$$

and, by definition, the invariant polynomials of the closed-loop system are 1, 1, $z^2(z-1)$ while the nonunit invariant polynomial of the matrices

$$\hat{C}_1 = \begin{bmatrix} z+1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z-1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1\ 0] = \begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\hat{C}_2 = [z-1][z+1] + [1\ 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = z^2$$

is evidently z^2.

The two polynomials do not coincide due to the nonminimal realization of $\boldsymbol{S}$ and there is no way of computing the actual invariant polynomials via the impulse response representations.

4.3. Assigning a characteristic and invariant polynomials by dynamical feedback

Having established an expression for the characteristic and invariant polynomials of the closed-loop system shown in Fig. 5 we are interested in solving the problem of assigning desired characteristic or invariant polynomials to this system. Such a problem is sometimes referred to as the pole assignment problem since, in fact, we are assigning desired eigenvalues (poles) to the closed-loop system matrix.

The pole assignement by state-variable feedback has been solved in [22; 43]. We recall that given a system (4.1) there exists a state feedback $\mathbf{u}_k = -\mathbf{L}\mathbf{x}_k$ such that $\det(z\mathbf{I}_n - \mathbf{A} + \mathbf{BL})$ is a preassigned monic polynomial of degree n belonging to $\mathfrak{F}[z]$ if and only if system (4.1) is completely reachable.

Using a constant output feddback $\mathbf{u}_k = -\mathbf{J}\mathbf{y}_k$ we cannot make $\det(z\mathbf{I}_n - \mathbf{A} + \mathbf{BJC})$ equal to an arbitrary monic polynomial of degree n belonging to $\mathfrak{F}[z]$ even under the stronger assumption that system (4.1) be a minimal realization [11; 16].

Thus we are naturally led to use a dynamical output feedback [37] realized as a controller (4.2), see Fig. 5. This problem is formally defined as follows.

(4.28) Given a system $\mathscr{S}$ which is a minimal realization of

$$S = \hat{B}_1\hat{A}_2^{-1} = \hat{A}_1^{-1}\hat{B}_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}.$$

Find a controller $\mathscr{R}$ which is a minimal realization of some

$$R \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the characteristic polynomial of the closed-loop system in Fig. 5 be equal to a given nonzero monic polynomial $\hat{c} \in \mathfrak{F}[z]$.

The dynamical feedback, however, can do much more than to assign a characteristic polynomial. This problem will be shown to be a special case of a more general problem of assigning desired *invariant* polynomials to the closed-loop system. By this way we assign not only a characteristic polynomial (it is the product of all invariant polynomials) but we endow the closed-loop system with a desired structure.

The formal formulation is as follows.

(4.29) Given a system $\mathscr{S}$ which is a minimal realization of

$$S = \hat{B}_1\hat{A}_2^{-1} = \hat{A}_1^{-1}\hat{B}_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}.$$

Find a controller $\mathscr{R}$ which is a minimal realization of some

$$R \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the invariant polynomials of the closed-loop system in Fig. 5 be equal to a given set of nonzero monic polynomials $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_s \in \mathfrak{F}[z]$, where $\hat{c}_k \mid \hat{c}_{k+1}$, $k = 1, 2, \ldots, s-1$ and

$$s = \sum_{k=1}^{s} \partial\hat{c}_k .$$

The dimension of the closed-loop system is $\sum_{k=1}^{s} \partial \hat{c}_k$ and it must be equal to the number of given invariant polynomials; hence $s = \sum_{k=1}^{s} \partial \hat{c}_k$.

Theorem 4.3. *Problem* (4.29) *has a solution if and only if either the linear Diophantine equation*

$$X_1 \hat{A}_2 + Y_2 \hat{B}_1 = \hat{C}_1 \tag{4.30}$$

has a solution X_1^0, Y_2^0 *satisfying*

$$\begin{aligned} &\partial \det X_1^0 = s - \partial \det \hat{A}_2 , \\ &\partial (\operatorname{adj} X_1^0)\, Y_2^0 \leqq \partial \det X_1^0 . \\ &X_1^0 \text{ and } Y_2^0 \text{ left coprime} \end{aligned} \tag{4.31}$$

or the linear Diophantine equation

$$\hat{A}_1 X_2 + \hat{B}_2 Y_1 = \hat{C}_2 \tag{4.32}$$

has a solution X_2^0, Y_1^0 *satisfying*

$$\begin{aligned} &\partial \det X_2^0 = s - \partial \det \hat{A}_1 \\ &\partial Y_1^0 \operatorname{adj} X_2^0 \leqq \partial \det X_2^0 \\ &X_2^0 \text{ and } Y_1^0 \text{ right coprime,} \end{aligned} \tag{4.33}$$

where $\hat{C}_1 \in \mathfrak{F}_{m,m}[z]$ *and* $\hat{C}_2 \in \mathfrak{F}_{l,l}[z]$ *are matrices having their nonunit invariant polynomials equal to the nonunit polynomials among* $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_s$.

The controller is not unique, in general, and all controllers are obtained as minimal realizations of

$$\boldsymbol{R} = X_1^{0-1} Y_2^0$$

for all $\hat{C}_1$ *or as minimal realizations of*

$$\boldsymbol{R} = Y_1^0 X_2^{0-1}$$

for all $\hat{C}_2$.

Proof. The proof is trivial in view of Theorem 4.2. It just remains to check whether $\mathcal{R}$ is a system according to our definition. Indeed, the second condition in (4.31) makes

$$\boldsymbol{R} = X_1^{0-1} Y_2^0 = \frac{(\operatorname{adj} X_1^0) Y_2^0}{\det X_1^0}$$

physically realizable while the third condition in (4.31) guarantees that $\mathscr{R}$ be a minimal realization of $\boldsymbol{R}$. Then $\partial \det X_1^0 = \delta \boldsymbol{R}$ and the first condition in (4.31) reads

$$\sum_{k=1}^{s} \partial \hat{c}_k = \delta \boldsymbol{S} + \delta \boldsymbol{R} ,$$

which is relation (4.5).

Conditions (4.33) play the same role for the solution X_2^0, Y_1^0 of equation (4.32). □

The requirement that $\mathscr{R}$ be a minimal realization of $\boldsymbol{R}$ certainly restricts the class of all controllers yielding given invariant polynomials $\hat{c}_k$, $k = 1, 2, \ldots, s$ but it is an essential restriction because otherwise the $\hat{c}_k$'s would not be given by Theorem 4.2.

Since $\hat{C}_1 \in \mathfrak{F}_{m,m}[z]$ and $\hat{C}_2 \in \mathfrak{F}_{l,l}[z]$ and their nonunit invariant polynomials equal, it is seen that the number of given *nonunit* invariant polynomials must not exceed min (l, m).

It can also be seen that the matrices $\hat{C}_1$ and $\hat{C}_2$ are given uniquely by $\hat{c}_k$, $k = 1, 2, \ldots$ $\ldots$, min (l, m) up to their associates.

Equations (4.30) and (4.32) can be put into the unified form (1.5) by writting

(4.34)
$$Y \begin{bmatrix} \hat{A}_2 \\ \hat{B}_1 \end{bmatrix} = \hat{C}_1 ,$$

$$[\hat{A}_1 \ \ \hat{B}_2] \, X = \hat{C}_2 ,$$

where

(4.35)
$$X = \begin{bmatrix} X_2 \\ Y_1 \end{bmatrix}, \quad Y = [X_1 \ \ Y_2] .$$

Then the results developed for (1.5) can be applied to solve equations (4.30) and (4.32).

Corollary 4.1. *Problem* (4.28) *has a solution if and only if either equation* (4.30) *has a solution* X_1^0, Y_2^0 *satisfying*

(4.36)
$$\partial \det X_1^0 = \partial \hat{c} - \partial \det \hat{A}_2 ,$$

$$\partial (\operatorname{adj} X_1^0) \, Y_2^0 \leqq \partial \det X_1^0 ,$$

$$X_1^0 \textit{ and } Y_2^0 \textit{ left coprime} ,$$

or equation (4.32) *has a solution* X_2^0, Y_1^0 *satisfying*

(4.37)
$$\partial \det X_2^0 = \partial \hat{c} - \partial \det \hat{A}_1 ,$$

$$\partial \, Y_1^0 \operatorname{adj} X_2^0 \leqq \partial \det X_2^0 ,$$

$$X_2^0 \textit{ and } Y_1^0 \textit{ right coprime},$$

where $\hat{C}_1 \in \mathfrak{F}_{m,m}[z]$ *and* $\hat{C}_2 \in \mathfrak{F}_{l,l}[z]$ *are matrices such that*

$$\hat{c} = \det \hat{C}_1 = \det \hat{C}_2 ,$$

up to units of $\mathfrak{F}[z]$.

The controller is not unique, in general, and all controllers are obtained as minimal realizations of

$$\boldsymbol{R} = X_1^{0^{-1}} Y_2^0$$

for all $\hat{C}_1$ *or as minimal realizations of*

$$\boldsymbol{R} = Y_1^0 X_2^{0^{-1}}$$

for all $\hat{C}_2$.

Proof. Since the characteristic polynomial is the product of all invariant polynomials, problem (4.28) is a special case of problem (4.29). The matrices $\hat{C}_1$ and $\hat{C}_2$ just will not be given by their invariant polynomials but only by the characteristic polynomial irrespective of their structure. □

This looser condition admits a wider choice of the $\hat{C}_1$ and $\hat{C}_2$ not confined to associated matrices and, therefore, one can expect that a solution will exist in more cases.

Example 4.3. Given a minimal realization of

$$\boldsymbol{S} = \frac{\begin{bmatrix} z & 0 \\ z & z+1 \end{bmatrix}}{z(z+1)} = \begin{bmatrix} 0 & z(z+1) \\ 1 & z \end{bmatrix}^{-1} \begin{bmatrix} z & z+1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} z & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} z(z+1) & z+1 \\ -z(z+1) & -z \end{bmatrix}^{-1}$$

over $\mathfrak{R}$, solve problem (4.29) for

$$\hat{c}_1 = 1 ,$$

$$\hat{c}_2 = z + 1 ,$$

$$\hat{c}_3 = z(z+1) .$$

Observe that $\hat{c}_1 \mid \hat{c}_2 \mid \hat{c}_3$, that $\sum_{k=1}^{3} \partial \hat{c}_k = 3$, and that $\min (l, m) = 2$ as required. Consider e.g. equation (4.32) and choose

(4.38) $$\hat{C}_2 = \begin{bmatrix} z+1 & 0 \\ 0 & z(z+1) \end{bmatrix} ,$$

i.e. equation (4.32) becomes

$$(4.39) \qquad \begin{bmatrix} 0 & z(z+1) \\ 1 & z \end{bmatrix} X_2 + \begin{bmatrix} z & z+1 \\ 1 & 1 \end{bmatrix} Y_1 = \begin{bmatrix} z+1 & 0 \\ 0 & z(z+1) \end{bmatrix}.$$

We rewrite (4.39) into the form

$$\begin{bmatrix} 0 & z(z+1) & z & z+1 \\ 1 & z & 1 & 1 \end{bmatrix} X = \begin{bmatrix} z+1 & 0 \\ 0 & z(z+1) \end{bmatrix}$$

and since

$$\begin{bmatrix} 0 & z(z+1) & z & z+1 \\ 1 & z & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & z+1 & -(z+1) \\ 0 & 1 & -z & 0 \end{bmatrix}^{-1},$$

equation (4.39) is equivalent to the set of polynomial equations

$$\bar{x}_{11} = 0, \qquad \bar{x}_{12} = z(z+1),$$

$$\bar{x}_{21} = z+1, \quad \bar{x}_{22} = 0$$

by Theorem 1.1.

The general solution of (4.39) is then

$$X = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & z+1 & -(z+1) \\ 0 & 1 & -z & 0 \end{bmatrix} \begin{bmatrix} 0 & z(z+1) \\ z+1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & z+1 & -(z+1) \\ 0 & 1 & -z & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}.$$

by (1.13) and (1.14), where $t_{ij} \in \Re[z]$ arbitrary. Hence by (4.35)

$$X_2 = \begin{bmatrix} 0 & z(z+1) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} -(z+1) & 0 \\ z+1 & 0 \end{bmatrix} + \begin{bmatrix} z+1 & -(z+1) \\ z & 0 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}.$$

Now we have to take the polynomials t_{ij} so as to obtain a solution satisfying (4.33). First of all,

$$\partial \det X_2 = s - \partial \det \hat{A}_1 = 3 - 2 = 1 .$$

Thus all polynomials t_{ij} such that

$$\det X_2 = -t_{11}t_{22} + t_{21}t_{12} - z(z+1)\, t_{21}$$

is a polynomial of degree 1 are acceptable. Let us choose for simplicity

(4.40)
$$t_{11} = 1 , \quad t_{12} \text{ arbitrary} ,$$
$$t_{21} = 0 , \quad t_{22} = \tau_0 + \tau_1 z , \quad \tau_1 \neq 0 ;$$

then

$$\det X_2 = -\tau_0 - \tau_1 z$$

and

$$X_2 = \begin{bmatrix} -1 & z(z+1) + \tau_0 + \tau_1 z - t_{12} \\ 0 & \tau_0 + \tau_1 z \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0 & (z+1)\, t_{12} - (z+1)(\tau_0 + \tau_1 z) \\ 1 & -z t_{12} \end{bmatrix}.$$

Computing

$$Y_1 \operatorname{adj} X_2 = \begin{bmatrix} 0 & -(z+1)\, t_{12} + (z+1)(\tau_0 + \tau_1 z) \\ \tau_0 + \tau_1 z & -z(z+1) - \tau_0 - \tau_1 z + (z+1)\, t_{12} \end{bmatrix}.$$

the t_{12} must be of the form

$$t_{12} = \varphi_0 + z$$

and

$$\tau_1 = 1$$

in order that the second condition (4.33) may be satisfied. Hence

(4.41)
$$X_2^0 = \begin{bmatrix} -1 & z^2 + z - (\varphi_0 - \tau_0) \\ 0 & z + \tau_0 \end{bmatrix},$$

$$Y_1^0 = \begin{bmatrix} 0 & (\varphi_0 - \tau_0)\, z + (\varphi_0 - \tau_0) \\ 1 & -z^2 - \varphi_0 z \end{bmatrix}$$

and it remains to guarantee that the X_2^0 and Y_1^0 be right coprime. Since

(4.42)
$$\begin{bmatrix} X_2^0 \\ Y_1^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & z + \tau_0 \\ 0 & (\varphi_0 - \tau_0)\, z - (\varphi_0 - \tau_0) \\ 0 & -(\varphi_0 - 1)\, z - (\varphi_0 - \tau_0) \end{bmatrix} \begin{bmatrix} 1 & -z^2 - z + (\varphi_0 - \tau_0) \\ 0 & 1 \end{bmatrix}$$

we have to exclude

$$\tau_0 = 1 ,$$
$$\tau_0 = \varphi_0 .$$

For these particular values the invariant polynomials of the matrix (4.42) would be different from unity and hence the X_2^0 and Y_1^0 would not be right coprime. For example, if $\tau_0 = 1$, we have

$$X_2 = \begin{bmatrix} -1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \varphi_0 - 1 \\ 0 & z + 1 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 1 & -z - (\varphi_0 - 1) \\ 0 & \varphi_0 - 1 \end{bmatrix} \begin{bmatrix} 1 & \varphi_0 - 1 \\ 0 & z + 1 \end{bmatrix}.$$

Therefore, the required controller is a minimal realization of

$$\boldsymbol{R} = Y_1^0 X_2^{0^{-1}} ,$$

where X_2^0 and Y_1^0 are given by (4.41) for any τ_0, $\varphi_0 \in \mathfrak{R}$, $\tau_0 \neq 1$, $\tau_0 \neq \varphi_0$. However, other controllers exist because the choices (4.38) and (4.40) are not the most general ones.

Example 4.4. Given the system over $\mathfrak{Q}$ which is a minimal realization of

$$\boldsymbol{S} = \frac{\begin{bmatrix} z \\ z - 1 \end{bmatrix}}{z(z-2)} = \begin{bmatrix} z \\ z - 1 \end{bmatrix} [z(z-2)]^{-1}$$
$$= \begin{bmatrix} z(z-2) & -z(z-2) \\ -(z-1) & z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

solve problem (4.29) for

$$\hat{c}_1 = 1 ,$$
$$\hat{c}_2 = 1 ,$$
$$\hat{c}_3 = z^3 - 2z .$$

We observe that $\hat{c}_1 \mid \hat{c}_2 \mid \hat{c}_3$, that $\sum_{k=1}^{3} \partial \hat{c}_k = 3$, and min $(l, m) = 1$. Let us first choose

$$\hat{C}_2 = \begin{bmatrix} 1 & 0 \\ 0 & z^3 - 2z \end{bmatrix}.$$

Then equation (4.32) becomes

$$\begin{bmatrix} z(z-2) & -z(z-2) \\ -(z-1) & z \end{bmatrix} X_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} Y_1 = \begin{bmatrix} 1 & 0 \\ 0 & z^3 - 2z \end{bmatrix}$$

or

$$\begin{bmatrix} z(z-2) & -z(z-2) & 1 \\ -(z-1) & z & 0 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & z^3 - 2z \end{bmatrix},$$

where

$$X = \begin{bmatrix} X_2 \\ Y_1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} z(z-2) & -z(z-2) & 1 \\ -(z-1) & z & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -z \\ 0 & 1 & 1-z \\ 1 & 0 & z(z-2) \end{bmatrix}^{-1},$$

we have

$$\bar{x}_{11} = 1, \quad \bar{x}_{12} = 0,$$

$$\bar{x}_{21} = 0, \quad \bar{x}_{22} = z^3 - 2z$$

and the general solution reads

$$X = \begin{bmatrix} 0 & 1 & -z \\ 0 & 1 & 1-z \\ 1 & 0 & z(z-2) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^3 - 2z \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -z \\ 0 & 1 & 1-z \\ 1 & 0 & z(z-2) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ t_{11} & t_{12} \end{bmatrix}$$

for arbitrary $t_{ij} \in \mathfrak{Q}[z]$. Hence

$$X_2 = \begin{bmatrix} 0 & z^3 - 2z \\ 0 & z^3 - 2z \end{bmatrix} - \begin{bmatrix} z \\ z-1 \end{bmatrix} [t_{11} \;\; t_{12}],$$

$$Y_1 = [1 \;\; 0] + z(z-2)[t_{11} \;\; t_{12}]$$

and the t_{ij}' s should be chosen so that conditions (4.33) are met.

First,

$$\partial \det X_2 = 3 - 2 = 1.$$

Since

$$\det X_2 = -(z^3 - 2z)\, t_{11},$$

it is seen that no such t_{11} exists.

It does not mean, however, that the problem has no solution. We can choose e.g.

$$\hat{C}_2 = \begin{bmatrix} 1 & 0 \\ 1 & z^3 - 2z \end{bmatrix}$$

and start again. Equation (4.32) will have the general solution

$$X_2 = \begin{bmatrix} 1 & z^3 - 2z \\ 1 & z^3 - 2z \end{bmatrix} - \begin{bmatrix} z \\ z-1 \end{bmatrix} [t_{11} \;\; t_{12}]$$

$$Y_1 = [1 \;\; 0] + [z(z-2)][t_{11} \;\; t_{12}]$$

for any $t_{ij} \in \mathfrak{Q}[z]$.

Again, $\partial \det X_2 = 1$. Since

$$\det X_2 = t_{12} - (z^3 - 2z)\, t_{11}\,,$$

we have to set

$$t_{11} \text{ arbitrary}\,,$$
$$t_{12} = (z^3 - 2z)\, t_{11} + \tau_0 + \tau_1 z\,, \quad \tau_1 \neq 0$$

to obtain

$$\det X_2 = \tau_0 + \tau_1 z\,.$$

Computing

$$Y_1 \operatorname{adj} X_2 =$$
$$= \begin{bmatrix} (1 - \tau_1)\, z^3 + (\tau_1 - \tau_0)\, z^2 + (\tau_0 + \tau_1 - 2)\, z + \tau_0 + (1 - z)(z^3 - 2z)\, t_{11} \\ -(1 - \tau_1)\, z^3 - (\tau_1 - \tau_0)\, z^2 - (\tau_0 - 2)\, z + z(z^3 - 2z)\, t_{11} \end{bmatrix}'$$

we must take

$$t_{11} = 0\,,$$
$$\tau_1 = 1\,, \quad \tau_0 = \tau_1\,,$$

to satisfy $\partial Y_1 \operatorname{adj} X_2 \leqq \partial \det X_2 = 1$.

Then

$$X_2^0 = \begin{bmatrix} 1 & z^3 - z^2 - 3z \\ 1 & z^3 - z^2 - 2z + 1 \end{bmatrix},$$

$$Y_1^0 = [1 \;\; z^3 - z^2 - 2z]$$

are right coprime and, therefore, a minimal realization of

$$\boldsymbol{R} = Y_1^0 X_2^{0^{-1}} = \frac{[1 \;\; z]}{z + 1}$$

is a solution to our problem.

This solution is not the only one, however. For example, take

$$\hat{C}_1 = z^3 - 2z$$

and solve equation (4.30) which becomes

$$X_1[z(z - 2)] + Y_2 \begin{bmatrix} z \\ z - 1 \end{bmatrix} = z^3 - 2z\,, \tag{4.43}$$

or

$$Y \begin{bmatrix} z(z - 2) \\ z \\ z - 1 \end{bmatrix} = z^3 - 2z\,.$$

Write

$$\begin{bmatrix} z(z-2) \\ z \\ z-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & z-1 \\ 1 & 1 & -z \\ z-1 & 0 & -z^2+2z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and hence equation (4.43) reduces to

$$\bar{y}_{11} = z^2 - 2z .$$

The general solution reads

$$Y = \begin{bmatrix} z^3 - 2z & t_{11} & t_{12} \end{bmatrix} \begin{bmatrix} -1 & 0 & z-1 \\ 1 & 1 & -z \\ z-1 & 0 & -z^2+2z \end{bmatrix}$$

for any $t_{ij} \in \mathfrak{Q}[z]$ and

$$X_1 = -z^3 + 2z + \begin{bmatrix} t_{11} & t_{12} \end{bmatrix} \begin{bmatrix} 1 \\ z-1 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0 & (z-1)(z^3-2z) \end{bmatrix} + \begin{bmatrix} t_{11} & t_{12} \end{bmatrix} \begin{bmatrix} 1 & -z \\ 0 & -z^2+2z \end{bmatrix}.$$

Again $\partial \det X_1 = 1$, i.e. we have to take

$$t_{11} = z^3 - 2z - (z-1)\, t_{12} + \tau_0 + \tau_1 z , \quad \tau_1 \neq 0 ,$$
$$t_{12} \text{ arbitrary}$$

to obtain

$$X_1 = \tau_0 + \tau_1 z .$$

Computing

$$(\operatorname{adj} X_1)\, Y_2 =$$
$$= \begin{bmatrix} z^3 + (\tau_1 - 2)\, z + \tau_0 - (z-1)\, t_{12} & -z^3 - \tau_1 z^2 - (\tau_0 - 2)\, z + z t_{12} \end{bmatrix}$$

the condition $\partial(\operatorname{adj} X_1)\, Y_2 \leqq 1$ will yield

$$t_{12} = z^2 + z + \sigma_0 ,$$
$$\tau_1 = 1 .$$

Then

$$X_1^0 = \tau_0 + z ,$$
$$Y_2^0 = \begin{bmatrix} -\sigma_0 z + (\sigma_0 + \tau_0) & (\sigma_0 - \tau_0 + 2)\, z \end{bmatrix}$$

are left coprime if and only if

$$\tau_0^2 - (2\sigma_0 + 3)\, \tau_0 - \sigma_0 \neq 0$$

and other controllers that solve our problem can be taken as minimal realizations of

$$R = X_1^{0^{-1}} Y_2^0 = \frac{[-\sigma_0 z + (\sigma_0 + \tau_0) \quad (\sigma_0 - \tau_0 + 2) z]}{z + \tau_0}$$

In particular, $\tau_0 = 0$, $\sigma_0 = -1$ gives the controller considered in Example 4.1.

Example 4.5. Given the system which is a minimal realization of

$$S = \frac{\begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}}{z^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix}^{-1}$$

over the field $\mathfrak{Z}_2$, show that there is no controller which assigns to the closed-loop system the invariant polynomials

$$\hat{c}_1 = 1 \,,$$
$$\hat{c}_2 = 1 \,,$$
$$\hat{c}_3 = z^3 \,.$$

We start with equation (4.32). Let

$$\hat{C}_2 = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

be any matrix over $\mathfrak{Z}_2[z]$ whose invariant polynomials are 1, z^3. Then

$$\begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix} X_2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Y_1 = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{4.44}$$

and the general solution becomes

$$X_2 = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} - \begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} t_{21} & t_{22} \\ t_{41} & t_{42} \end{bmatrix}$$

for arbitrary $t_{ij} \in \mathfrak{Z}_2[z]$.

Since

$$\partial \det X_2 = s - \partial \det \hat{A}_1 = 3 - 3 = 0 \,,$$

we have to confine ourselves to those t_{ij} which gives

$$\det X_2 = t_{11} t_{12} - t_{21} t_{22} = 1 \,.$$

Further the requirement

$$\partial Y_1 \operatorname{adj} X_2 \leqq \partial \det X_2 = 0$$

implies that

$$\det\left(Y_1 \operatorname{adj} X_2\right) = 1\,. \tag{4.45}$$

However,

$$Y_1 \operatorname{adj} X_2 = \begin{bmatrix} c_{11} - z^2 t_{11} & c_{12} - z^2 t_{12} \\ c_{21} - z t_{21} & c_{22} - z t_{22} \end{bmatrix} \begin{bmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{bmatrix} =$$

$$= \begin{bmatrix} c_{11} t_{22} - c_{12} t_{21} - z^2 & c_{12} t_{11} - c_{11} t_{12} \\ c_{21} t_{22} - c_{22} t_{21} & c_{22} t_{11} - c_{21} t_{12} - z \end{bmatrix}$$

and

$$\det\left(Y_1 \operatorname{adj} X_2\right) =$$

$$= z^3 + \left(c_{21} t_{12} - c_{22} t_{11}\right) z^2 + \left(c_{12} t_{21} - c_{11} t_{22}\right) z + \det \hat{C}_2\,.$$

Since $\det \hat{C}_2 = z^3$, we obtain $z \mid \det\left(Y_1 \operatorname{adj} X_2\right)$, a contradiction to (4.45). Hence no solution X_2^0, Y_1^0 exists regardless of $\hat{C}_2$.

Now consider equation (4.30). Since $l = m$, the $\hat{C}_1$ may be taken as $\hat{C}_2'$ without any lost of generality. Then the equation

$$X_1 \begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix} + Y_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \hat{C}_2'$$

is the transposed equation (4.32) and it can have no solution either.

We conclude that there is no controller making the closed-loop invariant polynomials equal to $\hat{c}_1 = 1$, $\hat{c}_2 = 1$, $\hat{c}_3 = z^3$.

Example 4.6. Consider again the system from Example 4.5. We will show here that the characteristic polynomial $\hat{c} = z^3$ can be assigned even though the invariant polynomials 1, 1, z^3 cannot.

Let us choose (this is the crucial step)

$$\hat{C}_2 = \begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix}.$$

Then we are to solve the equation

$$\begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix} X_2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Y_1 = \begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix}$$

the general solution of which reads

$$X_2 = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

for arbitrary $t_{ij} \in \mathfrak{Z}_2[z]$.

The t_{ij} must be choosen so that

$$\partial \det X_2 = \partial \hat{c} - \partial \det \hat{A}_1 = 3 - 3 = 0\,,$$

that is,

$$t_{11}t_{22} - t_{21}t_{12} = 1\,.$$

Computing

$$Y_1 \operatorname{adj} X_2 = \begin{bmatrix} z^2 t_{22} - z^2 & -z^2 t_{12} \\ z t_{21} & z t_{11} - z \end{bmatrix},$$

it is seen that the only choice satisfying $\partial Y_1 \operatorname{adj} X_2 \leq \partial \det X_2 = 0$ is

$$t_{11} = 1\,, \quad t_{12} = 0\,,$$
$$t_{21} = 0\,, \quad t_{22} = 1\,.$$

Then

$$X_2^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y_1^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the minimal realization of

$$R = Y_1^0 X_2^{0^{-1}} = 0$$

solves our problem.

Example 4.7. Given a minimal realization of

$$S = \frac{1}{z^2(z-1)}$$

over $\mathfrak{R}$, try to solve problem (4.28) for $\hat{c} = z^3(z - 0{\cdot}5)$.

We are to solve the equation

$$z^2(z-1)\,X + Y = z^3(z - 0{\cdot}5)$$

where $X = X_1 = X_2$, $Y = Y_1 = Y_2$. Its general solution is evidently

$$X = t\,,$$
$$Y = z^3(z - 0{\cdot}5) - z^2(z-1)\,t$$

for any $t \in \mathfrak{R}[z]$.

Since $\partial X = \partial \hat{c} - \partial \hat{a} = 4 - 3 = 1$, we have to take $t = \tau_0 + \tau_1 z$, $\tau_1 \neq 0$. Then

$$X = \tau_0 + \tau_1 z\,,$$
$$Y = (1 - \tau_1)\,z^4 + (\tau_1 - \tau_0 - 0{\cdot}5)\,z^3 + \tau_0 z^2$$

and no choice of τ_0, τ_1 will give $\partial Y \geq \partial X$. Hence no controller exists for $\hat{c} = z^3(z - 0{\cdot}5)$.

Example 4.8. The requirement that $\mathscr{R}$ be a minimal realization of $\boldsymbol{R}$ certainly restricts the class of controllers yielding a given characteristic polynomial. It may happen that no such controller exists whereas there are nonminimal realizations of $\boldsymbol{R}$ that solve the problem.

Consider a minimal realization of

$$S = \frac{1}{z^2}$$

over $\mathfrak{R}$ and solve problem (4.28) for

$$\hat{c} = z^3 .$$

Equations (4.30) and (4.32) read

$$z^2 X + Y = z^3$$

and give the general solution

$$X = z + t ,$$
$$Y = -z^2 t ,$$

for arbitrary $t \in \mathfrak{R}[z]$.

Since $\partial X = \partial \hat{c} - \partial \hat{a} = 3 - 2 = 1$, we have to take $t = \tau_0 + \tau_1 z$, $\tau_1 \neq -1$. Then

$$X = (1 + \tau_1) z + \tau_0 ,$$
$$Y = -\tau_1 z^3 - \tau_0 z^2$$

and the only choice to get $\partial Y \leqq \partial X$ is $\tau_0 = 0$, $\tau_1 = 0$. Then, however, $X^0 = z$, $Y^0 = 0$ and we have destroyed the primeness of X^0 and Y^0 because $(z, 0) = z$.

We conclude that no minimally realized controller exists that would assign the polynomial $\hat{c} = z^3$. Indeed, $\mathbf{R} = 0/z = 0$ would have the minimal realization $\mathscr{R} = \{0, 0, 0, 0\}$ and it would yield $\hat{c} = z^2$.

On the other hand, there are nonminimal realizations of $\mathbf{R} = 0$, e.g. the $\mathscr{R} = \{0, 0, 1, 0\}$, that do yield the desired polynomial $\hat{c} = z^3$. They cannot be found on the basis of the impulse response description, however. The resulting feedback system is degenerated, see Fig. 9.

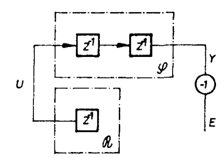

Fig. 9. The degenerated closed-loop system from Example 4.8.

Quite similarly, we can pose the problems of assigning a given pseudocharacteristic polynomial or pseudoinvariant polynomials. The formal definitions are as follows.

(4.46) Given a system $\mathscr{S}$ which is a minimal realization of

$$S = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\} .$$

Find a controller $\mathscr{R}$ which is a minimal realization of some

$$R \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the pseudocharacteristic polynomial of the closed-loop system in Fig. 5 be equal modulo a unit of $\mathfrak{F}[z^{-1}]$ to a given nonzero polynomial $c \in \mathfrak{F}[z^{-1}]$, where $(c, z^{-1}) = 1$.

(4.47) Given a system $\mathscr{S}$ which is a minimal realization of

$$S = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}.$$

Find a controller $\mathscr{R}$ which is a minimal realization of some

$$R \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the pseudoinvariant polynomials of the closed-loop system in Fig. 5 be equal modulo units of $\mathfrak{F}[z^{-1}]$ to a given set of nonzero polynomials $c_1, c_2, \ldots, c_s \in \mathfrak{F}[z^{-1}]$, where $(c_k, z^{-1}) = 1$ for $k = 1, 2, \ldots, s$, $c_k \mid c_{k+1}$ for $k = 1, 2, \ldots, s - 1$ and $s \geqq \sum_{k=1}^{s} \partial c_k$.

Since the dimension of the closed-loop system must be equal to the number of given invariant polynomials and $\partial c_k \leqq \partial \hat{c}_k$, we obtain $s \geqq \sum_{k=1}^{s} \partial c_k$.

Theorem 4.4. *Problem (4.47) has a solution if and only if either the linear Diophantine equation*

(4.48) $$X_1 A_2 + Y_2 B_1 = C_1$$

has a solution X_1^0, Y_2^0 satisfying

(4.49) $$(\det X_1^0, z^{-1}) = 1,$$

$$X_1^0 \text{ and } Y_2^0 \text{ left coprime}$$

or the linear Diophantine equation

(4.50) $$A_1 X_2 + B_2 Y_1 = C_2$$

has a solution X_2^0, Y_1^0 satisfying

(4.51) $$(\det X_2^0, z^{-1}) = 1,$$

$$X_2^0 \text{ and } Y_1^0 \text{ right coprime},$$

where $C_1 \in \mathfrak{F}_{m,m}[z^{-1}]$ and $C_2 \in \mathfrak{F}_{l,l}[z^{-1}]$ are matrices having their nonunit invariant polynomials equal to the nonunit polynomials among $c_1, c_2, \ldots, c_s$.

The controller is not unique, in general, and all controllers are obtained as minimal realizations of

$$\boldsymbol{R} = X_1^{0-1} Y_2^0$$

for all C_1 *or as minimal realizations of*

$$\boldsymbol{R} = Y_1^0 X_2^{0-1}$$

for all C_2.

Proof. The proof is trivial in view of the fact that the nonunit invariant polynomials of the matrices C_1 and C_2 are equal to the nonunit pseudoinvariant polynomials of the closed-loop system. It just remains to check whether $\mathscr{R}$ is a system according to our definition. Indeed, the first condition in (4.49) and (4.51) makes $\boldsymbol{R}$ physically realizable while the second condition in (4.49) and (4.51) guarantees that $\mathscr{R}$ be a minimal realization of $\boldsymbol{R}$. □

Since C_1 belongs to $\mathfrak{F}_{m,m}[z^{-1}]$, C_2 belongs to $\mathfrak{F}_{l,l}[z^{-1}]$, and their nonunit invariant polynomials equal, it is seen that the number of given nonunit pseudoinvariant polynomials must not exceed min (l, m).

Again, the matrices $\hat{C}_1$ and $\hat{C}_2$ are given uniquely by c_k, $k = 1, 2, \ldots, s$ up to their associates.

Corollary 4.2. *Problem* (4.46) *has a solution if and only if either equation* (4.48) *has a solution* X_1^0, Y_2^0 *satisfying*

$$(4.52) \qquad (\det X_1^0, z^{-1}) = 1\,, \quad X_1^0 \text{ and } Y_2^0 \text{ right coprime,}$$

or equation (4.50) *has a solution* X_2^0, Y_1^0 *satisfying*

$$(4.53) \qquad (\det X_2^0, z^{-1}) = 1\,, \quad X_2^0 \text{ and } Y_1^0 \text{ right coprime}\,,$$

where $C_1 \in \mathfrak{F}_{m,m}[z^{-1}]$ *and* $C_2 \in \mathfrak{F}_{l,l}[z^{-1}]$ *are matrices such that*

$$c = \det C_1 = \det C_2$$

up to units of $\mathfrak{F}[z^{-1}]$.

The controller is not unique, in general, and all controlers are obtained as minima realizations of

$$\boldsymbol{R} = X_1^{0-1} Y_2^0$$

for all C_1 *or as minimal realizations of*

$$\boldsymbol{R} = Y_1^0 X_2^{0-1}$$

for all C_2.

Proof. Since the pseudocharacteristic polynomial is the product of all pseudo-invariant polynomials, problem (4.46) is a special case of problem (4.47). The matrices C_1 and C_2 just will not be given by their invariant polynomials but only by the characteristic polynomial irrespective of their structure. □

The degree of the pseudocharacteristic polynomial is not equal to the dimension of the system and hence no counterpart of the very restrictive first condition in (4.36) or (4.37) is necessary. Moreover, the pseudocharacteristic polynomial determines the characteristic polynomial uniquely up to a power of the indeterminate z. Therefore, if a desired characteristic polynomial happens not to be assignable, we may try to assign the corresponding pseudocharacteristic polynomial $c = z^{-\partial \hat{c}}\hat{c}$ at the expense of increasing the characteristic polynomial $\hat{c}$ by an appropriate power of z. In fact, equations (4.48) and (4.50) have always a solution because the matrices A_2 and B_1 are right coprime and the matrices A_1 and B_2 are left coprime. It just becomes a matter of satisfying conditions (4.52) or (4.53).

Example 4.9. Consider again the system from Example 4.7. Inasmuch as the characteristic polynomial $\hat{c} = z^3(z - 0{\cdot}5)$ cannot be assigned, we will try to solve problem (4.46) for

$$c = z^{-4}[z^3(z - 0.5)] = 1 - 0.5z^{-1}.$$

We write

$$S = \frac{z^{-3}}{1 - z^{-1}}$$

and hence equations (4.48) and (4.50) become

$$(1 - z^{-1})X + z^{-3}Y = 1 - 0{\cdot}5z^{-1},$$

where $X = X_1 = X_2$, $Y = Y_1 = Y_2$. The general solution is evidently

$$X = 1 + 0{\cdot}5z^{-1} + 0{\cdot}5z^{-2} + z^{-3}t,$$
$$Y = 0{\cdot}5 - (1 - z^{-1})\,t$$

for any $t \in \Re[z^{-1}]$.

This solution satisfies $(X, z^{-1}) = 1$, for all t. We just have to avoid certain t's, e.g.

$$t \neq -0{\cdot}5,$$
$$t \neq \tau_0 + 2(1 - \tau_0)\,z^{-1}, \quad \tau_0 \in \Re, \tag{4.54}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

to guarantee that $(X, Y) = 1$. Thus the controller is a minimal realization of

$$R = \frac{0{\cdot}5 - (1 - z^{-1})\,t}{1 + 0{\cdot}5z^{-1} + 0{\cdot}5z^{-2} + z^{-3}t}$$

for any t meeting (4.54).

The characteristic polynomial of the closed-loop system then becomes

$$\hat{c}_0 = z^4(z - 0{\cdot}5) \quad \text{if} \quad t = 0\,,$$
$$= z^{n+5}(z - 0{\cdot}5) \quad \text{if} \quad \partial t = n \geqq 0\,.$$

Thus the choice $t = 0$, i.e.

$$R = \frac{0{\cdot}5}{1 + 0{\cdot}5z^{-1} + 0{\cdot}5z^{-2}}$$

gives the best assignable approximation of $\hat{c} = z^3(z - 0{\cdot}5)$.

4.4. Stability conditions

As mentioned at the beginning of the chapter the closed-loop system need not be a minimal realization even if both $\mathscr{S}$ and $\mathscr{R}$ are. Then the impulse response matrices $K_{W/Y}$, $K_{W/E}$, $K_{W/U}$ or $K_{V/Y}$, $K_{V/D}$, $K_{V/U}$ do not fully describe the closed-loop system any more. Specifically, this impulse response matrices may not reveal the actual system dynamics or, even worse, they may conceal the system instability. Otherwise speaking, stability of this impulse response matrices does not generally imply stability of the closed-loop system [33].

To illustrate the difficulties arising in the closed-loop system stability analysis, we consider

Example 4.10. Given the configuration shown in Fig. 7, where $\mathscr{S}$ is a minimal realization of

$$S = \frac{[z \quad 1 - z]}{z(z - 1)}$$

and $\mathscr{R}$ is a minimal realization of

$$R = \frac{\begin{bmatrix} z - 1 \\ z \end{bmatrix}}{z}\,,$$

both over the field $\Re$ valuated by (2.25).

Let the external input W be applied. Then all impulse responses of the closed-loop system, viz.

$$K_{W/Y} = SR(I_l + SR)^{-1} = [0]\,,$$
$$K_{W/E} = (I_l + SR)^{-1} = [1]\,,$$
$$K_{W/U} = R(I_l + SR)^{-1} = \frac{\begin{bmatrix} z - 1 \\ z \end{bmatrix}}{z}$$

are stable and one might get the impression that the closed-loop system is stable. This is false, however. The characteristic polynomial of the system is given by Theorem 4.1 as

$$\hat{c} = \det\left[z(z-1)\,z + 0\right] = z^2(z-1)$$

and it is *not* stable.

What has happened? A minimal realization of $\boldsymbol{S}$ is

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

and that of $\boldsymbol{R}$ becomes

$$\mathbf{F} = [0], \qquad \mathbf{G} = [1],$$

$$\mathbf{H} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then using (4.20), (4.24) and (4.21) we can check that the closed-loop system is not a minimal realization of any impulse response matrix considered above. Hence the closed-loop system contains certain parts which cannot be determined from the impulse response matrices, and they caused instability.

Our next task is, therefore, to find additional conditions for the impulse response matrices of the closed-loop system that would guarantee the system stability. To do so, we shall denote

$$(4.55) \qquad \begin{aligned} \boldsymbol{K}_1 &= \boldsymbol{K}_{W/Y} = \boldsymbol{SR}(\boldsymbol{I}_l + \boldsymbol{SR})^{-1} \in \mathfrak{F}_{l,l}\{z^{-1}\}\,. \\ \boldsymbol{K}_2 &= -\boldsymbol{K}_{V/D} = \boldsymbol{RS}(\boldsymbol{I}_m + \boldsymbol{RS})^{-1} \in \mathfrak{F}_{m,m}\{z^{-1}\}\,. \end{aligned}$$

Theorem 4.5. *Given the closed-loop system shown in Fig. 5, where $\mathscr{S}$ is a minimal realization of*

$$\boldsymbol{S} = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}$$

and $\mathscr{R}$ is a minimal realization of

$$\boldsymbol{R} = S_1 R_2^{-1} = R_1^{-1} S_2 \in \mathfrak{F}_{m,l}\{z^{-1}\}\,,$$

where $\mathfrak{F}$ is an arbitrary field with valuation $\mathscr{V}$. Then the characteristic polynomial of the closed-loop system is stable (with respect to $\mathscr{V}$) if and only if the impulse response matrices $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ have the form

$$(4.56) \qquad \begin{aligned} \boldsymbol{K}_1 &= B_1 \boldsymbol{M}_1\,, & K_2 &= \boldsymbol{M}_2 B_2\,, \\ I_l - \boldsymbol{K}_1 &= N_1 A_1\,, & Im - \boldsymbol{K}_2 &= A_2 N_2\,. \end{aligned}$$

where $M_1 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$, $N_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ *and* $M_2 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$, $N_2 \in \mathfrak{F}^+_{m,m}\{z^{-1}\}$ *satisfy the equations*

$$B_1 M_1 + N_1 A_1 = I_l , \quad A_2 N_2 + M_2 B_2 = I_m . \tag{4.57}$$

Proof. The stability of the characteristic polynomial of the closed-loop system is equivalent to stability of the pseudocharacteristic polynomial

$$c = \det C_1 = \det C_2 .$$

Necessity: Let c be stable. Using (4.19) and (4.55) we have

$$K_1 = K_{W/Y} = B_1 C_1^{-1} S_2 ,$$
$$I_l - K_1 = K_{W/E} = R_2 C_2^{-1} A_1 ,$$
$$K_2 = -K_{V/D} = S_1 C_2^{-1} B_2 ,$$
$$I_m - K_2 = K_{V/U} = A_2 C_1^{-1} R_1 .$$

Denoting

$$M_1 = C_1^{-1} S_2 , \quad M_2 = S_1 C_2^{-1} , \quad N_1 = R_2 C_2^{-1} , \quad N_2 = C_1^{-1} R_1 , \tag{4.58}$$

the K_1 and K_2 have indeed the form (4.56). By the assumption that c is stable the M_1, N_1 and M_2, N_2 are stable, i.e. they respectively belong to $\mathfrak{F}^+_{m,l}\{z^{-1}\}$, $\mathfrak{F}^+_{l,l}\{z^{-1}\}$ and $\mathfrak{F}^+_{m,l}\{z^{-1}\}$, $\mathfrak{F}^+_{m,m}\{z^{-1}\}$, and since

$$K_1 + (I_l - K_1) = I_l ,$$
$$(I_m - K_2) + K_2 = I_m ,$$

they satisfy equations (4.57).

Sufficiency: Let

$$K_1 = B_1 M_1 , \qquad K_2 = M_2 B_2 ,$$
$$I_l - K_1 = N_1 A_1 , \quad I_m - K_2 = A_2 N_2 ,$$

where

$$M_1 = C_1^{-1} S_2 \in \mathfrak{F}^+_{m,l}\{z^{-1}\} , \quad M_2 = S_1 C_2^{-1} \in \mathfrak{F}^+_{m,l}\{z^{-1}\} ,$$
$$N_1 = R_2 C_2^{-1} \in \mathfrak{F}^+_{l,l}\{z^{-1}\} , \quad N_2 = C_1^{-1} R_1 \in \mathfrak{F}^+_{m,m}\{z^{-1}\} ,$$

and suppose to the contrary of what is to be proved that c has an unstable factor e, $c = c_0 e$. Then matrices $E_1 \in \mathfrak{F}_{m,m}[z^{-1}]$ and $E_2 \in \mathfrak{F}_{l,l}[z^{-1}]$ exist such that

$$C_1 = E_1 C_{10} , \quad C_2 = C_{20} E_2$$

and

$$e = \det E_1 = \det E_2 .$$

Due to the stability of M_1 and N_2 the E_1 must be cancelled in both M_1 and N_2, i.e. the R_1 and S_2 must have the form

$$R_1 = E_1 R_{10}\,, \quad S_2 = E_1 S_{20}\,.$$

Similarly, due to the stability of N_1 and M_2 the E_2 must be cancelled in both N_1 and M_2 i.e. the S_1 and R_2 must have the form

$$S_1 = S_{10} E_2\,, \quad R_2 = R_{20} E_2\,.$$

By definition, R_1 and S_2 are left coprime and S_1 and R_2 are right coprime. Hence E_1 is a unit of $\mathfrak{F}_{m,m}[z^{-1}]$ and E_2 is a unit of $\mathfrak{F}_{l,l}[z^{-1}]$. It follows that e is a unit of $\mathfrak{F}[z^{-1}]$ and as such it is stable with respect to arbitrary valuation, contradicting our hypothesis. In turn, the c is stable. □

The above theorem specifies just all possible impulse response matrices K_1 and K_2 that yield a stable closed-loop system. Note that conditions (4.56) involve matrices over $\mathfrak{F}[z^{-1}]$ rather than $\mathfrak{F}[z]$. This is highly purposeful and enables to state that M_1, M_2 and N_1, N_2 are *arbitrary* matrices over $\mathfrak{F}^+\{z^{-1}\}$ satisfying (4.57). If the conditions (4.56) were stated in terms of matrices over $\mathfrak{F}[z]$, the M_1, N_1 and M_2, N_2, apart from being stable, would have to make the K_1, $I_l - K_1$ and $K_2, I_m - K_2$ physically realizable. The synthesis procedure would then be unnecessary involved.

It should also be stressed that both N_1 and N_2 are invertible. Indeed, by the assumption on including the delay into $\mathscr{S}$, we have $z^{-1} \mid B$ and hence $z^{-1} \mid B_1$, $z^{-1} \mid B_2$. Then $I_l - K_1$ and $I_m - K_2$ are units of $\mathfrak{F}^+_{l,l}\{z^{-1}\}$ and $\mathfrak{F}^+_{m,m}\{z^{-1}\}$ respectively, and as such they are invertible. Since A_1 and A_2 are invertible, the claim follows by (4.56).

Corollary 4.3. *The matrices* M_1, M_2 *and* N_1, N_2 *defined in* (4.58) *satisfy the following mutual relations*

$$A_2 M_1 = M_2 A_1\,, \qquad (4.59)$$
$$B_1 N_2 = N_1 B_2\,.$$

Proof. The identities

$$R(I_l + SR)^{-1} = (I_m + RS)^{-1} R\,,$$
$$(I_l + SR)^{-1} S = S(I_m + RS)^{-1}$$

can be directly verified. Then

$$K_{W/U} = S_1 C_2^{-1} A_1 = A_2 C_1^{-1} S_2\,,$$
$$K_{V/Y} = B_1 C_1^{-1} R_1 = R_2 C_2^{-1} B_2\,.$$

Taking the definitions in (4.58) into account, relations (4.59) follow. □

If the system $\mathscr{S}$ is *stable*, the statement of Theorem 4.5 greatly simplifies.

Corollary 4.4. *Given the closed-loop system shown in Fig. 5, where $\mathscr{S}$ and $\mathscr{R}$ have the same properties as in Theorem 4.5 but, in addition, the $\mathscr{S}$ is stable. Then the characteristic polynomial of the closed-loop system is stable if and only if the matrix K_1 has the form*

$$K_1 = B_1 M_1 ,$$

where M_1 is an arbitrary element of $\mathfrak{F}^+_{m,l}\{z^{-1}\}$.

Proof. The condition is evidently necessary. To prove sufficiency, observe that $\mathscr{S}$ stable implies that $\det A_1 = \det A_2 \in \mathfrak{F}[z^{-1}]$ is a stable polynomial. Hence A_1^{-1} is a unit of $\mathfrak{F}^+_{l,l}\{z^{-1}\}$ and A_2^{-1} is a unit of $\mathfrak{F}^+_{m,m}\{z^{-1}\}$.

By Corollary 4.3, $A_2 M_1 = M_2 A_1$ and, therefore, M_1 and M_2 are associates in $\mathfrak{F}^+_{m,l}\{z^{-1}\}$. Otherwise speaking, M_1 arbitrary implies that M_2 is also arbitrary to within its associates.

Further set

$$N_{10} = N_1 A_1 , \quad N_{20} = A_2 N_2 .$$

Then N_{10} and N_1 are associates in $\mathfrak{F}^+_{l,l}\{z^{-1}\}$ and N_{20} and N_2 are associates in $\mathfrak{F}^+_{m,m}\{z^{-1}\}$. With this notation, equations (4.57) become

$$B_1 M_1 + N_{10} = I_l ,$$
$$N_{20} + M_2 B_2 = I_m$$

and it is seen that

$$N_{10} = I_l - B_1 M_1 , \quad N_{20} = I_m - M_2 B_2$$

are stable for any M_1 and M_2. Hence also N_1 and N_2 are stable and the hypotheses of Theorem 4.5 are satisfied. It follows that the closed-loop system is stable. □

In other words, for a stable system $\mathscr{S}$ the condition $K_1 = B_1 M_1$ alone already implies all the remaining conditions. This is a striking illustration of how the stability assumption is restrictive.

Example 4.11. Given the system $\mathscr{S}$ which is a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1})^2 \end{bmatrix}}{1 - 2z^{-1}} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} \begin{bmatrix} 1 - 2z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

over the field $\mathfrak{R}$ valuated by (2.25), find all possible impulse response matrices K_1 and K_2 that yield a stable closed-loop system.

We are to solve the equations

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} M_1 + N_1 \begin{bmatrix} 1 - 2z^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 - 2z^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_2 + M_2 \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The general solutions become

$$\boldsymbol{M}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (1-2z^{-1})\,\boldsymbol{t}_{11} & \boldsymbol{t}_{12} \\ \boldsymbol{t}_{21} & \boldsymbol{t}_{22} \end{bmatrix},$$

$$\boldsymbol{N}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} z^{-1}\boldsymbol{t}_{11} & z^{-1}\boldsymbol{t}_{12} \\ z^{-1}\boldsymbol{t}_{21} & z^{-1}(1-2z^{-1})\,\boldsymbol{t}_{22} \end{bmatrix}$$

and

$$\boldsymbol{N}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} z^{-1}\boldsymbol{v}_{11} & z^{-1}\boldsymbol{v}_{12} \\ z^{-1}\boldsymbol{v}_{21} & z^{-1}(1-2z^{-1})\,\boldsymbol{v}_{22} \end{bmatrix},$$

$$\boldsymbol{M}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (1-2z^{-1})\,\boldsymbol{v}_{11} & \boldsymbol{v}_{12} \\ \boldsymbol{v}_{21} & \boldsymbol{v}_{22} \end{bmatrix}$$

for arbitrary elements $\boldsymbol{t}_{ij}$ and $\boldsymbol{v}_{ij}$ of $\mathfrak{R}^+\{z^{-1}\}$.

In order that $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ may be properly generated, these solutions must satisfy mutual conditions (4.59). It follows that

$$\begin{bmatrix} (1-2z^{-1})^2\,\boldsymbol{t}_{11} & (1-2z^{-1})\,\boldsymbol{t}_{12} \\ \boldsymbol{t}_{21} & \boldsymbol{t}_{22} \end{bmatrix} = \begin{bmatrix} (1-2z^{-1})^2\,\boldsymbol{v}_{11} & \boldsymbol{v}_{12} \\ (1-2z^{-1})\,\boldsymbol{v}_{21} & \boldsymbol{v}_{22} \end{bmatrix}$$

and

$$\begin{bmatrix} z^{-2}\boldsymbol{v}_{11} & z^{-2}\boldsymbol{v}_{12} \\ z^{-2}(1-2z^{-1})\,\boldsymbol{v}_{21} & z^{-2}(1-2z^{-1})^2\,\boldsymbol{v}_{22} \end{bmatrix} = \begin{bmatrix} z^{-2}\boldsymbol{t}_{11} & z^{-2}(1-2z^{-1})\,\boldsymbol{t}_{12} \\ z^{-2}\boldsymbol{t}_{21} & z^{-2}(1-2z^{-1})^2\,\boldsymbol{t}_{22} \end{bmatrix}$$

that is,

$$\boldsymbol{v}_{11} = \boldsymbol{t}_{11}\,, \quad \boldsymbol{v}_{12} = (1-2z^{-1})\,\boldsymbol{t}_{12}\,,$$
$$(1-2z^{-1})\,\boldsymbol{v}_{21} = \boldsymbol{t}_{21}\,, \quad \boldsymbol{v}_{22} = \boldsymbol{t}_{22}\,.$$

Thus

$$\boldsymbol{M}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (1-2z^{-1})\,\boldsymbol{t}_{11} & \boldsymbol{t}_{12} \\ (1-2z^{-1})\,\boldsymbol{v}_{21} & \boldsymbol{v}_{22} \end{bmatrix},$$

$$\boldsymbol{N}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} z^{-1}\boldsymbol{t}_{11} & z^{-1}\boldsymbol{t}_{12} \\ z^{-1}(1-2z^{-1})\,\boldsymbol{v}_{21} & z^{-1}(1-2z^{-1})\,\boldsymbol{v}_{22} \end{bmatrix},$$

and

$$\boldsymbol{N}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} z^{-1}\boldsymbol{t}_{11} & z^{-1}(1-2z^{-1})\,\boldsymbol{t}_{12} \\ z^{-1}\boldsymbol{v}_{21} & z^{-1}(1-2z^{-1})\,\boldsymbol{v}_{22} \end{bmatrix},$$

$$\boldsymbol{M}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (1-2z^{-1})\,\boldsymbol{t}_{11} & (1-2z^{-1})\,\boldsymbol{t}_{12} \\ \boldsymbol{v}_{21} & \boldsymbol{v}_{22} \end{bmatrix},$$

for arbitrary $\boldsymbol{t}_{11}$, $\boldsymbol{t}_{12}$ and $\boldsymbol{v}_{21}$, $\boldsymbol{v}_{22}$ belonging to $\mathfrak{R}^+\{z^{-1}\}$.

All admissible K_1 have the form

$$K_1 = \begin{bmatrix} 2z^{-1} + z^{-1}(1 - 2z^{-1})\,t_{11} & z^{-1}t_{12} \\ z^{-1}(1 - 2z^{-1})^2\, v_{21} & z^{-1}(1 - 2z^{-1})\, v_{22} \end{bmatrix}$$

and all admissible K_2 have the form

$$K_2 = \begin{bmatrix} 2z^{-1} + z^{-1}(1 - 2z^{-1})\,t_{11} & z^{-1}(1 - 2z^{-1})^2\, t_{12} \\ z^{-1}v_{21} & z^{-1}(1 - 2z^{-1})\, v_{22} \end{bmatrix}$$

on using (4.56).

In particular, note that only the first (or the second) equation (4.57) alone is not sufficient to guarantee stability, even though the system $\mathscr{S}$ is diagonal! Indeed, the matrices

$$M_1 = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix}$$

satisfy the first equation (4.57) but they yield the controller

$$R = \begin{bmatrix} 2(1 - 2z^{-1}) & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 - 2z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -(1 - 2z^{-1}) \\ 1 & 2(1 - 2z^{-1}) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and the pseudocharacteristic polynomial

$$c = \det \begin{bmatrix} z^{-1} & -(1 - 2z^{-1}) \\ 1 - 2z^{-1} & 2(1 - 2z^{-1} \end{bmatrix} = \det \begin{bmatrix} 1 - 2z^{-1} & 0 \\ -z^{-1}(1 - 2z^{-1}) & 1 \end{bmatrix} = 1 - 2z^{-1}\,,$$

which is *not* stable.

An interesting interpretation of the above results is as follows.

Theorem 4.6. *Given the closed-loop system shown in Fig. 5, where $\mathscr{S}$ is a minimal realization of*

$$S = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}$$

and $\mathscr{R}$ is a minimal realization of

$$R = S_1 R_2^{-1} = R_1^{-1} S_2 \in \mathfrak{F}_{m,l}\{z^{-1}\}\,.$$

Write

$$M_1 = C_1^{-1} S_2 = M_{11}^{-1} M_{12}\,, \quad M_2 = S_1 C_2^{-1} = M_{21} M_{22}^{-1}\,,$$

$$N_1 = R_2 C_2^{-1} = N_{11} N_{12}^{-1}\,, \quad N_2 = C_1^{-1} R_1 = N_{21}^{-1} N_{22}$$

where matrices M_{11} and M_{12} as well as M_{21} and N_{22} are left coprime while matrices M_{21} and M_{22} as well as N_{11} and N_{12} are right coprime.

Let

(4.60) D_{11} = *greatest common left divisor of* S_1 *and* A_2 ,

D_{12} = *greatest common right divisor of* R_1 *and* B_2 ,

D_{21} = *greatest common left divisor of* B_1 *and* R_2 ,

D_{22} = *greatest common right divisor of* A_1 *and* S_2 .

Then

(4.61)
$$c = \det D_{11} \,.\, \det M_{11} = \det N_{12} \,.\, \det D_{12} = \\ = \det D_{21} \,.\, \det N_{21} = \det M_{22} \,.\, \det D_{22}$$

up to units of $F[z^{-1}]$.

Proof. We shall prove the first two identities in (4.61), the remaining ones can be proved analogously.

By definition,

$$S_1 = D_{11}S_{10} ,$$
$$A_2 = D_{11}A_{20} .$$

Note that

(4.62)
$$A_2^{-1}S_1R_2^{-1} = A_2^{-1}R_1^{-1}S_2 .$$

Since D_{11} is cancelled on the left-hand side of (4.62), a matrix $F_{11} \in \mathfrak{F}_{m,m}[z^{-1}]$ such that $\det F_{11} = \det D_{11}$ must be cancelled on the right-hand side of (4.62). Hence F_{11} is a greatest common right divisor of R_1A_2 and S_2, and

(4.63)
$$C_1 = R_1A_2 + S_2B_1$$

implies that F_{11} is also a greatest common left divisor of C_1 and S_2.

Then

$$M_1 = C_1^{-1}S_2 = M_{11}^{-1}M_{12}$$

yields

$$c = \det C_1 = \det F_{11} \det M_{11} = \det D_{11} \det M_{11}$$

up to units of $\mathfrak{F}[z^{-1}]$.

Similarly,

$$R_1 = R_{10}D_{12} ,$$
$$B_2 = B_{20}D_1$$

by definition. Note that

(4.64)
$$B_2R_1^{-1}S_2 = B_2S_1R_2^{-1} .$$

Since D_{12} is cancelled on the lef-hand side of (4.64), a matrix $G_{12} \in F_{l,l}[z^{-1}]$ such that det G_{12} = det D_{12} must be cancelled on the right-hand side of (4.64). Hence G_{12} is a greatest common right divisor of B_2S_1 and R_2, and

$$C_2 = A_1R_2 + B_2S_1 \tag{4.65}$$

implies that G_{12} is also a greatest common right divisor of C_2 and R_2.
Then

$$N_1 = R_2C_2^{-1} = N_{11}N_{12}^{-1}$$

yields

$$c = \det C_2 = \det N_{12} \det G_{12} = \det N_{12} \det D_{12} .$$

up to units of $\mathfrak{F}[z^{-1}]$. □

We recall that if $l = m = 1$ (single-input single-output system) then

$$SR = RS = \frac{b}{a}\frac{s}{r}$$

and the polynomials $D_{11} = D_{22} = (a, s)$ and $D_{12} = D_{21} = (b, r)$ can be interpreted [33] as the "zero-pole" cancellations, i.e. as factors cancelled from the numerator and denominator polynomials in the cascade $\mathcal{SR} = \mathcal{RS}$.

In the multivariable case, we have

$$SR = B_1A_2^{-1}S_1R_2^{-1} = A_1^{-1}B_2R_1^{-1}S_2 ,$$
$$RS = S_1R_2^{-1}B_1A_2^{-1} = R_1^{-1}S_2A_1^{-1}B_2$$

and, therefore, matrices D_{11}, D_{12}, D_{21} and D_{22} in (4.60) can be interpreted as the *matrix* "zero-pole" cancellations between the numerator and denominator matrices in the cascades $\mathcal{SR}$ and $\mathcal{RS}$. Whenever any of these calcellations occurs the closed-loop system is not a minimal realization of the respective impulse response matrix.

In view of this interpretation we can say that the closed-loop system is stable if and only if both K_1 and K_2 are stable and no unstable "zero-pole" matrix cancellations occur. In fact, Theorem 4.5 guarantees the closed-loop stability just by prohibiting such cancellations.

We have to make distinction between the "zero-pole" calcellations defined above, which are cancellations between *polynomial* matrices, and the cancellations of *rational* matrices in the cascades $\mathcal{SR}$ or $\mathcal{RS}$. Example:

$$S = \frac{[z^{-1} \quad -z^{-1}(z^{-1} - 2)]}{1 - z^{-1}} =$$

$$= [1 - z^{-1}]^{-1} [z^{-1} \quad -z^{-1}(z^{-1} - 2)] = [z^{-1} \quad 0] \begin{bmatrix} 1 - z^{-1} & z^{-1} - 2 \\ 0 & 1 \end{bmatrix}^{-1} ,$$

$$R = \frac{\begin{bmatrix} 1 - z^{-1} \\ 1 \end{bmatrix}}{z^{-1} - 2} = \begin{bmatrix} 1 - z^{-1} \\ 1 \end{bmatrix} [z^{-1} - 2]^{-1} = \begin{bmatrix} 0 & z^{-1} - 2 \\ 1 & z^{-1} - 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We can write

$$S = \begin{bmatrix} z^{-1} & -\dfrac{z^{-1}}{1 - z^{-1}} \end{bmatrix} \begin{bmatrix} \dfrac{1}{1 - z^{-1}} & 0 \\ 0 & z^{-1} - 2 \end{bmatrix},$$

$$R = \begin{bmatrix} \dfrac{1}{1 - z^{-1}} & 0 \\ 0 & z^{-1} - 2 \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{1}{z^{-1} - 2} \\ 1 \end{bmatrix},$$

that is, the rational matrix

$$\begin{bmatrix} \dfrac{1}{1 - z^{-1}} & 0 \\ 0 & z^{-1} - 2 \end{bmatrix}$$

cancels in the cascade $\mathscr{S}\mathscr{R}$, yet no "zero-pole" cancellations occur!

Example 4.12. Consider the systems $\mathscr{S}$ and $\mathscr{R}$ over the field $\mathfrak{R}$ that are minimal realizations of

$$S = \frac{[z^{-1}(1 - z^{-1}) \quad -z^{-1}(z^{-1} - 2)]}{(1 - z^{-1})(z^{-1} - 2)} =$$

$$= [(1 - z^{-1})(z^{-1} - 2)]^{-1} [z^{-1}(1 - z^{-1}) \quad - z^{-1}(z^{-1} - 2)] =$$

$$= [z^{-1} \quad 0] \begin{bmatrix} -(1 - z^{-1})(z^{-1} - 2) & -(z^{-1} - 2) \\ (1 - z^{-1})(z^{-1} - 2) & -(1 - z^{-1}) \end{bmatrix}^{-1}$$

and

$$R = \frac{\begin{bmatrix} z^{-1} - 2 \\ 1 - z^{-1} \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} -(1 - z^{-1}) & -(1 - z^{-1}) \\ -(1 - z^{-1}) & z^{-1} - 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} z^{-1} - 2 \\ 1 - z^{-1} \end{bmatrix} [1 - z^{-1}]^{-1}$$

respectively and analyze the "zero-pole" cancellations.

We have

$$A_2 = \begin{bmatrix} -(1 - z^{-1})(z^{-1} - 2) & -(z^{-1} - 2) \\ (1 - z^{-1})(z^{-1} - 2) & -(1 - z^{-1}) \end{bmatrix} =$$

$$= \begin{bmatrix} z^{-1} - 2 & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} \begin{bmatrix} -(1 - z^{-1}) & -1 \\ z^{-1} - 2 & -1 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} z^{-1} - 2 \\ 1 - z^{-1} \end{bmatrix} = \begin{bmatrix} z^{-1} - 2 & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$B_2 = [z^{-1}(1 - z^{-1}) \;\; -z^{-1}(z^{-1} - 2)] = [z^{-1} \;\; -z^{-1}(z^{-1} - 2)]\begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

$$R_1 = \begin{bmatrix} -(1 - z^{-1}) & -(1 - z^{-1}) \\ -(1 - z^{-1}) & z^{-1} - 2 \end{bmatrix} = \begin{bmatrix} -1 & -(1 - z^{-1}) \\ -1 & z^{-1} - 2 \end{bmatrix}\begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_1 = [z^{-1} \;\; 0],$$

$$R_2 = [1 - z^{-1}],$$

$$A_1 = [(1 - z^{-1})(z^{-1} - 2)],$$

$$S_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and hence

$$D_{11} = \begin{bmatrix} z^{-1} - 2 & 0 \\ 0 & 1 - z^{-1} \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_{21} = [1], \qquad D_{22} = [1]$$

are the "zero-pole" matrix cancellations.

Indeed,

$$C_1 = \begin{bmatrix} z^{-1} & -(1 - z^{-1}) \\ -(1 - z^{-1})(z^{-1} - 2) & 0 \end{bmatrix}$$

$$C_2 = [(1 - z^{-1})^2 (z^{-1} - 2)]$$

$$c = (1 - z^{-1})^2 (z^{-1} - 2),$$

$$\boldsymbol{M}_1 = \frac{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}{1 - z^{-1}}, \quad \boldsymbol{N}_1 = \frac{1}{(1 - z^{-1})(z^{-1} - 2)},$$

$$\boldsymbol{M}_2 = \frac{\begin{bmatrix} z^{-1} - 2 \\ 1 - z^{-1} \end{bmatrix}}{(1 - z^{-1})^2 (z^{-1} - 2)},$$

$$\boldsymbol{N}_2 = \frac{\begin{bmatrix} (1 - z^{-1})^2 & -(1 - z^{-1})(z^{-1} - 2) \\ -(1 - z^{-1})(z^{-2} - 4z^{-1} + 2) & (z^{-1} - 2)(1 - 3z^{-1} + z^{-2}) \end{bmatrix}}{(1 - z^{-1})^2 (z^{-1} - 2)},$$

$$\boldsymbol{K}_1 = [0], \quad I_l - \boldsymbol{K}_1 = [1],$$

$$\boldsymbol{K}_2 = \frac{\begin{bmatrix} z^{-1}(1 - z^{-1})(z^{-1} - 2) & -z^{-1}(z^{-1} - 2)^2 \\ z^{-1}(1 - z^{-1})^2 & -z^{-1}(1 - z^{-1})(z^{-1} - 2) \end{bmatrix}}{(1 - z^{-1})^2 (z^{-1} - 2)},$$

$$I_m - K_2 = \frac{\begin{bmatrix} (1 - z^{-1})(z^{-1} - 2)(1 - 2z^{-1}) & z^{-1}(z^{-1} - 2)^2 \\ -z^{-1}(1 - z^{-1})^2 & (1 - z^{-1})(z^{-1} - 2) \end{bmatrix}}{(1 - z^{-1})^2 (z^{-1} - 2)}$$

and the closed-loop system is a nonminimal realization of both K_1 and $I_l - K_1$ while it is a minimal realization of K_2 and $I_m - K_2$. Note that

$$\det D_{11} = (1 - z^{-1})(z^{-1} - 2),$$
$$\det D_{12} = 1 - z^{-1}$$

and

$$\det M_{11} = \frac{c}{\det D_{11}} = 1 - z^{-1}$$

$$\det N_{12} = \frac{c}{\det D_{12}} = (1 - z^{-1})(z^{-1} - 2).$$

4.5. The existence of a stabilizing feedback

We have seen that given a system $\mathscr{S}$ it is not always possible to make the closed-loop characteristic polynomial equal to an arbitrary polynomial. The question now is whether or not the characteristic polynomial can be made stable. The affirmative answer is plausible but the author is not aware of any direct proof.

Theorem 4.7. *Given the system $\mathscr{S}$ as a minimal realization of*

$$S = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}$$

where $\mathfrak{F}$ is an arbitrary field with valuation $\mathscr{V}$, then a controller $\mathscr{R}$ which is a minimal realization of some

$$R \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

always exists such that the closed-loop system shown in Fig. 5 is stable (with respect to $\mathscr{V}$).

Proof. We recall (2.4) that

$$B_1 = E_1 \operatorname{diag}\{b_1, b_2, \dots, b_r, 0, \dots, 0\},$$
$$A_2 = E_2^{-1} \operatorname{diag}\{a_1, a_2, \dots, a_r, 1, \dots, 1\},$$
$$A_1 = \operatorname{diag}\{a_1, a_2, \dots, a_r, 1, \dots, 1\} E_1^{-1},$$
$$B_2 = \operatorname{diag}\{b_1, b_2, \dots, b_r, 0, \dots, 0\} E_2.$$

Hence equations (4.57) are equivalent to the set of polynomial equations

$$b_i \hat{\boldsymbol{m}}^1_{ij} + \hat{\boldsymbol{n}}^1_{ij} a_j = \delta_{ij}\,, \quad i, j = 1, 2, \ldots, l\,,$$
$$a_p \hat{\boldsymbol{n}}^2_{pq} + \hat{\boldsymbol{m}}^2_{pq} b_q = \delta_{pq}\,, \quad p, q = 1, 2, \ldots, m\,,$$

where $b_k = 0,\ a_k = 1$ for $k > r$ and

$$\begin{aligned} \delta_{kn} &= 1 \quad \text{for} \quad k = n\,, \\ &= 0 \quad \text{for} \quad k \neq n\,. \end{aligned}$$

These equations have a solution if and only if $(a_k, b_n) \mid \delta_{kn}$ for all k, n and this condition is always satisfied since $(a_k, b_k) = 1$ by definition.

Further, mutual conditions (4.59) are equivalent to the polynomial equations

$$a_i \hat{\boldsymbol{m}}^1_{ij} = \hat{\boldsymbol{m}}^2_{ij} a_j\,, \quad i = 1, 2, \ldots, m\,, \quad j = 1, 2, \ldots, l\,,$$
$$b_p \hat{\boldsymbol{n}}^2_{pq} = \hat{\boldsymbol{m}}^2_{pq} b_q\,, \quad p = 1, 2, \ldots, l\,, \quad q = 1, 2, \ldots, m\,,$$

which can always be satisfied.

Therefore, elements $\boldsymbol{M}_1 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$, $\boldsymbol{N}_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ and $\boldsymbol{M}_2 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$, $\boldsymbol{N}_2 \in \mathfrak{F}^+_{m,m}\{z^{-1}\}$ always exist that satisfy equations (4.57) and (4.59). Then the impulse response matrices

$$\boldsymbol{K}_1 = B_1 \boldsymbol{M}_1\,, \quad \boldsymbol{K}_2 = \boldsymbol{M}_2 B_2$$

satisfy the hypothesis of Theorem 4.5 and hence the closed-loop system is stable. □

All stabilizing controllers $\mathcal{R}$ are given as minimal realizations of

$$\boldsymbol{R} = \boldsymbol{M}_2 \boldsymbol{N}_1^{-1} = \boldsymbol{N}_2^{-1} \boldsymbol{M}_1\,. \tag{4.66}$$

Indeed, using (4.19) and (4.56),

$$B_1 \boldsymbol{M}_1 = \boldsymbol{K}_1 = \boldsymbol{SR}(\boldsymbol{I}_l + \boldsymbol{SR})^{-1} = \boldsymbol{SR}(\boldsymbol{I}_l - \boldsymbol{K}_1) = B_1 A_2^{-1} \boldsymbol{R} \boldsymbol{N}_1 A_1$$

and hence

$$\boldsymbol{R} = A_2 \boldsymbol{M}_1 A_1^{-1} \boldsymbol{N}_1^{-1} = \boldsymbol{M}_2 A_1 A_1^{-1} \boldsymbol{N}_1^{-1} = \boldsymbol{M}_2 \boldsymbol{N}_1^{-1}$$

by (4.59). Similarly, using (4.19) and (4.56),

$$\boldsymbol{M}_2 B_2 = \boldsymbol{K}_2 = (\boldsymbol{I}_m + \boldsymbol{RS})^{-1}\, \boldsymbol{RS} = (\boldsymbol{I}_m = \boldsymbol{K}_2)\, \boldsymbol{RS} = A_2 \boldsymbol{N}_2 \boldsymbol{R} A_1^{-1} B_2$$

and hence

$$\boldsymbol{R} = \boldsymbol{N}_2^{-1} A_2^{-1} \boldsymbol{M}_2 A_1 = \boldsymbol{N}_2^{-1} A_2^{-1} A_2 \boldsymbol{M}_1 = \boldsymbol{N}_2^{-1} \boldsymbol{M}_1$$

by (4.59).

Example 4.13. Given the system $\mathcal{S}$ as a minimal realization of

$$S = \begin{bmatrix} z^{-1} & \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} \begin{bmatrix} 1 - 2z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

over the field $\mathfrak{R}$ valuated by (2.25), find all stabilizing controllers.

The system has been considered in Example 4.11. All stabilizing controllers are given by (4.66) as minimal realizations of

$$R = M_2 N_1^{-1} = N_2^{-1} M_1 ,$$

where

$$M_1 = \begin{bmatrix} 2 + (1 - 2z^{-1}) t_{11} & t_{12} \\ (1 - 2z^{-1}) v_{21} & v_{22} \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1 - z^{-1} t_{11} & - z^{-1} t_{12} \\ - z^{-1}(1 - 2z^{-1}) v_{21} & 1 - z^{-1}(1 - 2z^{-1}) v_{22} \end{bmatrix}$$

and

$$N_2 = \begin{bmatrix} 1 - z^{-1} t_{11} & - z^{-1}(1 - 2z^{-1}) t_{12} \\ - z^{-1} v_{21} & 1 - z^{-1}(1 - 2z^{-1}) v_{22} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 2 + (1 - 2z^{-1}) t_{11} & (1 - 2z^{-1}) t_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

for arbitrary t_{11}, t_{12} and v_{21}, v_{22} belonging to $\mathfrak{R}^+\{z^{-1}\}$.

5. CLOSED-LOOP CONTROL

5.1. Problem formulation

This chapter is devoted to the synthesis of optimal closed-loop control systems. The configuration of the closed-loop system considered here is shown in Fig. 10. The $\mathcal{S}$ denotes the system to be controlled, $\mathcal{R}$ is the controller, and W is a given

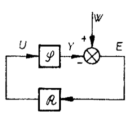

Fig.10. The closed-loop control configuration.

reference sequence. The fundamental properties of the closed-loop system have been discussed in Chapter 4, now we concentrate on solving the optimal control problems.

Roughly speaking, the closed-loop optimal control consists in the following. Given a system $\mathcal{S}$, find a controller $\mathcal{R}$ such that the closed-loop system is stable and an optimality criterion is minimized. The same optimality criteria as for the open-lopp control will be considered here, viz. the stable time optimal control, the finite time optimal control, and the least squares control.

The basic and most important condition is that the closed-loop system be stable. It makes it possible to counteract disturbances appearing anywhere in the control loop simply by making them decay exponentially.

It is appropriate to make the following remark at this early stage of development. The controller $\mathcal{R}$ couples the $\boldsymbol{E}$ and $\boldsymbol{U}$ as

(5.1) $$\boldsymbol{U} = \boldsymbol{R}\boldsymbol{E}\,.$$

One might think of closing the loop by simply feeding back the error of the optimal *open-loop* control to get the closed-loop system, i.e. finding any transfer function matrix R satisfying (5.1) with $\boldsymbol{U}$ and $\boldsymbol{E}$ obtained via the methods discussed in Chapter 3. This is not acceptable, however. The resulting controller need not exist or need not be physically realizable. To make the matters worse, if such a physically realizable controller does exist, it may not yield a stable closed-loop system. By (4.66), only the controllers given as minimal realizations of

$$\boldsymbol{R} = \boldsymbol{M}_2 \boldsymbol{N}_1^{-1} = \boldsymbol{N}_2^{-1} \boldsymbol{M}_1\,,$$

where $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ and $\boldsymbol{N}_1$, $\boldsymbol{N}_2$, satisfy the hypothesis of Theorem 4.5, will create a stable closed-loop system. Thus special synthesis procedures have to be developed to produce the closed-loop optimal control systems.

Theorem 4.5 itself suggests that first all possible closed-loop transfer function matrices yielding a stable system should be determined and then the remaining degrees of freedom should be used to minimize some criterion.

The exact formulation of the optimal control problems is given below.

(5.2) *Stable time optimal control problem*:

Given a system $\mathcal{S}$ which is a minimal realization of

$$\boldsymbol{S} = \frac{\boldsymbol{B}}{a} \in \mathfrak{F}_{l,m}\{z^{-1}\}\,, \quad \boldsymbol{B} \neq 0\,,$$

and a reference sequence

$$\boldsymbol{W} = \frac{\boldsymbol{Q}}{p} \in \mathfrak{F}_{l,1}\{z^{-1}\}\,, \quad \boldsymbol{Q} \neq 0\,.$$

Find a controller $\mathcal{R}$ which is a minimal realization of some

$$\mathcal{R} \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the closed-loop system is stable, the control sequence $\boldsymbol{U}$ is stable, and the error sequence $\boldsymbol{E}$ vanishes in a minimum time $k_{\min}$ and thereafter.

(5.3) *Finite time optimal control problem*:

Given a system $\mathscr{S}$ which is a minimal realization of

$$S = \frac{B}{a} \in \mathfrak{F}_{l,m}\{z^{-1}\}, \quad B \neq 0,$$

and a reference sequence

$$W = \frac{Q}{p} \in \mathfrak{F}_{l,1}\{z^{-1}\}, \quad Q \neq 0.$$

Find a controller $\mathscr{R}$ which is a minimal realization of some

$$\mathscr{R} \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the closed-loop system is stable, the control sequence $\boldsymbol{U}$ is finite, and the error sequence $\boldsymbol{E}$ vanishes in a minimum time $k_{\min}$ and thereafter.

(5.4) *Least squares control problem*:

Given a system $\mathscr{S}$ which is a minimal realization of

$$S = \frac{B}{a} \in \mathfrak{F}_{l,m}\{z^{-1}\}, \quad B \neq 0,$$

and a reference sequence

$$W = \frac{Q}{p} \in \mathfrak{F}_{l,1}\{z^{-1}\}, \quad Q \neq 0.$$

Find a controller $\mathscr{R}$ which is a minimal realization of some

$$\boldsymbol{R} \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the closed-loop system is stable, the control sequence $\boldsymbol{U}$ is stable, and the quadratic norm $\|\boldsymbol{E}\|^2$ of the error sequence $\boldsymbol{E}$ is minimized.

It is to be noted that the control sequence $\boldsymbol{U}$ is required to be stable in all control problems. This is rather a strict assumption motivated by physical realizability of the optimal control. However, an optimal control which is bounded instead of stable may be well acceptable in the engineering practice. This it to be born in mind when applying the synthesis procedure.

It is also essential that both $\mathscr{S}$ and $\mathscr{R}$ be *minimal realizations* of $\boldsymbol{S}$ and $\boldsymbol{R}$, respectively. Otherwise the actual closed-loop system characteristic polynomial would be different from $\hat{c} = \det \hat{C}_1 = \det \hat{C}_2$ and the method of synthesis could not guarantee a stable closed-loop system.

It is easy and transparent to find a minimal realization of S when $\mathscr{S}$ is a single-input single-output system. However, the problem becomes quite difficult for multivariable systems. For instance, realizing each element s_{ij} of S or r_{ij} of R separately almost always leads to a nonminimal realization and the general procedure described in Chapter 2 is recommended.

An interesting feature of the closed-loop control is the inherent nonuniquenes of the optimal controller. More specifically, the optimal control and error sequences are, as a rule, unique but they are generated by many and many controllers. Hence the closed-loop system transfer function K_1 and the characteristic polynomial are not unique, either. This phenomenon makes the synthesis depend upon somewhat arbitrary choices and, therefore, more complicated and less suited for machine processing. On the other hand, it leaves more room for the engineer to realize the synthesized system according to additional requirements. The author is not aware of any systematic description of this effect in the literature. In fact the closed-loop optimal control problems (5.2), (5.3), and (5.4) have never been solved in general. The only exception is the solution for single-variable systems in [30, 31, 32, 34] and a very restricted solution of multivariable problems (5.2), (5.3) in [55] and (5.4) in [60].

5.2. Stable time optimal control problem

Let $\mathfrak{F}$ be an arbitrary field with valuation $\mathscr{V}$ and write

$$S = \frac{B}{a} = B_1 A_2^{-1} = A_1^{-1} B_2 \,,$$

$$\operatorname{rank} B_1 = \operatorname{rank} B_2 = r$$

and

$$B_1 = B_1^- B_1^+ \,.$$

By the definition of B_1^- in (2.30) we have

$$B_1^- = [B_{11}^- \;\; 0]$$

where

$$B_{11}^- \in \mathfrak{F}_{l,r}[z^{-1}] \,, \quad 0 \in \mathfrak{F}_{l,m-r}[z^{-1}] \text{ and } \operatorname{rank} B_{11}^- = r \,.$$

We also write

$$Q = Q^+ Q^-$$

where

$$Q^- = \begin{bmatrix} q^- \\ 0 \end{bmatrix}$$

with

$$q^- \in \mathfrak{F}_{1,1}[z^{-1}], 0 \in \mathfrak{F}_{l-1,1}[z^{-1}]$$

and denote

$$Q_1^+ = Q^+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{Q}{q^-} .$$

For convenience, let

$$A_1 \frac{Q}{p} = \frac{F}{p_0} ,$$

where $(p_0, F) = 1$ and write

$$F = F^+ F^- ,$$

where

$$F^- = \begin{bmatrix} f^- \\ 0 \end{bmatrix}$$

with $f^- \in F_{1,1}[z^{-1}]$, $0 \in \mathfrak{F}_{l-1,1}[z^{-1}]$, and denote

$$f^- = f_0^- q^- .$$

Then we have the following result.

Theorem 5.1. *Problem* (5.2) *has a solution if and only if the linear Diophantine equation*

(5.5) $$B_{11}^- X + Y p f_0^- = Q_1^+$$

has a solution X^0, Y^0 *such that* $\partial Y^0 = \min$ *subject to matrices* M_1, N_1 *and* M_2, N_2 *exist in* $\mathfrak{F}_{m,l}^+\{z^{-1}\}$, $\mathfrak{F}_{l,l}^+\{z^{-1}\}$ *and* $\mathfrak{F}_{m,l}^+\{z^{-1}\}$, $\mathfrak{F}_{m,m}^+\{z^{-1}\}$ *respectively and satisfy the following equations*

(5.6) $$B_1 M_1 + N_1 A_1 = I_l , \quad A_2 N_2 + M_2 B_2 = I_m ,$$

(5.7) $$A_2 M_1 = M_2 A_1 , \quad B_1 N_2 = N_1 B_2 ,$$

(5.8) $$M_{11} = X^0 , \quad B_1^+ M_1 Q^+ = \begin{bmatrix} M_{11} & M_{22} \\ M_{21} & M_{22} \end{bmatrix} , \quad N_{11} = Y^0 p_0 , \quad N_1 F^+ = [N_{11} \; N_{12}]$$

and also subject to

(5.9) $$U = M_2 \frac{F}{p_0} ,$$

belongs to $\mathfrak{F}_{m,1}^+\{z^{-1}\}$.

The optimal controller is not unique, in general, and all optimal controllers are given as minimal realizations of

$$R = M_2 N_1^{-1} = N_2^{-1} M_1 .$$

Moreover, U *is given by* (5.9) *and*

$$E = Y^0 f^- ,$$
$$k_{\min} = 1 + \partial Y^0 + \partial f^- .$$

Proof. The error is given as

$$E = K_{W/E} W = (I_l - K_1) W .$$

To guarantee a stable closed-loop system we have to set

$$I_l = K_1 = N_1 A_1 ,$$

where $N_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$. It follows that

$$E = N_1 A_1 \frac{Q}{p} = N_1 \frac{F}{p_0} = [N_{11} \ \ N_{12}] \frac{\begin{bmatrix} f^- \\ 0 \end{bmatrix}}{p_0} = N_{11} \frac{f^-}{p_0} ,$$

where

$$N_1 F^+ = [N_{11} \ \ N_{12}]$$

and

$$N_{11} \in \mathfrak{F}^+_{l,1}\{z^{-1}\} , \quad N_{12} \in \mathfrak{F}^+_{l,l-1}\{z^{-1}\} .$$

Since the error sequence is to vanish in a finite time and thereafter, E must be a matrix polynomial in $\mathfrak{F}_{l,1}[z^{-1}]$. Therefore,

$$N_{11} = Y p_0 , \tag{5.10}$$

where $Y \in \mathfrak{F}_{l,1}[z^{-1}]$ is a matrix polynomial to be specified later. This choice yields the error

$$E = Y f^- . \tag{5.11}$$

The error is also given as

$$E = W - K_1 W$$

and, in order to guarantee a stable system, we have to set

$$K_1 = B_1 M_1 ,$$

where $M_1 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$. Then

$$(5.12)\quad pE = Q - B_1 M_1 Q = Q - [B^-_{11}\ 0]\, B^+_1 M_1 Q^+ \begin{bmatrix} q^- \\ 0 \end{bmatrix} = Q - B^-_{11} M_{11} q^- ,$$

where

$$B^+_1 M_1 Q^+ = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

and

$$M_{11} \in \mathfrak{F}^+_{r,1}\{z^{-1}\},\ M_{12} \in \mathfrak{F}^+_{r,l-1}\{z^{-1}\},\ M_{21} \in \mathfrak{F}^+_{m-r,1}\{z^{-1}\},\ M_{22} \in \mathfrak{F}^+_{m-r,l-1}\{z^{-1}\} .$$

The E is a matrix polynomial whenever pE is so. It follows, that $B^-_{11} M_{11} q^-$ must be a matrix polynomial, too. This is effected by the choice

$$(5.13)\qquad M_{11} = X ,$$

where $X \in \mathfrak{F}_{r,1}[z^{-1}]$ is an unspecified matrix polynomial as yet.

In fact, substituting (5.11) into (5.12) we end up with equation (5.5) coupling the X and Y.

To guarantee the closed-loop stability, the M_1 and N_1 must satisfy the equation

$$B_1 M_1 + N_1 A_1 = I_l$$

in addition to (5.10) and (5.13), see Theorem 4.5. However, we must also solve the equation

$$A_2 N_2 + M_2 B_2 = I_m$$

for $M_2 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$ and $N_2 \in \mathfrak{F}^+_{m,m}\{z^{-1}\}$ and in order that the four matrices may be properly related they must further satisfy the mutual relations

$$A_2 M_1 = M_2 A_1 ,$$

$$B_1 N_2 = N_1 B_2 .$$

We must take, therefore, only those solutions of equation (5.5) that make the above specified M_1, N_1 and M_2, N_2 exist. Further, we must take only those solutions which make the control sequence

$$U = K_{W/U} W = A_2 M_1 \frac{Q}{p} = M_2 A_1 \frac{Q}{p} = M_2 \frac{F}{p_0}$$

stable, as required. And within this class we must further confine ourselves to those solutions which minimize the degree of E. Therefore, in view of (5.11), equation (5.5) is to be solved for a solution X^0, Y^0 such that $\partial Y^0 = \min$ subject to all stability requirements.

All optimal controllers are then obtained by (4.66) as minimal realizations of

$$R = M_1 N_1^{-1} = N_2^{-1} M_1 ,$$

where M_1, N_1 and M_2, N_2 satisfy (5.6), (5.7) and (5.8).

The optimal performance measure becomes

$$k_{\min} = 1 + \partial E = 1 + \partial Y^0 + \partial f^-$$

in view of (5.11). Since it is assumed that $z^{-1} \mid B$, we always have $Y^0 \neq 0$. □

Example 5.1. Given the system $\mathscr{S}$ over the field $\mathfrak{R}$ valuated by (2.25) as a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} & z^{-1} \\ z^{-1} & z^{-1} \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} & 0 \\ z^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & -1 \\ 0 & 1 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 - z^{-1} & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} & z^{-1} \\ 0 & 0 \end{bmatrix},$$

solve problem (5.2) for the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{1 - z^{-1}} .$$

We compute

$$B_{11}^- = \begin{bmatrix} z^{-1} \\ z^{-1} \end{bmatrix}, \quad Q^+ = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad q^- = 1 , \quad Q_1^+ = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$F^+ = \begin{bmatrix} 1 - z^{-1} & 0 \\ -2 & 1 \end{bmatrix}, \quad f^- = 1 , \quad f^- = 1$$

and hence equation (5.5) becomes

$$\begin{bmatrix} z^{-1} \\ z^{-1} \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{5.14}$$

Since the matrix

$$\begin{bmatrix} B_{11}^- & 0 \\ 0 & pf_0^- \end{bmatrix} = \begin{bmatrix} z^{-1} & 0 \\ z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix}$$

has the invariant polynomials 1, $z^{-1}(1 - z^{-1})$ and the matrix

$$\begin{bmatrix} B_{11}^{-} & Q_1^{+} \\ 0 & pf_0^{-} \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \\ z^{-1} & -1 \\ 0 & 1 - z^{-1} \end{bmatrix}$$

has the invariant polynomials 1, z^{-1}, equation (5.14) has no solution. Therefore, our problem has no solution.

Example 5.2. Consider a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} & z^{-1} \\ 0 & z^{-1}(1 - 2z^{-1})(z^{-1} - 2) \end{bmatrix}}{1 - z^{-1}} =$$

$$= \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1})(z^{-1} - 2) \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & -(1 - z^{-1}) \\ 0 & 1 - z^{-1} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} & z^{-1} \\ 0 & z^{-1}(1 - 2z^{-1})(z^{-1} - 2) \end{bmatrix}$$

over $\mathfrak{R}$ valuated by (2.25) and solve problem (5.2) for the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix}}{1 - z^{-1}}.$$

We shall first find all matrices M_1 and N_1 that satisfy the equation

$$B_1 M_1 + N_1 A_1 = I_l .$$

It is equivalent to the set of equations

$$z^{-1} m_{1,11} + n_{1,11}(1 - z^{-1}) = 1 ,$$

$$z^{-1} m_{1,12} + n_{1,12}(1 - z^{-1}) = 0 ,$$

$$z^{-1}(1 - 2z^{-1})(z^{-1} - 2)\, m_{1,21} + n_{1,21}(1 - z^{-1}) = 0 ,$$

$$z^{-1}(1 - 2z^{-1})(z^{-1} - 2)\, m_{1,22} + n_{1,22}(1 - z^{-1}) = 1$$

and

$$M_1 = \begin{bmatrix} m_{1,11} & m_{1,12} \\ m_{1,21} & m_{1,22} \end{bmatrix}, \quad N_1 = \begin{bmatrix} n_{1,11} & n_{1,12} \\ n_{1,21} & n_{1,22} \end{bmatrix}.$$

The general solution becomes

$$M_1 = \begin{bmatrix} 1 + (1 - z^{-1})\, t_{11} & (1 - z^{-1})\, t_{12} \\ (1 - z^{-1})\, t_{21} & 1 + (1 - z^{-1})\, t_{22} \end{bmatrix},$$

$$N_1 =$$

$$\begin{bmatrix} 1 - z^{-1} t_{11} & -z^{-1} t_{12} \\ -z^{-1}(1 - 2z^{-1})(z^{-1} - 2) t_{21} & 1 + 3z^{-1} - 2z^{-2} - z^{-1}(1 - 2z^{-1})(z^{-1} - 2) t_{22} \end{bmatrix}$$

for arbitrary $t_{ij} \in \Re^+\{z^{-1}\}$.

Further we shall solve the equation

$$A_2 N_2 + M_2 B_2 = I_m ,$$

which is equivalent to the set of equations

$$\begin{gathered} (1 - z^{-1}) n_{2,11} + m_{2,11} z^{-1} = 1 , \\ (1 - z^{-1}) n_{2,12} + m_{2,12} z^{-1}(1 - 2z^{-1})(z^{-1} - 2) = 0 , \\ (1 - z^{-1}) n_{2,21} + m_{2,21} z^{-1} = 0 , \\ (1 - z^{-1}) n_{2,22} + m_{2,22} z^{-1}(1 - 2z^{-1})(z^{-1} - 2) = 1 \end{gathered}$$

and

$$N_2 = \begin{bmatrix} n_{2,11} & n_{2,12} \\ n_{2,21} & n_{2,22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} , \quad M_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{2,11} & m_{2,12} \\ m_{2,21} & m_{2,22} \end{bmatrix} .$$

The general solution becomes

$$N_2 = \begin{bmatrix} 1 + z^{-1} v_{11} & 1 + z^{-1} v_{11} + z^{-1}(1 - 2z^{-1})(z^{-1} - 2) v_{12} \\ z^{-1} v_{21} & 1 + 3z^{-1} - 2z^{-2} + z^{-1} v_{21} + z^{-1}(1 - 2z^{-1})(z^{-1} - 2) v_{22} \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 - (1 - z^{-1})(v_{11} - v_{21}) & -1 - (1 - z^{-1})(v_{12} - v_{22}) \\ -(1 - z^{-1}) v_{21} & 1 - (1 - z^{-1}) v_{22} \end{bmatrix}$$

for arbitrary $v_{ij} \in \Re^+\{z^{-1}\}$.

In order that the mutual conditions

$$\begin{aligned} A_2 M_1 &= M_2 A_1 , \\ B_1 N_2 &= N_1 B_2 \end{aligned}$$

may be satisfied, we must take

$$v_{ij} = -t_{ij} , \quad i = 1, 2 ,$$

i.e. the matrices N_2 and M_2 become

$$N_2 = \begin{bmatrix} 1 - z^{-1} t_{11} & 1 - z^{-1} t_{11} - z^{-1}(1 - 2z^{-1})(z^{-1} - 2) t_{12} \\ - z^{-1} t_{21} & 1 + 3z^{-1} - 2z^{-2} - z^{-1} t_{21} - z^{-1}(1 - 2z^{-1})(z^{-1} - 2) t_{22} \end{bmatrix} ,$$

$$M_2 = \begin{bmatrix} 1 + (1 - z^{-1})(t_{11} - t_{21}) & -1 + (1 - z^{-1})(t_{12} - t_{22}) \\ (1 - z^{-1}) t_{21} & 1 + (1 - z^{-1}) t_{22} \end{bmatrix} .$$

Computing

$$Q = \begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix}, \quad Q^+ = \begin{bmatrix} 1 & 0 \\ 1 - z^{-1} & 1 \end{bmatrix}, \quad Q_1^+ = \begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix},$$

$$F = \begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix}, \quad F^+ = \begin{bmatrix} 1 & 0 \\ 1 - z^{-1} & 1 \end{bmatrix}, \quad q^- = f^- = p_0 = 1,$$

$$B_{11}^- = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix}, \quad B_1^+ = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} - 2 \end{bmatrix},$$

equation (5.5) becomes

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix}$$

and its general solution obtains as

$$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} (1 - z^{-1}),$$

$$Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}.$$

for arbitrary $t_1, t_2 \in \Re[z^{-1}]$.

Now we have to confine ourselves to those solutions M_1, N_1 and M_2, N_2 only that satisfy (5.8). Computing the $B_1^+ M_1 Q^+$ and $N_2 F^+$, equations (5.8) become

$$1 + (1 - z^{-1}) t_{11} + (1 - z^{-1})^2 t_{12} = 1 + (1 - z^{-1}) t_1 .$$

$$(1 - z^{-1})(z^{-1} - 2) + (1 - z^{-1})(z^{-1} - 2) t_{21} + (z^{-1} - 2)(1 - z^{-1})^2 t_{22} = \\ = (1 - z^{-1}) t_2 .$$

$$1 - z^{-1} t_{11} - z^{-1}(1 - z^{-1}) t_{12} = 1 - z^{-1} t_1 ,$$

$$(1 - z^{-1})(1 + 3z^{-1} - 2z^{-2}) - z^{-1}(1 - 2z^{-1}) . \\ . (z^{-1} - 2) t_{21} - z^{-1}(1 - 2z^{-1})(z^{-1} - 2)(1 - z^{-1}) t_{22} = \\ = 1 - z^{-1}(1 - 2z^{-1}) t_2$$

and yield

$$t_{11} + (1 - z^{-1}) t_{12} = t_1 , \\ (z^{-1} - 2)\left[1 + t_{21} + (1 - z^{-1}) t_{22}\right] = t_2 . \tag{5.16}$$

We have to further choose only such solutions X^0, Y^0 of (5.15) that minimize ∂Y^0 while satisfying (5.16). It follows that $t_1 = 0$, $t_2 = 0$ and, in turn,

$$t_{11} = -(1 - z^{-1}) t_{12} , \\ t_{21} = -1 - (1 - z^{-1}) t_{22} .$$

Hence

$$X^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and, eventually,

$$(5.17) \qquad M_1 = \begin{bmatrix} -1 - (1 - z^{-1})^2 t_{12} & (1 - z^{-1}) t_{12} \\ -(1 - z^{-1}) - (1 - z^{-1})^2 t_{22} & 1 + (1 - z^{-1}) t_{22} \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1 + z^{-1}(1 - z^{-1}) t_{12} & -z^{-1} t_{12} \\ z^{-1}(1 - 2z^{-1})(z^{-1} - 2) + & 1 + 3z^{-1} - 2z^{-2} - \\ + z^{-1}(1 - 2z^{-1})(z^{-1} - 2)(1 - z^{-1}) t_{22} & -z^{-1}(1 - 2z^{-1})(z^{-1} - 2) t_{22} \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 1 + z^{-1}(1 - z^{-1}) t_{12} & 1 + z^{-1}(1 - z^{-1}) t_{12} - z^{-1}(1 - 2z^{-1})(z^{-1} - 2) t_{12} \\ z^{-1} + z^{-1}(1 - z^{-1}) t_{22} & 1 + 4z^{-1} - 2z^{-2} + z^{-1}(1 - z^{-1}) t_{22} - z^{-1}(1 - 2z^{-1})(z^{-1} - 2) t_{22} \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 2 - z^{-1} - (1 - z^{-1})^2 (t_{12} - t_{22}) & -1 + (1 - z^{-1})(t_{12} - t_{22}) \\ -(1 - z^{-1}) - (1 - z^{-1})^2 t_{22} & 1 + (1 - z^{-1}) t_{22} \end{bmatrix}.$$

Since the control

$$U = M_2 \frac{F}{p_0} = M_2 \begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is stable, all optimal controllers are given as minimal realizations of

$$(5.18) \qquad R = M_2 N_1^{-1} = N_2^{-1} M_1$$

where M_1, N_1 and M_2, N_2 are given by (5.17).

The resulting error becomes

$$E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k_{\min} = 1 .$$

We recall that the same system has been considered in Example 3.7 for the open-loop control. We have obtained exactly the same U and E. One might get the idea to bypass the above computations and find an optimal controller $\mathscr{R}$ simply as

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = R \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

that is,

$$R = \begin{bmatrix} t_3 & 1 - t_3 \\ t_4 & -t_4 \end{bmatrix}$$

where t_3, t_4 are arbitrary elements of $\Re\{z^{-1}\}$. This is impossible, however, since not all controllers created in this way will yield a stable closed-loop system. For example, $t_3 = 1$, $t_4 = 0$ gives

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

and

$$c = \det C_1 = \det \begin{bmatrix} 1 & -(1 - z^{-1}) \\ 0 & 1 - z^{-1} \end{bmatrix} = 1 - z^{-1}$$

is not stable. Only the controllers having form (5.18) are acceptable.

Example 5.3. Consider a minimal realization of

$$S = \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} = \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} [1]^{-1} =$$

$$= \begin{bmatrix} 1 & 0 \\ -\sqrt{2}\,(1 - z^{-1}) & 1 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix}$$

over the field $\Re$ valuated by (2.25) and solve problem (5.2) for the reference sequence

$$W = \frac{\begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ -1 \end{bmatrix}}{z^{-1} - 2}.$$

We first find a stabilizing feedback.

The first equation (5.6) becomes

$$\begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} M_1 + N_1 \begin{bmatrix} 1 & 0 \\ -\sqrt{2}\,(1 - z^{-1}) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and it is equivalent to the set of equations

$$z^{-1} m_{1,11} + n_{1,11} = 1\,, \quad z^{-1} m_{1,12} + n_{1,12} = 0\,,$$
$$n_{1,21} = 0\,, \quad n_{1,22} = 1\,,$$

where

$$M_1 = [m_{1,11} \;\; m_{1,12}] \begin{bmatrix} 1 & 0 \\ -\sqrt{2}\,(1 - z^{-1}) & 1 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1 & 0 \\ \sqrt{2}\,(1 - z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} n_{1,11} & n_{1,12} \\ n_{1,21} & n_{1,22} \end{bmatrix}.$$

The general solution is

$$M_1 = [t_{11} - \sqrt{2}\,(1 - z^{-1})\, t_{12} \quad t_{12}]$$

$$N_1 = \begin{bmatrix} 1 - z^{-1} t_{11} & -z^{-1} t_{12} \\ \sqrt{2}\,(1 - z^{-1}) - \sqrt{2}\, z^{-1}(1 - z^{-1})\, t_{11} & 1 - \sqrt{2}\, z^{-1}(1 - z^{-1}) t_{12} \end{bmatrix}$$

for arbitrary $t_{11}, t_{12} \in \Re^{+}\{z^{-1}\}$.

The other equation (5.6) becomes

$$[1]\, N_2 + M_2 \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix} = [1]$$

the general solution being

$$N_2 = 1 + z^{-1} v_{11}\,,$$
$$M_2 = \begin{bmatrix} -v_{11} & -v_{12} \end{bmatrix},$$

for arbitrary v_{11}, $v_{12} \in \mathfrak{R}^+\{z^{-1}\}$.

Mutual conditions (5.7) then necessitate

$$v_{11} = -t_{11}\,, \quad v_{12} = -t_{12}\,,$$

i.e.

$$N_2 = 1 - z^{-1} t_{11}\,, \quad M_2 = \begin{bmatrix} t_{11} & t_{12} \end{bmatrix}.$$

Now we shall seek for optimality. We compute

$$Q = \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ -1 \end{bmatrix}, \quad Q^+ = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 \\ -1 & 1 \end{bmatrix}, \quad Q_1^+ = \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ -1 \end{bmatrix},$$

$$F = \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ z^{-1} - 2 \end{bmatrix}, \quad F^+ = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 \\ z^{-1} - 2 & 1 \end{bmatrix},$$

$$q^- = 1\,, \quad f^- = 1\,, \quad p_0 = z^{-1} - 2$$

and equation (5.5) reads

$$\begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} X + Y(z^{-1} - 2) = \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ -1 \end{bmatrix}.$$

Consulting Example 3.3 we obtain the general solution as

$$X = \frac{1}{2\sqrt{2}} + t_1(z^{-1} - 2)\,,$$

$$Y = \begin{bmatrix} -\dfrac{1}{2\sqrt{2}} \\ \dfrac{1 + z^{-1}}{2} \end{bmatrix} - \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} [t_1]$$

for any $t_1 \in \mathfrak{R}^-[z^{-1}]$.

Conditions (5.8) can now be written as

$$\frac{1}{\sqrt{2}} t_{11} + (z^{-1} - 2)\, t_{12} = \frac{1}{2\sqrt{2}} + (z^{-1} - 2)\, t_1 ,$$

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} z^{-1} t_{11} - z^{-1}(z^{-1} - 2)\, t_{12} = \frac{1}{2\sqrt{2}} (z^{-1} - 2) - z^{-1}(z^{-1} - 2)\, t_1 ,$$

$$-1 - z^{-1}(1 - z^{-1})\, t_{11} - \sqrt{2}\, z^{-1}(1 - z^{-1})(z^{-1} - 2)\, t_{12} =$$
$$= -1 - 0{\cdot}5z^{-1} + 0{\cdot}5z^{-2} - \sqrt{2}\, z^{-1}(1 - z^{-1})(z^{-1} - 2)\, t_1$$

and they yield

$$t_{11} = 0{\cdot}5$$
$$t_{12} = t_1 .$$

Now we can minimize the degree of Y by taking $t_1 = 0$. It follows that

$$X^0 = \frac{1}{2\sqrt{2}} , \quad Y^0 = \begin{bmatrix} -\dfrac{1}{2\sqrt{2}} \\ \dfrac{1 + z^{-1}}{2} \end{bmatrix}$$

and

$$M_1 = [0{\cdot}5 \;\; 0] ,$$
$$N_1 = \begin{bmatrix} 1 - 0{\cdot}5z^{-1} & 0 \\ \sqrt{2}\,(1 - z^{-1})(1 - 0{\cdot}5z^{-1}) & 1 \end{bmatrix} ,$$
$$N_2 = 1 - 0{\cdot}5z^{-1} ,$$
$$M_2 = [0{\cdot}5 \;\; 0] .$$

The control sequence

$$U = \frac{1}{2\sqrt{2}} \frac{1}{z^{-1} - 2}$$

is stable and the optimal controller is given as a minimal realization of

$$R = [0{\cdot}5 \;\; 0] \begin{bmatrix} 1 - 0{\cdot}5z^{-1} & 0 \\ \sqrt{2}\,(1 - z^{-1})(1 - 0{\cdot}5z^{-1}) & 1 \end{bmatrix}^{-1} =$$
$$= [1 - 0{\cdot}5z^{-1}]^{-1}\, [0{\cdot}5 \;\; 0] = \frac{[-1 \;\; 0]}{z^{-1} - 1}$$

and it is unique. The resulting error becomes

$$E = \begin{bmatrix} -\dfrac{1}{2\sqrt{2}} \\ 0{\cdot}5 + 0{\cdot}5z^{-1} \end{bmatrix} , \quad k_{\min} = 2 .$$

Example 5.4. Given the system over $\mathfrak{R}$ valuated by (2.25) that is a minimal realization of

$$S = \frac{[z^{-1} \;\; z^{-1}]}{1 - z^{-1}} = [z^{-1} \;\; 0] \begin{bmatrix} 1 - z^{-1} & -1 \\ 0 & 1 \end{bmatrix} =$$

$$= [1 - z^{-1}]^{-1} \, [z^{-1} \;\; z^{-1}],$$

solve problem (5.2) for the reference sequence

$$W = \frac{1 - 2z^{-1}}{1 - z^{-1}}.$$

As usual, we shall solve first the equations

$$[z^{-1} \;\; 0] \, M_1 + N_1[1 - z^{-1}] = [1],$$

$$\begin{bmatrix} 1 - z^{-1} & -1 \\ 0 & 1 \end{bmatrix} N_2 + M_2[z^{-1} \;\; z^{-1}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and obtain

$$M_1 = \begin{bmatrix} 1 + (1 - z^{-1}) \, t_{11} \\ t_{21} \end{bmatrix}, \quad N_1 = [1 - z^{-1} t_{11}),$$

$$N_2 = \begin{bmatrix} 1 + z^{-1} v_{11} & 1 + z^{-1} v_{11} \\ z^{-1} v_{21} & 1 + z^{-1} v_{21} \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 - (1 - z^{-1}) \, v_{11} + v_{21} \\ -v_{21} \end{bmatrix}$$

for arbitrary $t_{ij}, v_{ij} \in \mathfrak{R}^+\{z^{-1}\}$.

Mutual conditions (5.7) necessitate to set

$$t_{11} = -v_{11},$$

$$t_{21} = -(1 - z^{-1}) \, v_{21}.$$

Then

$$M_1 = \begin{bmatrix} 1 - (1 - z^{-1}) \, v_{11} \\ -(1 - z^{-1}) \, v_{21} \end{bmatrix}, \quad N_1 = [1 + z^{-1} v_{11}].$$

Now we compute

$$B_1^- = [z^{-1} \;\; 0], \qquad B_{11}^- = z^{-1}, \quad r = 1,$$

$$Q = 1 - 2z^{-1}, \qquad F = 1 - 2z^{-1},$$

$$Q^+ = Q_1^+ = F^+ = 1, \quad q^- = f^- = 1 - 2z^{-1}, \quad p_0 = 1,$$

and equation (5.5) reads

$$z^{-1}X + Y(1 - z^{-1}) = 1 .$$

The general solution is

$$X = 1 + (1 - z^{-1})\,t\,, \quad Y = 1 - z^{-1}t$$

for arbitrary $t \in \Re[z^{-1}]$.

Equations (5.8) give

$$\begin{aligned} 1 - (1 - z^{-1})\,v_{11} &= 1 + (1 - z^{-1})\,t\,, \\ 1 + z^{-1}v_{11} &= 1 - z^{-1}t \end{aligned}$$

and, hence

$$v_{11} = -t\,.$$

To minimize the degree of Y we set $t = 0$.

Then

$$X^{\circ} = 1\,, \quad Y^{0} = 1$$

and

$$\boldsymbol{M}_1 = \begin{bmatrix} 1 \\ -(1 - z^{-1}) & \boldsymbol{v}_{21} \end{bmatrix}, \quad N_1 = 1\,,$$

$$\boldsymbol{N}_2 = \begin{bmatrix} 1 & 1 \\ z^{-1}\boldsymbol{v}_{21} & 1 + z^{-1}\boldsymbol{v}_{21} \end{bmatrix}, \quad \boldsymbol{M}_2 = \begin{bmatrix} 1 + \boldsymbol{v}_{21} \\ - \boldsymbol{v}_{21} \end{bmatrix}$$

and the control sequence

$$\boldsymbol{U} = \begin{bmatrix} 1 + \boldsymbol{v}_{21} \\ - \boldsymbol{v}_{21} \end{bmatrix} (1 - 2z^{-1})$$

is stable, as required.

Thus all optimal controllers are given as minimal realizations of

$$\boldsymbol{R} = \begin{bmatrix} 1 + \boldsymbol{v}_{21} \\ - \boldsymbol{v}_{21} \end{bmatrix} [1]^{-1} = \begin{bmatrix} 1 & 1 \\ z^{-1}\boldsymbol{v}_{21} & 1 + \boldsymbol{v}_{21} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -(1 - z^{-1})\,\boldsymbol{v}_{21} \end{bmatrix} = \begin{bmatrix} 1 + \boldsymbol{v}_{21} \\ - \boldsymbol{v}_{21} \end{bmatrix}$$

and they yield the error sequence

$$\boldsymbol{E} = 1 - 2z^{-1}\,, \quad k_{\min} = 2\,.$$

It is worth examining how a particular choice of v_{21} affects the pseudocharacteristic polynomial of the closed-loop system.

Write

$$\boldsymbol{v}_{21} = \frac{b}{a}\,,$$

where $a, b \in \Re[z^{-1}]$, $(a, b) = 1$, a stable. Then

$$R = \begin{bmatrix} 1 + \dfrac{b}{a} \\ \\ -\dfrac{b}{a} \end{bmatrix} = \begin{bmatrix} a + b \\ -b \end{bmatrix} [a]^{-1}$$

and

$$c = \det C_1 = \det\left[(1 - z^{-1})\, a + z^{-1}(a + b) - z^{-1}b\right] = a\,.$$

5.3. Finite time optimal control problem

Let $\mathfrak{F}$ be an arbitrary field with valuation $\mathscr{V}$ and write

$$S = \frac{B}{a} = B_1 A_2^{-1} = A_2^{-1} B_2\,,$$

$$\operatorname{rank} B_1 = \operatorname{rank} B_2 = r\,.$$

By the definition of B_1 in (2.19) we have

$$B_1 = [B_{11}\ \ 0]$$

where

$$B_{11} \in \mathfrak{F}_{l,r}[z^{-1}]\,, \quad 0 \in \mathfrak{F}_{l,m-r}[z^{-1}]$$

and rank $B_{11} = r$.

We also write

$$Q = Q^+ Q^-\,,$$

where

$$Q^- = \begin{bmatrix} q^- \\ 0 \end{bmatrix}$$

with $q^- \in \mathfrak{F}_{1,1}[z^{-1}]$, $0 \in \mathfrak{F}_{l-1,1}[z^{-1}]$ and denote

$$Q_1^+ = Q^+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{Q}{q^-}\,.$$

For convenience, let

$$A_1 \frac{Q}{p} = \frac{F}{p_0}\,,$$

where $(p_0, F) = 1$ and write

$$F = F^+F^- ,$$

where

$$F^- = \begin{bmatrix} f^- \\ 0 \end{bmatrix}$$

with $f^- \in \mathfrak{F}_{1,1}[z^{-1}]$, $0 \in \mathfrak{F}_{l-1,1}[z^{-1}]$ and denote

$$f^- = f_0^- q^- .$$

Then we have the following result.

Theorem 5.2. *Problem* (5.3) *has a solution if and only if the linear Diophantine equation*

$$B_{11}X + Ypf_0^- = Q_1^+ \tag{5.19}$$

has a solution $X°$, Y^0 *such that* $\partial Y^0 =$ min *subject to matrices* M_1, N_1 and M_2, N_2 *exist in* $\mathfrak{F}^+_{m,l}\{z^{-1}\}$, $\mathfrak{F}^+_{l,l}\{z^{-1}\}$ and $\mathfrak{F}^+_{m,l}\{z^{-1}\}$, $\mathfrak{F}^+_{m,m}\{z^{-1}\}$ *respetively and satisfy the following equations*

$$B_1M_1 + N_1A_1 = I_l , \tag{5.20}$$

$$A_2N_2 + M_2B_2 = I_m ,$$

$$A_2M_1 = M_2A_1 , \tag{5.21}$$

$$B_1N_2 = N_1B_2 ,$$

$$M_{11} = X^0 , \quad M_1Q^+ = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} , \tag{5.22}$$

$$N_{11} = Y^0p_0 , \quad N_1F^+ = [N_{11} \quad N_{12}]$$

and subject to

$$U = M_2 \frac{F}{p_0} \tag{5.23}$$

belongs to $\mathfrak{F}^+_{m,1}[z^{-1}]$.

The optimal controller is not unique, in general, and all optimal controllers are given as minimal realizations of

$$R = M_2N_1^{-1} = N_2^{-1}M_1 .$$

Moreover, U is given by (5.23) and

$$E = Y^0 f^- ,$$

$$k_{\min} = 1 + \partial Y^0 + \partial f^- .$$

Proof. The error is given as

$$E = K_{W/E} = (I_l - K_1) W .$$

To guarantee a stable closed-loop system we have to set

$$I_l - K_1 = N_1 A_1 ,$$

where $N_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$. If follows that

$$E = N_1 A_1 \frac{Q}{p} = N_1 \frac{F}{p_0} = [N_{11} \;\; N_{12}] \frac{\begin{bmatrix} f^- \\ 0 \end{bmatrix}}{p_0} = N_{11} \frac{f^-}{p_0} ,$$

where

$$N_1 F^+ = [N_{11} \; N_{12}]$$

and

$$N_{11} \in \mathfrak{F}^+_{l,1}\{z^{-1}\} , \quad N_{12} \in \mathfrak{F}^+_{l,l-1}\{z^{-1}\} .$$

Since the error sequence is to vanish in a minimum time and thereafter, E must be a matrix polynomial in $\mathfrak{F}_{l,1}[z^{-1}]$. Therefore,

(5.24) $$N_{11} = Y p_0 ,$$

where $Y \in \mathfrak{F}_{l,1}[z^{-1}]$ is a matrix polynomial to be specified later. This choice yields the error

(5.25) $$E = Y f^- .$$

The error is also given as

$$E = W - K_1 W$$

and, in order to guarantee a stable system, we have to set

$$K_1 = B_1 M_1$$

where $M_1 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$. Then

(5.26) $$pE = Q - B_1 M_1 Q = Q - [B_{11} \;\; 0] \, M_1 Q^+ \begin{bmatrix} q^- \\ 0 \end{bmatrix} = Q - B_{11} M_{11} q^- ,$$

where

$$M_1 Q^+ = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

and $M_{11} \in \mathfrak{F}^+_{r,1}\{z^{-1}\}$, $M_{12} \in \mathfrak{F}^+_{r,l-1}\{z^{-1}\}$, $M_{21} \in \mathfrak{F}^+_{m-r,1}\{z^{-1}\}$, and $M_{22} \in$ $\in \mathfrak{F}^+_{m-r,l-1}\{z^{-1}\}$.

The E is a matrix polynomial whenever pE is so. It follows that $B_{11}M_{11}q^-$ must be a matrix polynomial, too. This is effected by the choice

$$M_{11} = X, \tag{5.27}$$

where $X \in \mathfrak{F}_{r,1}[z^{-1}]$ is an unspecified matrix polynomial as yet.

In fact, substituting (5.25) into (5.26) we end up with equation (5.19) coupling the X and Y.

To guarantee the closed-loop system stability the M_1 and N_1 must satisfy the equation

$$B_1 M_1 + N_1 A_1 = I_l$$

in addition to (5.24) and (5.27), see Theorem 4.5. However, we must also solve the equation

$$A_2 N_2 + M_2 B_2 = I_m$$

for $M_2 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$ and $N_2 \in \mathfrak{F}^+_{m,m}\{z^{-1}\}$ and in order that the four matrices may be properly related, they must further satisfy the mutual relations

$$A_2 M_1 = M_2 A_1,$$

$$B_1 N_2 = N_1 B_2.$$

We must take, therefore, only those solutions of equation (5.19) that make the above specified M_1, N_1 and M_2, N_2 exist. Further, we must take only those solutions which make the control sequence

$$U = K_{W/U} W = A_2 M_1 \frac{Q}{p} = M_2 A_1 \frac{Q}{p} = M_2 \frac{F}{p_0}$$

finite, as required. And within this class we must further confine ourselves to those solutions which minimize the degree of E. Therefore, in view of (5.25), equation (5.19) is to be solved for a solution X^0, Y^0 such that $\partial Y^0 = \min$ subject to all stability and finiteness requirements.

All optimal controllers are then obtained by (4.66) as minimal realization of

$$R = M_2 N_1^{-1} = N_2^{-1} M_1 ,$$

where M_1, N_1 and M_2, N_2 satisfy (5.20), (5.21), and (5.22). The optimal performance measure becomes

$$k_{\min} = 1 + \partial Y^0 + \partial f^-$$

in view of (5.25). Since it is assumed that $z^{-1} | B_1$ we always have $Y^0 \neq 0$.

Example 5.5 Let the system over the field $\mathfrak{R}$ valuated by (2.25) be given as a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1})^2 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix} \begin{bmatrix} 1 - z^1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

and solve problem (5.3) for the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ (1 - z^{-1})^2 \end{bmatrix}}{1 - z^{-1}} .$$

Closed-loop stability is guaranteed by solving equations (5.20) and (5.21). They give

$$M_1 = \begin{bmatrix} 1 + (1 - z^{-1})\, t_{11} & t_{12} \\ -(1 - z^{-1})\, v_{21} & -v_{22} \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1 - z^{-1} t_{11} & -z^{-1} t_{12} \\ z^{-1}(1 - z^{-1})\, v_{21} & 1 + z^{-1}(1 - z^{-1})\, v_{22} \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 1 - z^{-1} t_{11} & -z^{-1}(1 - z^{-1})\, t_{12} \\ z^{-1} v_{21} & 1 + z^{-1}(1 - z^{-1})\, v_{22} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 + (1 - z^{-1})\, t_{11} & (1 - z^{-1})\, t_{12} \\ -v_{21} & -v_{22} \end{bmatrix}$$

for arbitrary $t_{ij}, v_{ij} \in \mathfrak{R}^+\{z^{-1}\}$, similarly to Example 4.11.

We compute

$$Q = \begin{bmatrix} 1 \\ (1 - z^{-1})^2 \end{bmatrix}, \quad Q^+ = \begin{bmatrix} 1 & 0 \\ (1 - z^{-1})^2 & 1 \end{bmatrix}, \quad Q_1^+ = \begin{bmatrix} 1 \\ (1 - z^{-1})^2 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix}, \quad F^+ = \begin{bmatrix} 1 & 0 \\ 1 - z^{-1} & 1 \end{bmatrix},$$

$$q^- = 1 , \quad f^- = 1 , \quad p_0 = 1$$

and solve equation (5.19), which is

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} 1 \\ (1 - z^{-1})^2 \end{bmatrix}.$$

The general solution is obtained as

$$X = \begin{bmatrix} 1 + (1 - z^{-1})\, t_1 \\ -1 + t_2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 - z^{-1}t_1 \\ 1 - z^{-1}t_2 \end{bmatrix}$$

for arbitrary $t_1, t_2 \in \Re[z^{-1}]$, see Example 3.6.

Now we have to satisfy equations (5.22), which are

$$\begin{aligned} 1 + (1 - z^{-1})\, t_{11} + (1 - z^{-1})^2\, t_{12} &= 1 + (1 - z^{-1})\, t_1\,, \\ -(1 - z^{-1})\, v_{21} - (1 - z^{-1})^2\, v_{22} &= -1 + t_2\,, \\ 1 - z^{-1}t_{11} - z^{-1}(1 - z^{-1})\, t_{12} &= 1 - z^{-1}t_1\,, \\ 1 - z^{-1} + z^{-1}(1 - z^{-1})\, v_{21} + z^{-1}(1 - z^{-1})^2\, v_{22} &= 1 - z^{-1}t_2\,. \end{aligned}$$

It follows that

$$\begin{aligned} t_{11} + (1 - z^{-1})\, t_{12} &= t_1\,, \\ 1 - (1 - z^{-1})\, v_{21} - (1 - z^{-1})^2\, v_{22} &= t_2\,. \end{aligned} \tag{5.28}$$

At this stage we should take t_1, t_2 so as to make $\partial Y = \min$. The choice $t_1 = 0$, $t_2 = 0$ *totally* minimizes ∂Y, but it does not satisfy the second equation (5.28). Hence $\partial Y = \min$ *subject to* (5.28) is obtained when setting $t_1 = \tau_0$, $t_2 = 1$, $\tau_0 \in \Re$ arbitrary. Then

$$X^0 = \begin{bmatrix} (1 + \tau_0) - \tau_0 z^{-1} \\ 0 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} 1 - \tau_0 z^{-1} \\ 1 - z^{-1} \end{bmatrix}$$

and

$$\begin{aligned} t_{11} &= \tau_0 - (1 - z^{-1})\, t_{12}\,, \\ v_{21} &= -(1 - z^{-1})\, v_{22} \end{aligned}$$

yields

$$M_1 = \begin{bmatrix} (1 + \tau_0) - \tau_0 z^{-1} - (1 - z^{-1})^2\, t_{12} & t_{12} \\ (1 - z^{-1})\, v_{22} & -v_{22} \end{bmatrix}, \tag{5.29}$$

$$N_1 = \begin{bmatrix} 1 - \tau_0 z^{-1} + z^{-1}(1 - z^{-1})\, t_{12} & -z^{-1}t_{12} \\ -z^{-1}(1 - z^{-1})^2\, v_{22} & 1 + z^{-1}(1 - z^{-1})\, v_{22} \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 1 - \tau_0 z^{-1} + z^{-1}(1 - z^{-1})\, t_{12} & -z^{-1}(1 - z^{-1})\, t_{12} \\ -z^{-1}(1 - z^{-1})\, v_{22} & 1 + z^{-1}(1 - z^{-1})\, v_{22} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} (1 + \tau_0) - \tau_0 z^{-1} - (1 - z^{-1})^2\, t_{12} & (1 - z^{-1})\, t_{12} \\ (1 - z^{-1})\, v_{22} & -v_{22} \end{bmatrix}.$$

Since the control sequence

$$U = M_2 \begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix} = \begin{bmatrix} (1 + \tau_0) - \tau_0 z^{-1} \\ 0 \end{bmatrix}$$

is finite, as required, all optimal controllers are given as minimal realizations of

$$R = M_2 N_1^{-1} = N_2^{-1} M_1 ,$$

where M_1, N_1 and M_2, N_2 are given by (5.29).
The resulting error is

$$E = \begin{bmatrix} 1 - \tau_0 z^{-1} \\ 1 - z^{-1} \end{bmatrix}, \quad k_{\min} = 2 .$$

Example 5.6. Given a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} \\ z^{-2} \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} \\ z^{-2} \end{bmatrix} [1 - z^{-1}]^{-1} = \begin{bmatrix} 1 - z^{-1} & 0 \\ - z^{-1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix}$$

over $\mathfrak{R}$ valuated by (2.25), solve problem (5.3) for the reference sequence

$$W = \begin{bmatrix} 1 - z^{-1} \\ z^{-1} \end{bmatrix}.$$

To make the closed-loop system stable, we solve the equations

$$\begin{bmatrix} z^{-1} \\ z^{-2} \end{bmatrix} M_1 + N_1 \begin{bmatrix} 1 - z^{-1} & 0 \\ - z^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$[1 - z^{-1}]\, N_2 + M_2 \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix} = [1],$$

$$[1 - z^{-1}]\, M_1 = M_2 \begin{bmatrix} 1 - z^{-1} & 0 \\ - z^{-1} & 1 \end{bmatrix},$$

$$\begin{bmatrix} z^{-1} \\ z^{-2} \end{bmatrix} N_2 = N_1 \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix}.$$

They yield the general solutions

$$M_1 = \left[1 + (1 - z^{-1})\,t_{11} - z^{-1}t_{12} \quad t_{12}\right],$$

$$N_1 = \begin{bmatrix} 1 - z^{-1}t_{11} & -z^{-1}t_{12} \\ z^{-1} - z^{-2}t_{11} & 1 - z^{-2}t_{12} \end{bmatrix},$$

$$N_2 = 1 - z^{-1}t_{11},$$

$$M_2 = \left[1 + (1 - z^{-1})\,t_{11} \quad (1 - z^{-1})\,t_{12}\right]$$

for arbitrary $t_{11}, t_{12} \in \Re^+\{z^{-1}\}$.

Equation (5.19) becomes

$$\begin{bmatrix} z^{-1} \\ z^{-2} \end{bmatrix} X + Y = \begin{bmatrix} 1 - z^{-1} \\ z^{-1} \end{bmatrix}$$

and it has the general solution

$$X = -1 + t_1, \qquad Y = \begin{bmatrix} 1 - z^{-1}t_1 \\ z^{-1} + z^{-2} - z^{-2}t_1 \end{bmatrix}$$

for any $t_1 \in \Re[z^{-1}]$.

Now we are to satisfy equations (5.22), i.e.

$$1 - z^{-1} + (1 - z^{-1})^2\,t_{11} + z^{-2}t_{12} = -1 + t_1$$

and

$$(1 - z^{-1})^2 - z^{-1}(1 - z^{-1})^2\,t_{11} - z^{-3}t_{12} = 1 - z^{-1}t_1,$$

$$z^{-1} - z^{-2} + z^{-3} - z^{-2}(1 - z^{-1})^2\,t_{11} - z^{-4}t_{12} = z^{-1} + z^{-2} - z^{-2}t_1.$$

They necessitate the choice

$$(1 - z^{-1})^2\,t_{11} + z^{-2}t_{12} = z^{-1} - 2 + t_1,$$

where t_1 is to be taken such that $\partial Y = \min$.

It follows that $t_1 = 1$, and

$$t_{11} = -1 - z^{-1} + z^{-2}t$$

$$t_{12} = -1 + z^{-1} - (1 - z^{-1})^2\,t$$

for arbitrary $t \in \mathfrak{R}^+\{z^{-1}\}$. Therefore,

$$X^0 = 0\,, \qquad Y^0 = \begin{bmatrix} 1 - z^{-1} \\ z^{-1} \end{bmatrix}$$

and

$$M_1 = \begin{bmatrix} z^{-1} + z^{-1}(1 - z^{-1})\,t & -(1 - z^{-1}) - (1 - z^{-1})^2\, t \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1 + z^{-1} + z^{-2} - z^{-3}t & z^{-1} - z^{-2} + z^{-3}(1 - z^{-1})^2\, t \\ z^{-1} + z^{-2} + z^{-3} - z^{-4}t & 1 + z^{-2} - z^{-3} + z^{-2}(1 - z^{-1})^2\, t \end{bmatrix},$$

$$N_2 = 1 + z^{-1} + z^{-2} - z^{-3}t\,,$$

$$M_2 = \begin{bmatrix} z^{-2} + z^{-2}(1 - z^{-1})\,t & -(1 - z^{-1})^2 - (1 - z^{-1})^3\, t \end{bmatrix}.$$

The optimal controllers are given by

$$R = M_2 N_1^{-1} = N_2^{-1} M_1\,,$$

with the matrices M_1, N_1 and M_2, N_2 given above.
The optimal control is

$$U = \begin{bmatrix} 1 - z^{-1} \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} = 0$$

and the error

$$E = \begin{bmatrix} 1 - z^{-1} \\ z^{-1} \end{bmatrix}.$$

Note that

$$R = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

is not acceptable, since it does not stabilize the closed-loop system.

Example 5.7. It is important that both $\mathscr{S}$ and $\mathscr{R}$ be minimal realizations of their transfer function matrices. This example is to illustrate what might happen if this assumption is violated.

Consider again the problem solved in Example 5.6 and let the $\mathscr{S}$ be realized as $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is an elementwise realization of S, see Fig. 11.

Further choose $t = 0$ in (5.29). Then the controller is

$$R = \frac{[z^{-1} \quad -1 + z^{-1}]}{1 + z^{-1} + z^{-2}} \tag{5.30}$$

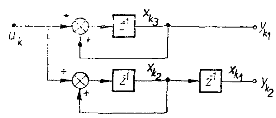

Fig. 11. An elementwise realization of $\boldsymbol{S}$ in Example 5.7.

and let it be realized as $\{\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}\}$, where

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \tag{5.31}$$

$$\mathbf{H} = [1 \ 0], \qquad \mathbf{J} = [0 \ -1].$$

This is a minimal realization of $\boldsymbol{R}$, see Fig. 12.

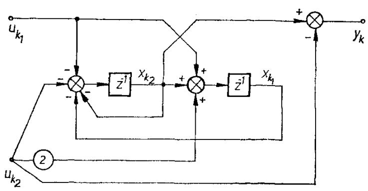

Fig. 12. A minimal realization of $\boldsymbol{R}$ in Example 5.7.

Then the characteristic polynomial of the closed-loop system becomes

$$\hat{c} = \det(zI_{n+p} - \mathbf{K}) = \det \begin{bmatrix} z & -1 & 0 & 0 & 0 \\ -1 & z-1 & 0 & -1 & 0 \\ -1 & 0 & z-1 & -1 & 0 \\ 2 & 0 & 1 & z & -1 \\ -1 & 0 & -1 & 1 & z+1 \end{bmatrix} =$$

$$= z^5 - z^4 - 4z^2 + 2$$

and it is *not* stable.

The trouble is due to a nonminimal realization of S. A nonminimal realization of R can cause the same sort of trouble. Consider a minimal realization $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ of S, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.32}$$

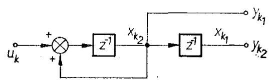

Fig. 13. A minimal realization of S in Example 5.7.

see Fig. 13, and let the $\mathscr{R}$ in (5.30) be realized as $\{\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{F}\}$ with

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

This is an elementwise realization, see Fig. 14.

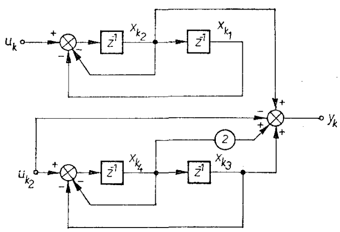

Fig. 14. An elementwise realization of $\mathscr{R}$ in Example 5.7.

Then the characteristic polynomial of the closed-loop system becomes

$$\hat{c} = \det\left(zI_{n+p} - \mathbf{K}\right) = \det \begin{bmatrix} z & -1 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & -1 & -1 & -2 \\ 0 & 0 & z & -1 & 0 & 0 \\ 1 & 0 & 1 & z+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & -1 \\ 0 & 1 & 0 & 0 & 1 & z+1 \end{bmatrix} =$$

$$= z\left(z^2 + z + 1\right)\left(z^3 + z^2 + 3z + 2\right)$$

and it is *not* stable, either.

On the other hand, taking the minimal realization (5.32) of $\mathscr{S}$ and the minimal realization (5.31) of $\mathscr{R}$, the characteristic polynomial becomes $\hat{c} = z^4$ and it can be computed via Theorem 4.1. Its stability is guaranteed by the method of synthesis.

Example 5.8. Consider the system $\mathscr{S}$ described by the infinite set of equations

$$\begin{aligned} x_{k+1,l} &= x_{k,l-1} + u_{k,l}, \quad x_{k,-1} = 0, \\ y_{k,l} &= x_{k,l}, \quad k, l = 0, 1, 2, \ldots, \end{aligned}$$

over the field $\mathfrak{R}$ valuated by (2.24). This is an infinite dimensional system over $\mathfrak{R}$.

To simplify its analysis, let us view it as a system over $\mathfrak{F} = \mathfrak{R}\{w^{-1}\}$, the field of rational functions over $\mathfrak{R}$ in the indeterminate w^{-1}. Indeed, making the identifications

$$\begin{aligned} x_k &= x_{k,0} + x_{k,1}w^{-1} + x_{k,2}w^{-2} + \ldots \in \mathfrak{F}, \\ u_k &= u_{k,0} + u_{k,1}w^{-1} + u_{k,2}w^{-2} + \ldots \in \mathfrak{F}, \end{aligned}$$

the system equations can be written as

$$\begin{aligned} x_{k+1} &= w^{-1}x_k + u_k, \\ y_k &= x_k \end{aligned}$$

and the $\mathscr{S}$ has dimension 1 over $\mathfrak{R}\{w^{-1}\}$.

The transfer function of $\mathscr{S}$ becomes

$$S = \frac{z^{-1}}{1 - w^{-1}z^{-1}} \tag{5.33}$$

by virtue of (2.1). The S is stable under the valuation (2.26), see Example 2.12, which is compatible with valuation (2.24).

To illustrate how Theorem 5.2 works, consider problem (5.3) for a minimal realization of (5.33) and the reference sequence

$$W = \frac{1}{1 - w^{-1}z^{-1}}.$$

The stability equations (5.20) and (5.21) reduce to the equation

$$z^{-1}M + N(1 - w^{-1}z^{-1}) = 1,$$

which has a solution

$$M \in \mathfrak{F}^+\{z^{-1}\} \quad \text{arbitrary},$$

$$N = \frac{1}{1 - w^{-1}z^{-1}} - \frac{z^{-1}}{1 - w^{-1}z^{-1}} M.$$

The optimality equation becomes

$$z^{-1}X + Y\left(1 - w^{-1}z^{-1}\right) = 1$$

and its general solution is

$$X = w^{-1} + \left(1 - w^{-1}z^{-1}\right)t\,,$$
$$Y = 1 - z^{-1}t$$

for arbitrary $t \in \mathfrak{F}[z^{-1}]$.

To minimize the degree of Y we set $t = 0$, i.e.

$$X^0 = w^{-1}\,, \quad Y^0 = 1$$

and, by virtue of (5.22), we obtain

$$\boldsymbol{M} = w^{-1}\,, \quad \boldsymbol{N} = 1\,.$$

Hence the optimal controller is given as a minimal realization of

$$\boldsymbol{R} = \frac{\boldsymbol{M}}{\boldsymbol{N}} = w^{-1}$$

and it yields

$$\boldsymbol{U} = w^{-1}\,, \quad \boldsymbol{E} = 1\,, \quad k_{\min} = 1\,.$$

This control law over $\mathfrak{R}\{w^{-1}\}$ can be inplemented over $\mathfrak{R}$ as shown in Fig. 15.

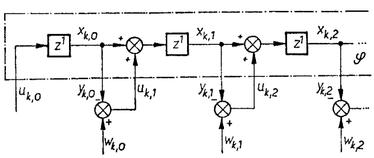

Fig. 15. The optimal control system in Example 5.8.

5.4. Least squares control problem

Let $\mathfrak{F}$ be a subfield of the field $\mathfrak{C}$ of complex numbers valuated by (2.25) and write

$$S = \frac{B}{a} = B_1A_2^{-1} = A_1^{-1}B_2\,,$$

$$\operatorname{rank} B_1 = \operatorname{rank} B_2 = r$$

and

$$B_1 = B_1^- B_1^+ .$$

By the definition of B_1^- in (2.30) we have

$$B_1^- = [B_{11}^- \ 0] ,$$

where $B_{11}^- \in \mathfrak{F}_{l,r}[z^{-1}]$, $0 \in \mathfrak{F}_{l,m-r}[z^{-1}]$ and rank $B_{11}^- = r$.
We also write

$$Q = Q^+ Q^- ,$$

where

$$Q^- = \begin{bmatrix} q^- \\ 0 \end{bmatrix}$$

with $q^- \in \mathfrak{F}_{1,1}[z^{-1}]$, $0 \in \mathfrak{F}_{l-1,1}[z^{-1}]$ and denote

$$Q^* = Q^+ \begin{bmatrix} q^{-\sim} \\ 0 \end{bmatrix} ,$$

see Chapter 2.
For convenience, let

$$A_1 \frac{Q}{p} = \frac{F}{p_0} ,$$

where $(p_0, F) = 1$. Write

$$F = F^+ F^- ,$$

where

$$F^- = \begin{bmatrix} f^{-\sim} \\ 0 \end{bmatrix}$$

with $f^- \in F_{1,1}[z^{-1}]$, $0 \in F_{l-1,1}[z^{-1}]$ and denote

$$f^- = f_0^- q^- .$$

Further, let

$$B_{11}^{-=\prime} B_{11}^- = (B_{11}^-)^{*=\prime} (B_{11}^-)^*$$

and denote

(5.34) $$d = \partial B_{11}^- - \partial (B_{11}^-)^* .$$

For notational convenience we shall denote

$$(B_{11}^-)^* = H .$$

Then we have the following result.

Theorem 5.3. *Let $\mathfrak{F}$ be a subfield of $\mathfrak{C}$ valuated by (2.25). Then problem (5.4) has a solution if and only if the linear Diophantine equation*

$$z^{-d}H^{\sim\prime}X + Ypf_0^- = B_{11}^{-\sim\prime}Q^*f_0^{-\sim} \tag{5.35}$$

has a solution X^0, Y^0 such that $\partial Y^0 = \min$, matrices $\boldsymbol{M}_1$, $\boldsymbol{N}_1$ and $\boldsymbol{M}_2$, $\boldsymbol{N}_2$ exist in $\mathfrak{F}^+_{m,l}\{z^{-1}\}$, $\mathfrak{F}^+_{l,l}\{z^{-1}\}$ and $\mathfrak{F}^+_{m,l}\{z^{-1}\}$, $\mathfrak{F}^+_{m,m}\{z^{-1}\}$ respectively and satisfy the equations

$$\begin{aligned} B_1\boldsymbol{M}_1 + \boldsymbol{N}_1A_1 &= I_l\,, \\ A_2\boldsymbol{N}_2 + \boldsymbol{M}_2B_2 &= I_m\,, \end{aligned} \tag{5.36}$$

$$\begin{aligned} A_2\boldsymbol{M}_1 &= \boldsymbol{M}_2A_1\,, \\ B_1\boldsymbol{N}_2 &= \boldsymbol{N}_1B_2\,, \end{aligned} \tag{5.37}$$

$$HM_{11}f^{-\sim} = X^\circ\,, \quad B_1^+\boldsymbol{M}_1Q^+ = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{bmatrix}, \tag{5.38}$$

$$B_{11}^{-\sim\prime}N_{11}f^{-\sim} = Y^0p_0\,, \quad N_1F^+ = [N_{11}\ N_{12}]\,,$$

and

$$\boldsymbol{U} = \boldsymbol{M}_2\frac{F}{p_0}\,, \tag{5.39}$$

$$\boldsymbol{E} = N_1\frac{F}{p_0} \tag{5.40}$$

belong to $\mathfrak{F}^+_{m,1}\{z^{-1}\}$ and $\mathfrak{F}^+_{l,1}\{z^{-1}\}$ respectively.

The optimal controller is not unique, in general, and all optimal controllers are given as minimal realizations of

$$\boldsymbol{R} = \boldsymbol{M}_2\boldsymbol{N}_1^{-1} = \boldsymbol{N}_2^{-1}\boldsymbol{M}_1\,.$$

*Moreover, **U** is given by (5.39), **E** is given by (5.40) and also satisfies*

$$B_{11}^{-\sim\prime}\boldsymbol{E} = Y^0\frac{f^-}{f^{-\sim}}\,,$$

and

$$\begin{aligned} \|\boldsymbol{E}\|^2_{\min} = &\langle((H^{\sim\prime})^{-1}\,Y^0)^{=\prime}\,((H^{\sim\prime})^{-1}\,Y^0)\rangle + \\ &+ \langle W^{=\prime}(I_l - B_{11}^-H^{-1}(H^{=\prime})^{-1}\,B_{11}^{-=\prime})\,W\rangle\,. \end{aligned}$$

Proof. In order to minimize $\|E\|^2$ we shall assume that E is stable whereby

$$\|E\|^2 = \langle E^{=\prime}E \rangle .$$

Then we manipulate the expression $\langle E^{=\prime}E \rangle$ so as to make the minimizing choice of R obvious.

Write

$$E = (I_l - K_1)\, W .$$

To guarantee a stable closed-loop system we have to set

$$K_1 = B_1 M_1 ,$$

where $M_1 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$. Then

$$E = W - [B_{11}^- \;\; 0]\, B_1^+ M_1 \frac{Q^+}{p} \begin{bmatrix} q^- \\ 0 \end{bmatrix} = W - B_{11}^- M_{11} \frac{q^-}{p} ,$$

where

$$B_1^+ M_1 Q^+ = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

and

$$M_{11} \in \mathfrak{F}^+_{r,1}\{z^{-1}\} , \quad M_{12} \in \mathfrak{F}^+_{r,l-1}\{z^{-1}\} , \quad M_{21} \in \mathfrak{F}^+_{m-r,1}\{z^{-1}\} ,$$

$$M_{22} \in \mathfrak{F}^+_{m-r,l-1}\{z^{-1}\} .$$

Then

$$E^{=\prime}E = W^{=\prime}W - W^{=\prime}B_{11}^- M_{11} \frac{q^-}{p} - \tag{5.41}$$

$$- \frac{q^{-=}}{\bar{p}} M_{11}^{=\prime} B_{11}^{-=\prime} W + \frac{q^{-=}}{\bar{p}} M_{11}^{=\prime} B_{11}^{-=\prime} B_{11}^- M_{11} \frac{q^-}{p} =$$

$$= \left((H^{=\prime})^{-1} B_{11}^{-=\prime} W - H M_{11} \frac{q^-}{p} \right)^{=\prime} \left((H^{=\prime})^{-1} B_{11}^{-=\prime} W - H M_{11} \frac{q^-}{p} \right) +$$

$$+ W^{=\prime}W - W^{=\prime}B_{11}^- H^{-1} (H^{=\prime})^{-1} B_{11}^{-=\prime} W$$

on completing the squares. Since the last two terms in (5.41) are independent of M_{11} (and hence M_1 and, in turn, R) the expression $\langle E^{=\prime}E \rangle$ attains its minimum for the same controller R as the expression $\langle E_1^{=\prime}E_1 \rangle$ does, where

$$E_1 = (H^{=\prime})^{-1} B_{11}^{-=\prime} W - H M_{11} \frac{q^-}{p} .$$

Further observe that

$$\langle \boldsymbol{E}_1^{=\prime} \boldsymbol{E}_1 \rangle = \left\langle \left(\boldsymbol{E}_1 \frac{f^{-\sim}}{f^{-}} \right)^{=\prime} \left(\boldsymbol{E}_1 \frac{f^{-\sim}}{f^{-}} \right) \right\rangle$$

because

$$\frac{f^{-=\sim} f^{-\sim}}{f^{-=} f^{-}} = \frac{z^{\partial f^{-\sim}} f^{-\sim\sim} f^{-\sim}}{z^{\partial f^{-}} f^{-\sim} f^{-}} = \frac{f^{-} f^{-\sim}}{f^{-\sim} f^{-}} = 1 .$$

Therefore,

$$\boldsymbol{E}_1 \frac{f^{-\sim}}{f^{-}} = (H^{=\prime})^{-1} B_{11}^{-=\prime} \frac{Q}{p} \frac{f^{-\sim}}{f^{-}} - H\boldsymbol{M}_{11} \frac{q^{-}}{p} \frac{f^{-\sim}}{f^{-}} =$$

$$= (H^{=\prime})^{-1} B_{11}^{-=\prime} \frac{Q^{+}}{p} \begin{bmatrix} q^{-} \\ 0 \end{bmatrix} \frac{f_0^{-\sim} q^{-\sim}}{f_0^{-} q^{-}} - H\boldsymbol{M}_{11} \frac{q^{-}}{p} \frac{f_0^{-\sim} q^{-\sim}}{f^{-} q^{-}} =$$

$$= (H^{=\prime})^{-1} B_{11}^{-=\prime} \frac{Q^{*} f_0^{-\sim}}{p f_0^{-}} - H\boldsymbol{M}_{11} \frac{f^{-\sim}}{p f_0^{-}} .$$

Using (2.28) and (5.34) we have

$$(H^{=\prime})^{-1} B_{11}^{-=\prime} = \frac{(H^{\sim\prime})^{-1} B_{11}^{-\sim\prime}}{z^{-d}} ,$$

and hence

$$\boldsymbol{E}_1 \frac{f^{-\sim}}{f^{-}} = \frac{(H^{\sim\prime})^{-1} B_{11}^{-\sim\prime} Q^{*} f_0^{-\sim}}{z^{-d} p f_0^{-}} - \frac{H\boldsymbol{M}_{11} f^{-\sim}}{p f_0^{-}} . \tag{5.42}$$

Now take the partial fraction expansion

$$\frac{(H^{\sim\prime})^{-1} B_{11}^{-\sim\prime} Q^{*} f_0^{-\sim}}{z^{-d} p f_0^{-}} = \frac{X}{p f_0^{-}} + \frac{(H^{\sim\prime})^{-1} Y}{z^{-d}}$$

of the first term on the right-hand side of (5.42). It follows that the X and Y are coupled by equation (5.35).

Collecting the terms gives us

$$\boldsymbol{E}_1 \frac{f^{-\sim}}{f^{-}} = \frac{(H^{\sim\prime})^{-1} Y}{z^{-d}} + \boldsymbol{A} , \tag{5.43}$$

where

(5.44) $$A = \frac{X}{pf_0^-} - \frac{HM_{1,1}f^{-\sim}}{pf_0^-}.$$

Hence, by virtue of (5.43),

(5.45) $$\left\langle \left(E_1\frac{f^{-\sim}}{f^-}\right)^{=\prime} \left(E_1\frac{f^{-\sim}}{f^-}\right)\right\rangle =$$

$$\left\langle \left(\frac{(H^{\sim\prime})^{-1}\,Y}{z^{-d}}\right)^{=\prime} \left(\frac{(H^{\sim\prime})^{-1}\,Y}{z^{-d}}\right)\right\rangle + \langle A^{=\prime}A\rangle +$$

$$+ \left\langle \left(\frac{(H^{\sim\prime})^{-1}\,Y}{z^{-d}}\right)^{=\prime} A\right\rangle + \left\langle A^{=\prime} \left(\frac{(H^{\sim\prime})^{-1}\,Y}{z^{-d}}\right)\right\rangle.$$

Any solution of equation (5.35) can be written in the form

(5.46) $$X = X^0 + D^{-1}Tpf_0^-,$$

(5.47) $$Y = Y^0 - z^{-d}H^{\sim\prime}D^{-1}T$$

by (1.19), where $D \in \mathfrak{F}_{r,r}[z^{-1}]$ is defined in (1.20) and $T \in \mathfrak{F}_{r,1}[z^{-1}]$ is arbitrary, and where

(5.48) $$\partial Y^0 < \partial z^{-d}H^{\sim\prime}.$$

Substituting (5.47) into (5.45) we obtain

$$\left\langle \left(E_1\frac{f^{-\sim}}{f^-}\right)^{=\prime} \left(E_1\frac{f^{-\sim}}{f^-}\right)\right\rangle =$$

$$= \left\langle \left(\frac{(H^{\sim\prime})^{-1}\,Y^0}{z^{-d}}\right)^{=\prime} \left(\frac{(H^{\sim\prime})^{-1}\,Y^0}{z^{-d}}\right)\right\rangle + \langle (D^{-1}T)^{=\prime}\,(D^{-1}T)\rangle -$$

$$- \left\langle \left(\frac{(H^{\sim\prime})^{-1}\,Y^0}{z^{-d}}\right)^{=\prime} (D^{-1}T)\right\rangle - \left\langle (D^{-1}T)^{=\prime} \left(\frac{(H^{\sim\prime})^{-1}\,Y^0}{z^{-d}}\right)\right\rangle +$$

$$+ \left\langle \left(\frac{(H^{\sim\prime})^{-1}\,Y^0}{z^{-d}}\right)^{=\prime} A\right\rangle + \left\langle A^{=\prime} \left(\frac{(H^{\sim\prime})^{-1}\,Y^0}{z^{-d}}\right)\right\rangle -$$

$$- \langle (D^{-1}T)^{=\prime}A\rangle - \langle A^{=\prime}(D^{-1}T)\rangle + \langle A^{=\prime}A\rangle.$$

The key observation is that

$$\left(\frac{(H^{\sim\prime})^{-1}\,Y^0}{z^{-d}}\right)^{=\prime} = z^{-[d+\partial H^{\sim\prime}-\partial Y^0]}H^{-1}(Y^0)^{\sim\prime}$$

is divisible by z^{-1} due to (5.48) and hence

$$\left\langle\left(\frac{(H^{\sim\prime})^{-1}\,Y^0}{z^{-d}}\right)^{=\prime}(D^{-1}T)\right\rangle = 0\,,$$

$$\left\langle\left(\frac{(H^{\sim\prime})^{-1}\,Y^0}{z^{-d}}\right)^{=\prime}A\right\rangle = 0\,.$$

Therefore,

$$\left\langle\left(E_1\frac{f^{-\sim}}{f^-}\right)^{=\prime}\left(E_1\frac{f^{-\sim}}{f^-}\right)\right\rangle = \langle((H^{\sim\prime})^{-1}\,Y^0)\,((H^{\sim\prime})^{-1}\,Y^0)\rangle +$$

$$+ \langle(A - D^{-1}T)^{=\prime}\,(A - D^{-1}T)\rangle\,.$$

The first term on the right-hand side of the above equation cannot be affected by any choice of $\boldsymbol{M}_{11}$ (and hence $\boldsymbol{R}$). The best we can do to minimize

(5.49) $$\left\langle\left(E_1\frac{f^{-\sim}}{f^-}\right)^{=\prime}\left(E_1\frac{f^{-\sim}}{f^-}\right)\right\rangle$$

is to set $A - D^{-1}T = 0$. By virtue of (5.44) we obtain

$$\frac{X}{pf_0^-} - \frac{H\boldsymbol{M}_{11}f^{-\sim}}{pf_0^-} - D^{-1}T = 0\,,$$

i.e.

$$X - D^{-1}Tpf_0^- = H\boldsymbol{M}_{11}f^{-\sim}\,.$$

But

$$X - D^{-1}Tpf_0^- = X^0$$

by (5.46) and hence (5.49) is minimized by setting

(5.50) $$H\boldsymbol{M}_{11}f^{-\sim} = X^0\,.$$

It means that

$$\|\boldsymbol{E}\|^2 = \langle \boldsymbol{E}^{=\prime}\boldsymbol{E}\rangle$$

is minimized by the same $\boldsymbol{M}_{11}$.

The error becomes

$$E = W - K_1 W = \frac{Q}{p} - \frac{B_{11}^- M_{11} q^-}{p} =$$

$$= \left(\frac{Q^+}{p}\begin{bmatrix} q^- \\ 0 \end{bmatrix} - \frac{B_{11}^- M_{11} q^-}{p}\right)\frac{q^{-\sim}}{q^{-\sim}} = \left(\frac{Q^*}{p} - \frac{B_{11}^- H^{-1} X^0}{p f_0^{-\sim}}\right)\frac{q^-}{q^{-\sim}}$$

by virtue of (5.50) and hence

$$(5.51)\qquad B_{11}^{-\sim\prime} E = \frac{B_{11}^{-\sim\prime} Q^* f_0^{-\sim} - z^{-d} H^{\sim\prime} X^0}{p f^{-\sim}} \frac{q^-}{q^{-\sim}} = Y^0 \frac{f^-}{f_0^{-\sim}}$$

on using (5.35).

To guarantee stability of the closed-loop system, we have to set

$$I_l - K_1 = N_1 A_1$$

for some $N_1 \in \mathfrak{F}_{l,l}^+\{z^{-1}\}$. Then

$$(5.52)\qquad B_{11}^{-\sim\prime} E = B_{11}^{-\sim\prime}(I_l - K_1) W = B_{11}^{-\sim\prime} N_1 A_1 \frac{Q}{p} =$$

$$= B_{11}^{-\sim\prime} N_1 \frac{F}{p_0} = B_{11}^{-\sim\prime}[N_{11}\ N_{12}] \frac{\begin{bmatrix} f^- \\ 0 \end{bmatrix}}{p_0} = \frac{B_{11}^{-\sim\prime} N_{11} f^-}{p_0} \frac{f^{-\sim}}{f^{-\sim}},$$

where

$$N_1 F^+ = [N_{11}\ N_{12}]$$

and $N_{11} \in \mathfrak{F}_{l,1}^+\{z^{-1}\}$, $N_{12} \in \mathfrak{F}_{l,l-1}^+\{z^{-1}\}$. By comparison of (5.51) and (5.52) we get

$$(5.53)\qquad B_{11}^{-\sim\prime} N_{11} f^{-\sim} = Y^0 p_0 .$$

The matrices M_1 and N_1 satisfy the equation

$$B_1 M_1 + N_1 A_1 = I_l .$$

However, we must also solve the equation

$$A_2 N_2 + M_2 B_2 = I_m$$

for M_2 and N_2, see Theorem 4.5, and satisfy the mutual relations

$$A_2M_1 = M_2A_1\,,$$

$$B_1N_2 = N_1B_2\,.$$

Since the M_1, N_1 and M_2, N_2 must be stable matrices, we must take only those solutions of equation (5.35) that, in addition to satisfying $\partial Y^0 < \partial z^{-d}H^{\sim\prime}$ and (5.50), (5.53), will make the M_1, N_1 and M_2, N_2 stable. Further, both the resulting control sequence

$$U = K_{W/U}W = A_2M_1\frac{Q}{p} = M_2A_1\frac{Q}{p} = M_2\frac{F}{p_0}$$

and the associated error sequence

$$E = K_{W/E}W = N_1A_1\frac{Q}{p} = N_1\frac{F}{p_0}\,,$$

must also be stable.

All optimal controllers are then obtained by (4.69) as minimal realizations of

$$R = M_2N_1^{-1} = N_2^{-1}M_1$$

where M_1, N_1 and M_2, N_2 satisfy (5.36), (5.37), and (5.38).

The optimal performance measure becomes

$$\|E\|^2_{\min} = \langle((H^{\sim\prime})^{-1}\,Y^0)\,((H^{\sim\prime})^{-1}\,Y^0)\rangle + \tag{5.54}$$

$$+\,\langle W^{=\prime}(I_l - B_{11}^{-}H^{-1}(H^{=\prime})^{-1}\,B_{11}^{-=\prime})\,W$$

by taking (5.41) into account. Note that when $r = l$ the B_{11}^{-} is invertible and, by definition, $B_{11}^{-}H^{-1}(H_{11}^{=\prime})^{-1}\,B_{11}^{-=\prime} = I_l$. Then (5.54) simplifies to

$$\|E\|^2_{\min} = \langle((H^{\sim\prime})^{-1}\,Y^0)^{=\prime}\,((H^{\sim\prime})^{-1}\,Y^0)\rangle\,.$$

Example 5.9. Consider a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}\end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}\end{bmatrix}\begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1}\end{bmatrix}^{-1}$$

over $\mathfrak{R}$ and solve problem (5.4) for the reference sequence

$$W = \frac{\begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}}{1 - z^{-1}} .$$

To make the closed-loop system stable we solve equations (5.36) and (5.37),

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} M_1 + N_1 \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} N_2 + M_2 \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} M_1 = M_2 \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix}$$

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} N_2 = N_1 \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} .$$

They give the general solutions

$$M_1 = M_2 = \begin{bmatrix} 1 + (1 - z^{-1}) t_{11} & (1 - z^{-1}) t_{12} \\ (1 - z^{-1}) t_{21} & 1 + (1 - z^{-1}) t_{22} \end{bmatrix} ,$$

$$N_1 = N_2 = \begin{bmatrix} 1 - z^{-1} t_{11} & - z^{-1} t_{12} \\ - z^{-1} t_{21} & 1 - z^{-1} t_{22} \end{bmatrix}$$

for arbitrary $t_{ij} \in \mathfrak{R}^+\{z^{-1}\}$.

Now we compute

$$B_{11}^- = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix}, \quad B_{11}^{-\sim\prime} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H^{\sim\prime} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad d = 1 ,$$

$$Q = \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}, \quad Q^+ = \begin{bmatrix} z^{-1} & 1 \\ 1 & 0 \end{bmatrix}, \quad q^- = 1 ,$$

$$F = \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}, \quad F^+ = \begin{bmatrix} z^{-1} & 1 \\ 1 & 0 \end{bmatrix}, \quad f^- = 1 , \quad p_0 = 1$$

and solve the optimality equation (5.35)

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} X + Y(1 - z^{-1}) = \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix},$$

which yields

$$X = \begin{bmatrix} 1 + (1 - z^{-1})\, t_1 \\ 1 + (1 - z^{-1})\, t_2 \end{bmatrix}, \quad Y = \begin{bmatrix} -z^{-1}t_1 \\ 1 - z^{-1}t_2 \end{bmatrix}$$

for arbitrary $t_1, t_2 \in \Re[z^{-1}]$. The solution X^0, Y^0 satisfying $\partial Y^0 < 1$ is obtained when setting $t_1 = 0,\ t_2 = 0$. Then

$$X^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Y^0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and equations (5.38) become

$$z^{-1} + z^{-1}(1 - z^{-1})\, \boldsymbol{t}_{11} + (1 - z^{-1})\, \boldsymbol{t}_{12} = 1\,,$$

$$1 + z^{-1}(1 - z^{-1})\, \boldsymbol{t}_{21} + (1 - z^{-1})\, \boldsymbol{t}_{22} = 1$$

and

$$z^{-1} - z^{-2}\boldsymbol{t}_{11} - z^{-1}\boldsymbol{t}_{12} = 0\,.$$

$$1 - z^{-2}\boldsymbol{t}_{21} - z^{-1}\boldsymbol{t}_{11} = 1\,.$$

They necessitate the choice

$$\boldsymbol{t}_{11} = \boldsymbol{v}_1\,, \quad \boldsymbol{t}_{12} = 1 - z^{-1}\boldsymbol{v}_1\,,$$

$$\boldsymbol{t}_{21} = \boldsymbol{v}_2\,, \quad \boldsymbol{t}_{22} = -z^{-1}\boldsymbol{v}_2$$

for arbitrary $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \Re^+\{z^{-1}\}$.

Therefore,

$$\boldsymbol{M}_1 = \boldsymbol{M}_2 = \begin{bmatrix} 1 + (1 - z^{-1})\, \boldsymbol{v}_1 & 1 - z^{-1} - z^{-1}(1 - z^{-1})\, \boldsymbol{v}_1 \\ (1 - z^{-1})\, \boldsymbol{v}_2 & 1 - z^{-1}(1 - z^{-1})\, \boldsymbol{v}_2 \end{bmatrix},$$

$$\boldsymbol{N}_1 = \boldsymbol{N}_2 = \begin{bmatrix} 1 - z^{-1}\boldsymbol{v}_1 & -z^{-1} - z^{-2}\boldsymbol{v}_1 \\ -z^{-1}\boldsymbol{v}_2 & 1 + z^{-2}\boldsymbol{v}_2 \end{bmatrix}$$

and all optimal controllers are given as minimal realizations of

$$\boldsymbol{R} = \boldsymbol{M}_2\boldsymbol{N}_1^{-1} = \boldsymbol{N}_2^{-1}\boldsymbol{M}_1 =$$

$$= \begin{bmatrix} \dfrac{1 + (1 - z^{-1})\, \boldsymbol{v}_1 + z^{-1}\boldsymbol{v}_2}{1 - z^{-1}\boldsymbol{v}_1} & 1 \\ \dfrac{\boldsymbol{v}_2}{1 - z^{-1}\boldsymbol{v}_1} & 1 \end{bmatrix}.$$

The resulting control is

$$U = M_2 \frac{F}{p_0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the error

$$E = N_1 \frac{F}{p_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \|E\|^2_{\min} = 0 + 1 = 1 .$$

Even if the system $\mathscr{S}$ is a very simple diagonal system, the optimal strategy requires a controller that cannot be made diagonal for any choice of v_1 and v_2. It follows that the optimal closed-loop system matrix K_1 cannot be diagonalized, either.

Example 5.10. Given a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} [1 - z^{-1}]^{-1} =$$

$$= \begin{bmatrix} 1 - z^{-1} & 0 \\ -\sqrt{2}\,(1 - z^{-1}) & 1 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix}$$

over the field $\mathfrak{R}$, solve problem (5.4) for the reference sequence

$$W = \frac{\begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix}}{z^{-1} - 2} .$$

The first equation (5.36) reads

$$\begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix} M_1 + N_1 \begin{bmatrix} 1 - z^{-1} & 0 \\ -\sqrt{2}\,(1 - z^{-1}) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and it is equivalent to the set of polynomial equations

$$z^{-1} m_{1,11} + n_{1,11}(1 - z^{-1}) = 1 , \quad z^{-1} m_{1,12} + n_{1,12} = 0 ,$$

$$n_{1,21}(1 - z^{-1}) = 0 , \qquad n_{1,22} = 1 ,$$

where

$$M_1 = [m_{1,11} \; m_{1,12}] \begin{bmatrix} 1 & 0 \\ -\sqrt{2}\,(1 - z^{-1}) & 1 \end{bmatrix} ,$$

$$N_1 = \begin{bmatrix} 1 & 0 \\ \sqrt{2}\,(1 - z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} n_{1,11} & n_{1,12} \\ n_{1,21} & n_{1,22} \end{bmatrix} .$$

The solution is

$$m_{1,11} = 1 + (1 - z^{-1})\, t_{11}\,, \quad m_{1,12} = 0 + t_{12}$$

and

$$n_{1,11} = 1 - z^{-1} t_{11}\,, \quad n_{1,12} = 0 - z^{-1} t_{12}\,,$$

$$n_{1,21} = 0\,, \qquad n_{1,22} = 1\,,$$

that is,

$$M_1 = \left[1 + (1 - z^{-1})\, t_{11} - \sqrt{2}\,(1 - z^{-1})\, t_{12} \quad t_{12}\right],$$

$$N_1 = \begin{bmatrix} 1 - z^{-1} t_{11} & -z^{-1} t_{12} \\ \sqrt{2}\,(1 - z^{-1}) - \sqrt{2}\, z^{-1}(1 - z^{-1})\, t_{11} & 1 - \sqrt{2}\, z^{-1}(1 - z^{-1})\, t_{12} \end{bmatrix}$$

for any $t_{11}, t_{12} \in \Re^+\{z^{-1}\}$. The second equation (5.36) becomes

$$\left[1 - z^{-1}\right] N_2 + M_2 \begin{bmatrix} z^{-1} \\ 0 \end{bmatrix} = [1]$$

and its solution is

$$N_2 = 1 + z^{-1} v_{11}\,,$$

$$M_2 = \left[1 - (1 - z^{-1})\, v_{11} \quad -v_{12}\right]$$

for any $v_{11}, v_{12} \in \Re^+\{z^{-1}\}$.

Mutual conditions (5.37) yield

$$1 - z^{-1} + (1 - z^{-1})^2\, t_{11} - \sqrt{2}\,(1 - z^{-1})^2\, t_{12}$$

$$= 1 - z^{-1} - (1 - z^{-1})^2\, v_{11} + \sqrt{2}\,(1 - z^{-1})\, v_{12}$$

$$(1 - z^{-1})\, t_{12} = -v_{12}$$

and hence

$$v_{11} = -t_{11}\,,$$

$$v_{12} = -(1 - z^{-1})\, t_{12}\,.$$

It follows that

$$N_2 = 1 - z^{-1} t_{11}\,,$$

$$M_2 = \left[1 + (1 - z^{-1})\, t_{11} \quad (1 - z^{-1})\, t_{12}\right].$$

Now compute

$$Q = \begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix}, \quad Q^{+} = \begin{bmatrix} 1 & 0 \\ 1 - z^{-1} & 1 \end{bmatrix}, \quad Q^{*} = \begin{bmatrix} 1 \\ 1 - z^{-1} \end{bmatrix}, \quad q^{-} = 1,$$

$$F = \begin{bmatrix} 1 - z^{-1} \\ (1 - \sqrt{2})(1 - z^{-1}) \end{bmatrix}, \quad F^{+} = \begin{bmatrix} 1 & 0 \\ 1 - \sqrt{2} & 1 \end{bmatrix},$$

$$f^{-} = f_0^{-} = 1 - z^{-1}, \quad p_0 = z^{-1} - 2,$$

$$B_{11}^{-} = \begin{bmatrix} z^{-1} \\ \sqrt{2}\, z^{-1}(1 - z^{-1}) \end{bmatrix}, \quad B_{11}^{-\sim\prime} = \begin{bmatrix} z^{-1} & \sqrt{2}\,(z^{-1} - 1) \end{bmatrix},$$

$$H = z^{-1} - 2, \quad H^{\sim\prime} = 1 - 2z^{-1}, \quad d = 1$$

and solve equation (5.35), which is

$$z^{-1}(1 - 2z^{-1})X + Y(z^{-1} - 2)(1 - z^{-1}) =$$

$$= (-\sqrt{2} + (1 + 2\sqrt{2})z^{-1} - \sqrt{2}\, z^{-2})(z^{-1} - 1).$$

We obtain

$$X = \frac{2 - \sqrt{2}}{6}(1 - z^{-1}) + (z^{-1} - 2)(1 - z^{-1})\, t_1$$

$$Y = -\frac{1}{\sqrt{2}} + \frac{2 + 2\sqrt{2}}{3} z^{-1} - z^{-1}(1 - 2z^{-1})\, t_1$$

for arbitrary $t_1 \in \Re[z^{-1}]$. The solution X^0, Y^0 with $\partial Y^0 < 2$ is

$$X^0 = \frac{2 - \sqrt{2}}{6}(1 - z^{-1}), \quad Y^0 = -\frac{1}{\sqrt{2}} + \frac{2 + 2\sqrt{2}}{3} z^{-1}$$

on setting $t_1 = 0$.

Now we have to satisfy equations (5.38). Computing

$$\boldsymbol{M}_{11} = [1 + (1 - z^{-1})\boldsymbol{t}_{11} + (1 - \sqrt{2})(1 - z^{-1})\boldsymbol{t}_{12}],$$

$$\boldsymbol{N}_{11} = \begin{bmatrix} 1 - z^{-1}\boldsymbol{t}_{11} - (1 - \sqrt{2})z^{-1}\boldsymbol{t}_{12} \\ 1 - \sqrt{2}\, z^{-1} - \sqrt{2}\, z^{-1}(1 - z^{-1})\boldsymbol{t}_{11} + (2 - \sqrt{2})z^{-1}(1 - z^{-1})\boldsymbol{t}_{12} \end{bmatrix},$$

we get

$$(z^{-1} - 2)\left[1 + (1 - z^{-1})\, t_{11} + (1 - \sqrt{2})(1 - z^{-1})\, t_{12}\right](z^{-1} - 1) =$$

$$= \frac{2 - \sqrt{2}}{6}(1 - z^{-1}),$$

$$[-\sqrt{2}\backslash + (3 + \sqrt{2})\, z^{-1} - 2z^{-2} + (2z^{-1} - 5z^{-2} + 2z^{-3})\, t_{11} +$$

$$+((2 - 2\sqrt{2})\, z^{-1} - (5 - 5\sqrt{2})\, z^{-2} + (2 - 2\sqrt{2})\, z^{-3})\, t_{12}] =$$

$$= \left(-\frac{1}{\sqrt{2}} + \frac{2 + 2\sqrt{2}}{3} z^{-1}\right)(z^{-1} - 2).$$

It can be seen that these equations cannot be satisfied by any stable rational functions t_{11} and t_{22}. Indeed, t_{11} and/or t_{12} would contain the factor $1 - z^{-1}$. Therefore, our problem has no solution.

Example 5.11. It is commonly asserted that when the system has poles on the stability boundary that are to be compensated in the least squares sence, the closed-loop system shown in Fig. 10 cannot be stable but has itself the same poles. This example illustrates that it need not always be true.

Consider a minimal realization of

$$S = \frac{0{\cdot}5z^{-2}}{1 - z^{-1}},$$

over $\mathfrak{R}$ and solve problem (5.4) for the reference sequence

$$W = \frac{1 - 0{\cdot}5z^{-1}}{1 + 0{\cdot}5z^{-1}}.$$

Solving the equation

$$0{\cdot}5z^{-2}M + N(1 - z^{-1}) = 1$$

we obtain

$$M = 2 + (1 - z^{-1})\, t, \quad N = 1 + z^{-1} - 0{\cdot}5z^{-2}t$$

for arbitrary $t \in \mathfrak{R}^+\{z^{-1}\}$.

Since

$$B_{11}^- = z^{-2}, \quad B_1^+ = 0{\cdot}5, \quad H = 1, \quad d = 2,$$

$$Q^* = 1 - 0{\cdot}5z^{-1}, \quad F^+ = 1, \quad Q^+ = 1 - 0{\cdot}5z^{-1},$$

$$q^- = 1, \quad f^- = f_0^- = 1 - z^{-1}, \quad p_0 = 1 + 0{\cdot}5z^{-1}$$

we have to solve the equation

$$z^{-2}X + Y(1 - z^{-1})(1 + 0{\cdot}5z^{-1}) = (1 - 0{\cdot}5z^{-1})(z^{-1} - 1)$$

and obtain

$$X = -0{\cdot}5(1 - z^{-1}) + (1 - z^{-1})(1 + 0{\cdot}5z^{-1})\,v\,,$$
$$Y = -(1 - z^{-1}) - z^{-2}v$$

for $v \in \Re[z^{-1}]$. The solution X^0, Y^0 for which $\partial Y^0 < 2$ reads

$$X^0 = -0{\cdot}5(1 - z^{-1})\,, \quad Y^0 = -(1 - z^{-1})$$

and equations (5.38) become

$$0{\cdot}5[2 + (1 - z^{-1})\,\boldsymbol{t}]\,(1 - 0{\cdot}5z^{-1})(z^{-1} - 1) = -0{\cdot}5(1 - z^{-1})\,,$$
$$[1 + z^{-1} - 0{\cdot}5z^{-2}\boldsymbol{t}]\,(z^{-1} - 1) = -(1 - z^{-1})(1 + 0{\cdot}5z^{-1})\,.$$

They yield

$$\boldsymbol{t} = -\frac{1}{1 - 0{\cdot}5z^{-1}}\,.$$

Then

$$\boldsymbol{M} = \frac{1}{1 - 0{\cdot}5z^{-1}}\,, \quad \boldsymbol{N} = \frac{1 + 0{\cdot}5z^{-1}}{1 - 0{\cdot}5z^{-1}}$$

and the optimal controller is unique and is given as a minimal realization of

$$\boldsymbol{R} = \frac{1}{1 + 0{\cdot}5z^{-1}}\,.$$

The pseudocharacteristic polynomial of the closed-loop system then becomes

$$C = (1 - z^{-1})(1 + 0{\cdot}5z^{-1}) + 0{\cdot}5z^{-2} = 1 - 0{\cdot}5z^{-1}\,,$$

which is stable. Further

$$\boldsymbol{U} = \frac{1 - z^{-1}}{1 + 0{\cdot}5z^{-1}}\,, \quad \boldsymbol{E} = 1 - z^{-1}\,, \quad \|\boldsymbol{E}\|^2_{\min} = 2\,.$$

Example 5.12. Given a minimal realization of

$$\boldsymbol{S} = [z^{-2} \;\; z^{-3}] = [z^{-2} \;\; 0]\begin{bmatrix}1 & -z^{-1}\\ 0 & 1\end{bmatrix}^{-1}$$

over $\mathfrak{R}$, solve problem (5.4) for the reference sequence

$$W = \frac{2 + 2z^{-1}}{2 - z^{-2}}.$$

Equations (5.36) and (5.37) become

$$[z^{-2} \ 0]\, M_1 + N_1 = 1\,,$$

$$\begin{bmatrix} 1 & -z^{-1} \\ 0 & 1 \end{bmatrix} N_2 + M_2[z^{-2} \ z^{-3}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -z^{-1} \\ 0 & 1 \end{bmatrix} M_1 = M_2\,,$$

$$[z^{-2} \ 0]\, N_2 = N_1[z^{-2} \ z^{-3}]\,.$$

The solution is

$$M_1 = \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix}, \quad N_1 = 1 - z^{-2}t_{11}\,,$$

$$N_2 = \begin{bmatrix} 1 - z^{-2}t_{11} & z^{-1} - z^{-3}t_{11} \\ -z^{-2}t_{21} & 1 - z^{-2}t_{21} \end{bmatrix}, \quad M_2 \begin{bmatrix} t_{11} - z^{-1}t_{21} \\ t_{21} \end{bmatrix}$$

for arbitrary $t_{11}, t_{21} \in \mathfrak{R}^+\{z^{-1}\}$.

The optimality equation (5.35) is

$$z^{-2}X + Y(2 - z^{-2}) = 2 + 2z^{-1}\,,$$

the solution being

$$X = 1 + z^{-1} + (2 - z^{-2})\,t\,,$$

$$Y = 1 + z^{-1} - z^{-2}t$$

for any $t \in \mathfrak{R}[z^{-1}]$. To satisfy $\partial Y^0 < 2$ we have to set $t = 0$. Then

$$X^0 = 1 + z^{-1}\,, \quad Y^0 = 1 + z^{-1}$$

and relations (5.38) become

$$t_{11}(2 + 2z^{-1}) = 1 + z^{-1}\,,$$

$$(1 - z^{-1}t_{11})(2 + 2z^{-1}) = (1 + z^{-1})(2 - z^{-2})\,.$$

They yield

$$t_{11} = 0{\cdot}5\,,$$

that is,

$$M_1 = \begin{bmatrix} 0{\cdot}5 \\ t_{21} \end{bmatrix}, \quad N_1 = 1 - 0{\cdot}5z^{-2}\,,$$

$$M_2 = \begin{bmatrix} 0{\cdot}5 & -z^{-1}t_{21} \\ & t_{21} \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 - 0{\cdot}5z^{-2} & z^{-1} - 0{\cdot}5z^{-3} \\ -z^{-2}t_{21} & 1 - z^{-3}t_{21} \end{bmatrix}.$$

Therefore, the optimal controllers are given as minimal realizations of

$$R = \frac{\begin{bmatrix} 0{\cdot}5 - z^{-1}t_{21} \\ t_{21} \end{bmatrix}}{1 - 0{\cdot}5z^{-2}}$$

and

$$U = \begin{bmatrix} 0{\cdot}5 - z^{-1}t_{21} \\ t_{21} \end{bmatrix} \frac{2 + 2z^{-1}}{2 - z^{-2}}\,, \quad E = 1 + z^{-1}\,, \quad \|E\|^2_{\min} = 2\,.$$

Note that the problem has a (stable) solution even though $q^- = 2 + 2z^{-1}$.

6. DECOUPLING A MULTIVARIABLE SYSTEM

6.1. The inverse system

Problems related to system invertibility are of basic importance in system theory. They have applications in system decoupling, decoding and signal recovering.

We first recall several algebraic concepts. Given a field $\mathfrak{F}$ and a matrix $A \in \mathfrak{F}_{m,m}$, the multiplicative *inverse* of A is defined as a matrix $A^{-1} \in \mathfrak{F}_{m,m}$ such that

$$A^{-1}A = AA^{-1} = I_m\,.$$

The inverse exists if and only if $\det A \neq 0$, or equivalently $\operatorname{rank} A = m$, and it is unique. It can be computed as

$$A^{-1} = \frac{\operatorname{adj} A}{\det A}\,, \tag{6.1}$$

where adj A is the adjoint of A, i.e. the matrix of $\mathfrak{F}_{m,m}$ whose (i, j)-th element is the cofactor of the (j, i)-th element of A, see [12]. When A^{-1} exists, the A is said to be invertible in $\mathfrak{F}_{m,m}$.

If $A \in \mathfrak{F}_{l,m}$, we can define more general kind of inverses. A matrix $\dagger A \in \mathfrak{F}_{m,l}$ such that

$$\dagger A A = I_m$$

is called the *left inverse* of A, while a matrix $A\dagger \in \mathfrak{F}_{m,l}$ such that

$$A A\dagger = I_l$$

is the *right inverse* of A.

The left inverse exists if and only if rank $A = m$ and all left inverses are given as

$$\dagger A = (CA)^{-1} C , \tag{6.2}$$

where C is a matrix in $\mathfrak{F}_{m,l}$ such that the CA is invertible in $\mathfrak{F}_{m,m}$.

The right inverse exists if and only if rank $A = l$ and all right inverses are given as

$$A\dagger = B(AB)^{-1} , \tag{6.3}$$

where B is a matrix in $\mathfrak{F}_{m,l}$ such that the AB is invertible in $\mathfrak{F}_{l,l}$.

When rank $A = l = m$, there is a unique inverse $\dagger A = A\dagger = A^{-1}$.

Given a matrix

$$A = A_0 + A_1 z^{-1} + A_2 z^{-2} + \ldots \in \mathfrak{F}_{m,m}\{z^{-1}\} ,$$

the multiplicative inverse of A is again a matrix $A^{-1} \in \mathfrak{F}_{m,m}\{z^{-1}\}$ such that

$$A^{-1} A = A A^{-1} = I_m . \tag{6.4}$$

By definition, the inverse exists if and only if A is a unit of $\mathfrak{F}_{m,m}\{z^{-1}\}$, that is, if and only if the A_0 is invertible in $\mathfrak{F}_{m,m}$. The inverse is unique and can be computed as shown in (6.1).

When $\boldsymbol{A} \in \mathfrak{F}_{l,m}\{z^{-1}\}$, the left inverse of $\boldsymbol{A}$ is a matrix $\dagger \boldsymbol{A} \in \mathfrak{F}_{m,l}\{z^{-1}\}$ such that

$$\dagger \boldsymbol{A} \boldsymbol{A} = I_m , \tag{6.5}$$

while the right inverse of $\boldsymbol{A}$ is a matrix $\boldsymbol{A}\dagger \in \mathfrak{F}_{m,l}\{z^{-1}\}$ such that

$$\boldsymbol{A}\boldsymbol{A}\dagger = I_l . \tag{6.6}$$

The left inverse exists if and only if the A_0 is left invertible in $\mathfrak{F}_{m,l}$. The left inverse is not unique and all left inverses are given by (6.2), where $\boldsymbol{C} \in \mathfrak{F}_{m,l}\{z^{-1}\}$ is such that the $\boldsymbol{CA}$ is invertible in $\mathfrak{F}_{m,m}\{z^{-1}\}$. The right inverse exists if and only if the A_0 is right invertible in $\mathfrak{F}_{m,l}$. The right inverse is not unique and all right inverses are given by (6.3), where $\boldsymbol{B} \in \mathfrak{F}_{m,l}\{z^{-1}\}$ is such that the $\boldsymbol{AB}$ is invertible in $\mathfrak{F}_{m,m}\{z^{-1}\}$.

Of course, if rank $A = l = m$ then $\dagger A = A\dagger = A^{-1}$ and the inverse is unique. If $l \neq m$ the existence of left inverse implies the nonexistence of right inverse, and vice versa.

Somewhat limited interest is attached to this intuitive notion of inverse in the system theory, however, since in a great number of cases no such inversion exists. For instance, if the system contains a delay $d > 0$, its transfer function matrix has order d and the inverse in the above sense does not exist. In this case the inverse belongs to $\mathfrak{F}_{m,l}(z^{-1})$ rather that to $\mathfrak{F}_{m,l}\{z^{-1}\}$, i.e. it is not physically realizable. Greater generality is obtained by considering "delayed" inverses defined below.

Given a system $\mathscr{S}$ over $\mathfrak{F}$ with impulse response matrix $S \in \mathfrak{F}_{l,m}\{z^{-1}\}$. Then any system $\mathscr{S}_1$ over $\mathfrak{F}$ whose impulse response matrix $S_1 \in \mathfrak{F}_{m,l}\{z^{-1}\}$ satisfies

(6.7) $$S_1 S = \operatorname{diag}\{z^{-L_1}, z^{-L_2}, \ldots, z^{-L_m}\}$$

for some nonnegative integers L_i, $i = 1, 2, \ldots, m$, is called a *delayed left inverse* of $\mathscr{S}$; any system $\mathscr{S}_2$ over $\mathfrak{F}$ whose impulse response matrix $S_2 \in \mathfrak{F}_{m,l}\{z^{-1}\}$ satisfies

(6.8) $$SS_2 = \operatorname{diag}\{z^{-R_1}, z^{-R_2}, \ldots, z^{-R_l}\}$$

for some nonnegative integers R_j, $j = 1, 2, \ldots, l$, is called a *delayed rihgt inverse* of $\mathscr{S}$.

Clearly, then, the cascade $\mathscr{S}_1\mathscr{S}$ acts as a pure delay of L_i time units in the i-th channel and the cascade $\mathscr{S}\mathscr{S}_2$ acts as a pure delay of R_j time units in the j-th channel. Otherwise speaking, the left inverse system is realized as a delayed left inverse system preceded by a bunch of L_i anticipators in the i-th channel, while the right inverse system is realized as a delayed right inverse system followed by a bunch of R_j anticipators in the j-th channel. It follows that the original input or output can be recovered by using the number of anticipators shown above.

It becomes a question of practical importance and theoretical interest to find a delayed inverse system which minimizes the number of anticipators required. Such an inverse will be called the *minimum-delay inverse*. We shall see below that the smallest numbers $L_1, L_2, \ldots, L_m$, denoted $l_1, l_2, \ldots, l_m$, are invariants of $\mathscr{S}$ with respect to left inversion and the smallest numbers $R_1, R_2, \ldots, R_l$, denoted $r_1, r_2, \ldots$ $\ldots, r_l$, are invariants of $\mathscr{S}$ with respect to right inversion. They can be interpreted as the inherent delays associated with the system, i.e. as the number of delayors which no realizable left (right) inverse can remove from the i-th (j-th) channel.

Write

(6.9) $$S = A_1^{-1}B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\},$$
$$\operatorname{rank} B_2 = r.$$

Then, by the definition of B_2 in (2.19),

(6.10)
$$B_2 = \begin{bmatrix} B_{21} \\ 0 \end{bmatrix},$$

where $B_{21} \in \mathfrak{F}_{r,m}[z^{-1}]$, $0 \in \mathfrak{F}_{l-r,m}[z^{-1}]$, and rank $B_{21} = r$.

If $r = m$, let

(6.11)
$$\det B_{21} = z^{-d} b_{20} ,$$

where $(z^{-d}, b_{20}) = 1$ and let $b_{2,ij}$, $i, j = 1, 2, \ldots, m$, be the elements of adj B_{21}. Further let

$$b_{2,ij} = z^{-d_{2,ij}} b'_{2,ij} , \quad i, j = 1, 2, \ldots, m ,$$

where $(z^{-d_{2,ij}}, b_{2,ij}) = 1$ and denote

(6.12)
$$z^{-d_{2i}} = (z^{-d_{2,i1}},\ z^{-d_{2,i2}}, \ldots, z^{-d_{2,im}}) .$$

That is, d_{2i} is the greatest common delay of the i-th row of adj B_{21}.

Write also

(6.13)
$$S = B_1 A_2^{-1} \in \mathfrak{F}_{l,m}\{z^{-1}\} ,$$
$$\text{rank } B_1 = r .$$

Then, by the definition of B_1 in (2.19),

(6.14)
$$B_1 = [B_{11}\ 0] ,$$

where $B_{11} \in \mathfrak{F}_{l,r}[z^{-1}]$, $0 \in \mathfrak{F}_{l,m-r}[z^{-1}]$, and rank $B_{11} = r$.

If $r = l$, let

(6.15)
$$\det B_{11} = z^{-d} b_{10} ,$$

where $(z^{-d}, b_{10}) = 1$ and let $b_{1,ij}$, $i, j = 1, 2, \ldots, l$, be the elements of adj B_{11}. Further let

$$b_{1,ij} = z^{-d_{1,ij}} b'_{1,ij} , \quad i, j = 1, 2, \ldots, l ,$$

where $(z^{-d_{1,ij}}, b'_{1,ij}) = 1$ and denote

(6.16)
$$z^{-d_{1j}} = (z^{-d_{1,1j}}, z^{-d_{1,2j}}, \ldots, z^{-d_{1,lj}}) .$$

That is, d_{1j} is the greatest common delay of the j-th column of adj B_{11}.

Since det B_{11} and det B_{21} are associates in $\mathfrak{F}\{z^{-1}\}$, we have $b_{10} = b_{20}$ up to units of $\mathfrak{F}\{z^{-1}\}$.

Theorem 6.1. *Let $\mathcal{S}$ be a (not necessarily minimal) realization of*

$$S = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\} .$$

Then a minimum-delay left inverse $\mathcal{S}_1$ of $\mathcal{S}$ exists if and only if

$$\text{rank } B_2 = m . \tag{6.17}$$

All minimum-delay left inverses are given as (not necessarily minimal) realizations of

$$S_1 = \frac{1}{b_{20}} \operatorname{diag} \left\{ \frac{1}{z^{-d_{21}}} , \frac{1}{z^{-d_{22}}}, \ldots, \frac{1}{z^{-d_{2m}}} \right\} [\operatorname{adj} B_{21}\ T] A_1 , \tag{6.18}$$

where $T \in \mathfrak{F}_{m,l-m}\{z^{-1}\}$ arbitrary.

The inherent delays of $\mathcal{S}$ with respect to left inversion are given as

$$l_i = d - d_{2i} , \quad i = 1, 2, \ldots, m . \tag{6.19}$$

Proof. To prove (6.17), let $\mathcal{S}_1$ be a minimum-delay left inverse of $\mathcal{S}$, i.e.

$$S_1 S = \operatorname{diag} \{ z^{-l_1}, z^{-l_2}, \ldots, z^{-l_m} \} .$$

Then rank $S = m$. Since rank $S =$ rank B_1, the necessity of (6.17) is apparent.

The sufficiency of (6.17) will be proved by construction. Let

$$\text{rank } B_2 (= \text{rank } B_{21}) = m ,$$

then

$$(\operatorname{adj} B_{21}) B_{21} = \det B_{21}$$

and

$$S_1 S = \frac{\det B_{21}}{b_{20}} \operatorname{diag} \left\{ \frac{1}{z^{-d_{21}}} , \frac{1}{z^{-d_{22}}} , \ldots , \frac{1}{z^{-d_{2m}}} \right\}$$

on using (6.18) and (6.9), (6.10). Noting (6.11) we obtain

$$S_1 S = \operatorname{diag} \left\{ \frac{z^{-d}}{z^{-d_{21}}} , \frac{z^{-d}}{z^{-d_{22}}} , \ldots , \frac{z^{-d}}{z^{-d_{2m}}} \right\} ;$$

hence $\mathscr{S}_1$ is a delayed left inverse of $\mathscr{S}$ for

$$L_i = d - d_{2i}\,, \quad i = 1, 2, \ldots, m\,.$$

Actually, it is a minimum-delay left inverse by virtue of the definition of d_{2i} and, therefore,

$$l_i = d - d_{2i}\,, \quad i = 1, 2, \ldots, m\,,$$

are the inherent delays. □

Theorem 6.2. *Let $\mathscr{S}$ be a (not necessarily minimal) realization of*

$$S = B_1 A_2^{-1} \in \mathfrak{F}_{l,m}\{z^{-1}\}\,.$$

Then a minimum-delay right inverse $\mathscr{S}_2$ of $\mathscr{S}$ exists if and only if

$$\text{rank}\; B_1 = l\,. \tag{6.20}$$

All minimum-delay right inverses are given as (not necessarily minimal) realizations of

$$S_2 = A_2 \begin{bmatrix} \text{adj}\; B_{11} \\ V \end{bmatrix} \text{diag} \left\{ \frac{1}{z^{-d_{11}}}\,, \; \frac{1}{z^{-d_{12}}}\,, \ldots, \frac{1}{z^{-d_{1l}}} \right\} \frac{1}{b_{10}}\,, \tag{6.21}$$

where $V \in \mathfrak{F}_{m-l,l}\{z^{-1}\}$ arbitrary.

The inherent delays of $\mathscr{S}$ with respect to right inversion are given as

$$r_j = d - d_{1j}\,, \quad j = 1, 2, \ldots, l\,. \tag{6.22}$$

Proof. To prove (6.20), let $\mathscr{S}_2$ be a minimum-delay right inverse of $\mathscr{S}$, i.e.

$$SS_2 = \text{diag}\,\{z^{-r_1}, z^{-r_2}, \ldots, z^{-r_l}\}\,.$$

Then rank $S = l$. Since rank S = rank B_1, the necessity of (6.20) is apparent.

The sufficiency of (6.20) will be proved by construction. Let

$$\text{rank}\; B_1 (= \text{rank}\; B_{11}) = l\,,$$

then

$$B_{11}(\text{adj}\; B_{11}) = \det B_{11}$$

and

$$SS_2 = \text{diag} \left\{ \frac{1}{z^{-d_{11}}}\,, \; \frac{1}{z^{-d_{12}}}\,, \ldots, \frac{1}{z^{-d_{1l}}} \right\} \frac{\det B_{11}}{b_{10}}$$

on using (6.21) and (6.13), (6.14). Noting (6.15) we obtain

$$SS_2 = \text{diag}\left\{\frac{z^{-d}}{z^{-d_{11}}},\ \frac{z^{-d}}{z^{-d_{12}}},\ \ldots,\ \frac{z^{-d}}{z^{-d_{1l}}}\right\};$$

hence S_2 is a delayed right inverse of S for

$$R_j = d - d_{1j}, \quad j = 1, 2, \ldots, l.$$

Actually, it is a minimum-delay right inverse by virtue of the definition of d_{1j} and, therefore,

$$r_j = d - d_{1j}, \quad j = 1, 2, \ldots, l,$$

are the inherent delays. □

It is to be noted that if S is a square nonsingular matrix, the minimum-delay left inverse system exists if and only if the minimum-delay right inverse system exists and both inverse systems are unique; however, they may be different in general. If S is not a square matrix, the existence of minimum-delay left inverse implies the non-existence of minimum-delay right inverse, and if either inverse system exists, it is not unique.

Moreover, it is clear that $l_i = 0$, $i = 1, 2, \ldots, m$, and $r_j = 0$, $j = 1, 1, \ldots, l$, implies the existence of the "instantaneous" inverse defined in (6.4) or (6.5), (6.6).

The following corollary may be useful.

Corollary 6.1. *Given an* $S \in \mathfrak{F}_{l,m}\{z^{-1}\}$, *where* $\mathfrak{F}$ *is an arbitrary field with valuation* $\mathscr{V}$. *Then the* S_1, *if it exists, is stable (with respect to* $\mathscr{V}$*) if and only if* b_{20} is stable and $T \in \mathfrak{F}^+_{m,l-m}\{z^{-1}\}$. *Similarly, the* S_2, *if it exists, is stable (with respect to* $\mathscr{V}$*) if and only if* b_{10} *is stable and* $V \in \mathfrak{F}^+_{m-l,l}\{z^{-1}\}$.

If $l = m$, *both* S_1 *and* S_2 *are stable if and only if* b_{10} *(or* b_{20}*) are stable.*

Proof. The proof is trivial in view of (6.18) and (6.21). If $l = m$, the matrices T in (6.18) and V in (6.21) disappear. Note that b_{10} and b_{20} are associates in $\mathfrak{F}\{z^{-1}\}$. □

It is of great importance in some applications to find a minimum-delay left or right inverse of *minimal dimension*. This may be a nontrivial problem when the inverse system is not unique. The explicit formulas (6.18) and (6.21) for the inverse systems are not convenient for systematic minimization of the system dimension. Instead, we shall employ the machinery of linear Diophantine equations.

Theorem 6.3. *Let* $\mathscr{S}$ *be a realization of*

$$S = \hat{B}_1 \hat{A}_2^{-1} \in \mathfrak{F}_{l,m}\{z^{-1}\}.$$

Then a minimum-dimension minimum-delay left inverse $\mathscr{S}_1$ *of* $\mathscr{S}$ *is given as a minimal realization of*

$$S_1 = (X_1^{0^{-1}})\, Y_2^0,$$

where X_1^0, Y_2^0 *is a solution of the linear Diophantine equation*

$$X_1 \operatorname{diag}\{z^{d_{21}}, z^{d_{22}}, \ldots, z^{d_{2m}}\} \hat{A}_2 - Y_2 z^d \hat{B}_1 = 0 \tag{6.23}$$

such that $\partial \det X_1^0 = \min$ *subject to*

$$\partial(\operatorname{adj} X_1^0)\, Y_2^0 \leqq \partial \det X_1^0\,, \tag{6.24}$$

X_1^0 *and* Y_2^0 *left coprime.*

Proof. Write $S = \hat{B}_1 \hat{A}_2^{-1}$ and $S_1 = X_1^{-1} Y_2$. Then

$$S_1 S = X_1^{-1} Y_2 \hat{B}_1 \hat{A}_2^{-1} = \operatorname{diag}\left\{\frac{1}{z^{l_1}}, \frac{1}{z^{l_2}}, \ldots, \frac{1}{z^{l_m}}\right\} =$$

$$= \frac{1}{z^d} \operatorname{diag}\{z^{d_{21}}\, z^{d_{22}}, \ldots, z^{d_{2m}}\}$$

by (6.19) and hence X_1, Y_2 is a solution of equation (6.23). This equation is to be solved for a solution X_1^0, Y_2^0 such that

$$\delta S_1 = \partial \det X_1^0 = \min$$

subject to physical realizability condition (6.24). □

Theorem 6.4. *Let* $\mathcal{S}$ *be a realization of*

$$S = \hat{A}_1^{-1} \hat{B}_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}\,.$$

Then a minimum-dimension minimum-delay right inverse $\mathcal{S}_2$ *of* $\mathcal{S}$ *is given as a minimal realization of*

$$S_2 = Y_1^0 (X_2^0)^{-1}\,,$$

where X_2^0, Y_1^0 *is a solution of the linear Diophantine equation*

$$\hat{A}_1 \operatorname{diag}\{z^{d_{11}}, z^{d_{12}}, \ldots, z^{d_{1l}}\} X_2 - z^d \hat{B}_2 Y_1 = 0 \tag{6.25}$$

such that $\partial \det X_2^0 = \min$ *subject to*

$$\partial Y_1^0 (\operatorname{adj} X_2^0) \leqq \partial \det X_2^0\,, \tag{6.26}$$

X_2^0 *and* Y_1^0 *right coprime.*

Proof. Write $S = \hat{A}_1^{-1}\hat{B}_2$ and $S_2 = Y_1 X_2^{-1}$. Then

$$SS_2 = \hat{A}_1^{-1}\hat{B}_2 Y_i X_2^{-1} = \operatorname{diag}\left\{\frac{1}{z^{r_1}}, \frac{1}{z^{r_2}}, \ldots, \frac{1}{z^{r_l}}\right\} =$$

$$= \frac{1}{z^d}\operatorname{diag}\{z^{d_{11}}, z^{d_{12}}, \ldots, z^{d_{1l}}\}$$

by (6.22) and hence X_2, Y_1 is a solution of equation (6.25). This equation is to be solved for a solution X_2^0, Y_1^0 such that

$$\delta S_2 = \partial \det X_2^0 = \min$$

subject to physical realizability condition (6.26). □

Equations (6.23) and (6.25) can be put to the unified form (1.5) by writing

$$Y\begin{bmatrix}\operatorname{diag}\{z^{d_{21}}, z^{d_{22}}, \ldots, z^{d_{2m}}\}\hat{A}_2 \\ -z^d\hat{B}_1\end{bmatrix} = 0 \tag{6.27}$$

and

$$[\hat{A}_1 \operatorname{diag}\{z^{d_{11}}, z^{d_{12}}, \ldots, z^{d_{1l}}\} \quad -z^d\hat{B}_2]\, X = 0 \tag{6.28}$$

respectively, where

$$X = \begin{bmatrix}X_2 \\ Y_1\end{bmatrix}, \quad Y = [X_1 \; Y_2].$$

Then the results developed in Chapter 1 can be applied to solve these equations.

It is to be noted that dimensions of $\mathscr{S}_1$ and $\mathscr{S}_2$ depend heavily on the numbers L_i and R_j. In Theorem 6.3 and Theorem 6.4 we assume that the inherent delays l_i and r_j are used, i.e. only the minimum-delay inverses are desired. However, considering delayed inverses with $L_i \geqq l_i$ or $R_j \geqq r_j$ may further reduce the inverse system dimension at the expense of increasing the delay.

Moreover, the minimal-dimension inverse is not unique, in general.

If we set

$$L_1 = L_2 = \ldots = L_m = L$$

or

$$R_1 = R_2 = \ldots = R_l = R,$$

we obtain the so called L-delay left inverse or R-delay right inverse. These special kinds of (nonminimum) delayed inverses have been extensively studied in [36; 52; 53]. The problem of minimal dimension of such inverses is solved in [36; 61].

Example 6.1. Given a realization of

$$S = \begin{bmatrix} 1 & z^{-1} \\ z^{-3} & z^{-2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -z^{-3} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & z^{-1} \\ 0 & z^{-2}(1 - z^{-2}) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ z^{-3} & z^{-2}(1 - z^{-2}) \end{bmatrix} \begin{bmatrix} 1 & -z^{-1} \\ 0 & 1 \end{bmatrix}^{-1}$$

over $\mathfrak{R}$, find minimum-delay left and right inverses.

We shall first find the inherent delays of $\mathscr{S}$. Since

$$B_{21} = \begin{bmatrix} 1 & z^{-1} \\ 0 & z^{-2}(1 - z^{-2}) \end{bmatrix}, \quad \operatorname{adj} B_{21} = \begin{bmatrix} z^{-2}(1 - z^{-2}) & -z^{-1} \\ 0 & 1 \end{bmatrix},$$

$$\det B_{21} = z^{-2}(1 - z^{-2}),$$

we have

$$d = 2,$$

$$d_{21} = 1, \quad d_{22} = 0$$

and hence the inherent delays of $\mathscr{S}$ with respect to left inversion are

$$l_1 = 1, \quad l_2 = 2.$$

Similarly,

$$B_{11} = \begin{bmatrix} 1 & 0 \\ z^{-3} & z^{-2}(1 - z^{-2} \end{bmatrix}, \quad \operatorname{adj} B_{21} = \begin{bmatrix} z^{-2}\ (1 - z^{-2} & 0 \\ -z^{-3} & 1 \end{bmatrix},$$

$$\det B_{21} = z^{-2}(1 - z^{-2})$$

and

$$d = 2,$$

$$d_{11} = 2, \quad d_{12} = 0$$

implies that the inherent delays of $\mathscr{S}$ with respect to right inversion are

$$r_1 = 0, \quad r_2 = 2.$$

Thus the the minimum-delay left inverse is a realization of

$$S_1 = \frac{1}{1 - z^{-2}} \begin{bmatrix} \dfrac{1}{z^{-1}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-2}(1 - z^{-2}) & -z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -z^{-3} & 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} z^{-1} & -1 \\ -z^{-3} & 1 \end{bmatrix}}{1 - z^{-2}}$$

by (6.18); the minimum-delay right inverse is a realization of

$$S_2 = \begin{bmatrix} 1 & -z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-2}(1 - z^{-1}) & 0 \\ -z^{-3} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{z^{-2}} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{1 - z^{-2}} =$$

$$= \frac{\begin{bmatrix} 1 & -z^{-1} \\ -z^{-1} & 1 \end{bmatrix}}{1 - z^{-2}}$$

by (6.21) and both impulse responses are unique but different.

If we choose $L = L_1 = L_2 = 2$, $R = R_1 = R_2 = 2$, we would obtain

$$S_1 = S_2 = \frac{\begin{bmatrix} z^{-3} & -z^{-1} \\ -z^{-3} & 1 \end{bmatrix}}{1 - z^{-2}},$$

but this is *not* a minimum-delay inverse.

Example 6.2. Given a realization of

$$S = \frac{\begin{bmatrix} z^{-1}(1 - z^{-1}) \\ z^{-2} \\ 1 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1}(1 - z^{-1}) \\ z^{-2} \\ 1 \end{bmatrix} [1 - z^{-1}]^{-1} =$$

$$= \begin{bmatrix} 0 & 0 & 1 - z^{-1} \\ 1 & 0 & -z^{-1}(1 - z^{-1}) \\ 0 & 1 & -z^{-2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

over $\mathfrak{R}$. Find all minimum-delay left inverses and also a minimum-delay left inverse of minimal dimension.

Since

$$B_{21} = 1, \quad \operatorname{adj} B_{21} = 1, \quad \det B_{21} = 1$$

we have

$$d = 0, \quad d_{21} = 0, \quad l_1 = 0.$$

Thus all minimum-delay left inverses are given by

$$S_1 = [1 \ T_1 \ T_2] \begin{bmatrix} 0 & 0 & 1 - z^{-1} \\ 1 & 0 & -z^{-1}(1 - z^{-1}) \\ 0 & 1 & -z^{-2} \end{bmatrix} =$$

$$= [T_1 \quad T_2 \quad (1 - z^{-1}) - z^{-1}(1 - z^{-1}) T_1 - z^{-2} T_2]$$

for arbitrary $T_1, T_2 \in \mathfrak{R}\{z^{-1}\}$, and the inverses are instantaneous.

To minimize the degree of S_1 we have to find appropriate T_1 and T_2. This can be done systematically by using Theorem 6.3. Write

$$S = \frac{\begin{bmatrix} z-1 \\ 1 \\ z^2 \end{bmatrix}}{z(z-1)} = \begin{bmatrix} z-1 \\ z \\ z^2 \end{bmatrix} [z(z-1)]^{-1} .$$

Equation (6.23) becomes

$$Y \begin{bmatrix} z(z-1) \\ -(z-1) \\ -1 \\ -z^2 \end{bmatrix} = 0 , \quad Y = [X_1 \ Y_2] ,$$

that is,

$$Y \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

and

$$Y = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & -(z-1) & 0 \\ 1 & 0 & z(z-1) & 0 \\ 0 & 0 & -z^2 & 1 \end{bmatrix} \bar{\bar{Y}} .$$

The general solution is

$$Y = [0 \ t_1 \ t_2 \ t_3] ,$$

i.e.

$$Y = [t_2 \quad t_1 \quad -(z-1)\,t_1 + z(z-1)\,t_2 - z^2 t_3 \ t_3]$$

for arbitrary $t_i \in \Re[z]$.

It follows that

$$X_1 = [t_2] ,$$

$$Y_2 = [t_1 \quad -(z-1)\,t_1 + z(z-1)\,t_2 - z^2 t_3 \quad t_3] .$$

The condition $\partial \det X_1 = \min$ calls for

$$t_2 = \tau_2 \neq 0 , \quad \tau_2 \in \Re .$$

Then the first condition (6.24) necessitates

$$t_1 = \tau_1 \in \mathfrak{R}\,, \quad t_3 = \tau_3 \in \mathfrak{R}$$

and

$$-(z-1)\,\tau_1 + z(z-1)\,\tau_2 - z^2\tau_3 = \tau_4\,.$$

Therefore,

$$\begin{aligned} \tau_1 &= -\tau_2\,, \\ \tau_3 &= \tau_2\,, \\ \tau_4 &= \tau_1 = -\tau_2 \end{aligned}$$

and the minimum-dimension minimum-delay left inverse is given as a minimal realization of

$$S_1 = [\tau_2]^{-1}\,[-\tau_2\ -\tau_2\ \tau_2] = [-1\ -1\ 1]\,.$$

Note that the inverse is unique and $\delta S_1 = 0$.

Of course, the existence of left inverse implies the nonexistence of right inverse of any kind.

Example 6.3. Consider a system $\mathscr{S}$ over $\mathfrak{Z}_2$ given by

$$S = \frac{[1\ 0]}{z} = [z]^{-1}\,[1\ 0]$$

and find the minimum-dimension minimum-delay right inverse $\mathscr{S}_2$ of $\mathscr{S}$.

Since

$$S = [z^{-1}\ 0]\,,$$

we have

$$B_{11} = z^{-1}\,, \quad \operatorname{adj} B_{11} = 1\,, \quad \det B_{11} = z^{-1}\,,$$

$$d = 1\,, \quad d_{11} = 0$$

and obtain

$$r_1 = 1\,.$$

No realizable inverse can remove this inherent delay.

To find a least dimension inverse, we solve equation (6.25),

$$[z\ -z\ 0]\,X = 0\,,$$

which is equivalent to

$$[z\ 0\ 0]\,\overline{X} = 0$$

and

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{X}.$$

It follows that

$$\bar{X} = \begin{bmatrix} 0 \\ t_1 \\ t_2 \end{bmatrix}, \quad X = \begin{bmatrix} t_1 \\ t_1 \\ t_2 \end{bmatrix}$$

for any $t_1, t_2 \in \mathfrak{Z}_2[z]$, and hence

$$X_2 = [t_1], \quad Y_1 = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}.$$

To satisfy ∂ det $X^2 = \min$, we set

$$t_1 = \tau_1 \in \mathfrak{Z}_2, \quad \tau_1 \neq 0.$$

Then physical realizability condition (6.26) necessitates $\partial t_2 \leqq \partial t_1$, i.e. $t_2 = \tau_2 \in \mathfrak{Z}_2$. As a result, the minimum-dimension minimum-delay right inverse is given as a minimal realization of

$$\boldsymbol{S}_2 = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} [\tau_1]^{-1} = \begin{bmatrix} 1 \\ \tau \end{bmatrix}, \quad \delta \boldsymbol{S}_2 = 0$$

for arbitrary $\tau \in \mathfrak{Z}_2$. Note that the inverse of minimal dimension need not be unique.

6.2. The decoupling problem

A multivariable system is a collection of coupled subsystems. Thus a particular input component may influence all output components. It would certainly be convenient for control purposes if a particular input component effected just the corresponding output component and all others left unaffected. This motivates the following definition.

(6.29) *Stable decoupling problem*

Given a closed-loop system configuration shown in Fig. 16, where $\mathscr{S}$ is a minimal realization of $\boldsymbol{S} \in \mathfrak{F}_{l,m}\{z^{-1}\}$ and $\mathfrak{F}$ is an arbitrary field with valuation $\mathscr{V}$. Consider the partition of the system output $\boldsymbol{Y}$ into l components

$$\boldsymbol{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_l \end{bmatrix}$$

and the corresponding partition of the reference input sequence

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_l \end{bmatrix}.$$

Find a controller $\mathscr{R}$ which is a minimal realization of some

$$R \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the closed-loop system is stable (with respect to $\mathscr{V}$) and the j-th reference input component w_j does not affect the output components y_i for $i \neq j$.

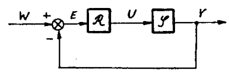

Fig. 16. Decoupled closed-loop system.

Since

$$Y = K_1 W,$$

the stable decoupling problem calls for a diagonal matrix K_1. In view of the expression

$$K_1 = SR(I_l + SR)^{-1}$$

it is intuitively apparent that $\mathscr{R}$ must be a kind of right inverse of $\mathscr{S}$ so that K_1 may be a diagonal matrix. This inverse will be more restricted, however, due to the requirement of closed-loop stability.

Write

$$S = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\},$$

where

(6.30) $$B_1 = [B_{11} \ 0]$$

and

$$B_{11} \in \mathfrak{F}_{l,r}[z^{-1}],$$
$$\operatorname{rank} B_{11} = r.$$

If $r = l$ let $b_{1,ij}$, $i, j = 1, 2, \ldots, l$, be the elements of adj B_{11}. Further let

$$b_{1j} = (b_{1,1j}, b_{1,2j}, \ldots, b_{1,lj})$$

and denote

$$\frac{\det B_{11}}{b_{1j}} = b_{0j}, \quad j = 1, 2, \ldots, l. \tag{6.31}$$

That is, b_{1j} is a greatest common divisor of the j-th column of adj B_{11}.

Similarly, let $a_{1,ij}$, $i, j = 1, 2, \ldots, l$, be the elements of adj A_1. Further let

$$a_{1i} = (a_{1,i1}, a_{1,i2}, \ldots, a_{1,il})$$

and denote

$$\frac{\det A_1}{a_{1i}} = a_{0i}, \quad i = 1, 2, \ldots, l. \tag{6.32}$$

That is, a_{1i} is a greatest common divisor of the i-th row of adj A_1.

Then we have the following result.

Theorem 6.5. *Problem* (6.29) *has a solution if and only if* rank $B_1 = l$ *and the linear Diophantine equation*

$$\operatorname{diag}\{b_{01}, \ldots, b_{0l}\}\, \boldsymbol{D}_1 + \boldsymbol{D}_2 \operatorname{diag}\{a_{01}, \ldots, a_{0l}\} = I_l \tag{6.33}$$

has a diagonal matrix solution $\boldsymbol{D}_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$, $\boldsymbol{D}_2 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ *such that matrices* $\boldsymbol{M}_1 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$, $\boldsymbol{N}_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ *and* $\boldsymbol{M}_2 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$, $\boldsymbol{N}_2 \in \mathfrak{F}^+_{m,m}\{z^{-1}\}$ *exist and satisfy the equations*

$$\begin{aligned} B_1\boldsymbol{M}_1 + \boldsymbol{N}_1 A_1 &= I_l, \\ A_2\boldsymbol{N}_2 + \boldsymbol{M}_2 B_2 &= I_m, \end{aligned} \tag{6.34}$$

$$\begin{aligned} A_2\boldsymbol{M}_1 &= \boldsymbol{M}_2 A_1, \\ B_1\boldsymbol{N}_2 &= \boldsymbol{N}_1 B_2 \end{aligned} \tag{6.35}$$

and

$$\begin{aligned} \boldsymbol{M}_{11} &= (\operatorname{adj} B_{11}) \operatorname{diag}\left\{\frac{1}{b_{11}}, \frac{1}{b_{12}}, \ldots, \frac{1}{b_{1l}}\right\} \boldsymbol{D}_1, \\ \boldsymbol{N}_1 &= \boldsymbol{D}_2 \operatorname{diag}\left\{\frac{1}{a_{11}}, \frac{1}{a_{12}}, \ldots, \frac{1}{a_{1l}}\right\} (\operatorname{adj} A_1), \\ \boldsymbol{M}_1 &= \begin{bmatrix} \boldsymbol{M}_{11} \\ \boldsymbol{M}_{21} \end{bmatrix}, \end{aligned} \tag{6.36}$$

where

$$\boldsymbol{M}_{11} \in \mathfrak{F}^+_{l,l}\{z^{-1}\}, \quad \boldsymbol{M}_{21} \in \mathfrak{F}^+_{m-l,l}\{z^{-1}\}.$$

The controller which stably decouples the closed-loop system is not unique and all such controllers are given as minimal realization of

$$R = M_2 N_1^{-1} = N_2^{-1} M_1 . \tag{6.37}$$

Moreover,

$$K_1 = \operatorname{diag}\{b_{01}, b_{02}, \ldots, b_{0l}\}\, D_1 , \tag{6.38}$$

$$I_l - K_1 = D_2 \operatorname{diag}\{a_{01}, a_{02}, \ldots, a_{0l}\} .$$

Proof. Necessity: Let the closed-loop system be decoupled and stable. Its stability implies, by Theorem 4.5, that matrices $M_1 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$, $N_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ and $M_2 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$, $N_2 \in \mathfrak{F}^+_{m,m}\{z^{-1}\}$ exist and satisfy (6.34) and (6.35). It follows that

$$K_1 = B_1 M_1 ,$$

$$I_l - K_1 = N_1 A_1 .$$

Write

$$K_1 = B_1 M_1 = [B_{11}\ 0] \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = B_{11} M_{11} ,$$

where $B_{11} \in \mathfrak{F}_{l,r}[z^{-1}]$, $\operatorname{rank} B_{11} = \operatorname{rank} B_1 = r$ and $M_{11} \in \mathfrak{F}^+_{r,l}\{z^{-1}\}$, $M_{21} \in F^+_{m-r,l}\{z^{-1}\}$. Thus K_1 can be a diagonal matrix only if $r = l$, i.e. only if $\operatorname{rank} B_1 = l$.

Then $B_{11} \in \mathfrak{F}_{l,l}[z^{-1}]$ is a nonsingular matrix and

$$B_{11} = \det B_{11} (\operatorname{adj} B_{11})^{-1} .$$

Hence the M_{11} must be of the form

$$M_{11} = (\operatorname{adj} B_{11}) \operatorname{diag}\left\{\frac{1}{b_{11}}, \frac{1}{b_{12}}, \ldots, \frac{1}{b_{1l}}\right\} D_1 ,$$

where $D_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ is a diagonal matrix. It follows that K_1 has the least possible predetermination

$$K_1 = \operatorname{diag}\{b_{01}, b_{02}, \ldots, b_{0l}\}\, D_1 .$$

The K_1 being diagonal, the $I_l - K_1$ is also diagonal (and, of course, nonsingular). We can write

$$A_1 = (\operatorname{adj} A_1)^{-1} \det A_1$$

and hence the N_1 must be of the form

$$N_1 = D_2 \operatorname{diag}\left\{\frac{1}{a_{11}}, \frac{1}{a_{12}}, \ldots, \frac{1}{a_{1l}}\right\}(\operatorname{adj} A_1),$$

where $D_2 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ is a diagonal (and also nonsingular) matrix. It follows that $I_l - K_1$ has the least possible predetermination

$$I_l - K_1 = D_2 \operatorname{diag}\{a_{01}, a_{02}, \ldots, a_{0l}\}.$$

We conclude that (6.33) and (6.36) hold.

Sufficiency: Let rank $B_1 = l$. Further let matrices $M_1 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$, $N_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ and $M_2 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}$, $N_2 \in \mathfrak{F}^+_{m,m}\{z^{-1}\}$ and diagonal matrices $D_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$, $D_2 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ exist and satisfy (6.33) through (6.36).

By virtue of (6.34) and (6.35) the closed-loop system is stable and

$$K_1 = B_1 M_1, \quad I_l - K_1 = N_1 A_1.$$

Now rank $B_1 = l$ implies that $B_{11} \in \mathfrak{F}_{l,l}[z^{-1}]$ is a nonsingular matrix and using (6.36) we obtain

$$K_1 = B_{11} M_{11} = B_{11}(\operatorname{adj} B_{11}) \operatorname{diag}\left\{\frac{1}{b_{11}}, \frac{1}{b_{12}}, \ldots, \frac{1}{b_{1l}}\right\} D_1 =$$

$$= \operatorname{diag}\{b_{01}, b_{02}, \ldots, b_{0l}\} D_1,$$

$$I_l - K_1 = N_1 A_1 = D_2 \operatorname{diag}\left\{\frac{1}{a_{11}}, \frac{1}{a_{12}}, \ldots, \frac{1}{a_{1l}}\right\}(\operatorname{adj} A_1) A_1 =$$

$$= D_2 \operatorname{diag}\{a_{01}, a_{02}, \ldots, a_{0l}\}.$$

Thus the K_1 (and also $I_l - K_1$) is a diagonal matrix, i.e. the closed-loop system is decoupled in addition to being stable.

To find all controllers which stably decouple the closed-loop system, we shall apply (4.66) and write

$$R = M_2 N_1^{-1} = N_2^{-1} M_1$$

where M_1, N_1 and M_2, N_2 are given by (6.33) through (6.36). □

It is to be noted that diagonal matrices $D_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ and $D_2 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$ exist and satisfy equation (6.33) if and only if

$$(a_{0i}, b_{0i}) = 1, \quad i = 1, 2, \ldots, l,$$

modulo units in $\mathfrak{F}^{+}\{z^{-1}\}$, that is, if and only if

$$(a_{0i}^{-}, b_{0i}^{-}) = 1\,, \quad i = 1, 2, \ldots, l\,,$$

modulo units in $\mathfrak{F}[z^{-1}]$.

When $l = m$ (the $\boldsymbol{S}$ is a square matrix) then $B_1 = B_{11}$, $\boldsymbol{M}_1 = \boldsymbol{M}_{11}$ and the first equation (6.34) is equivalent to equation (6.33) when relations (6.36) are taken into account. The decoupling controllers are then given simply as

$$(6.39) \qquad \boldsymbol{R} = \boldsymbol{M}_2\boldsymbol{N}_1^{-1} = A_2\boldsymbol{M}_1A_1^{-1}\boldsymbol{N}_1^{-1} =$$

$$= A_2(\operatorname{adj} B_{11}) \operatorname{diag}\left\{\frac{1}{b_{11}}, \ldots, \frac{1}{b_{1l}}\right\} \operatorname{diag}\left\{\frac{1}{a_{11}}, \ldots, \frac{1}{a_{1l}}\right\} \boldsymbol{D}_1\boldsymbol{D}_2^{-1}\,.$$

When $l \neq m$ we cannot avoid solving the first equation (6.34) because $\boldsymbol{D}_1$ specifies just the matrix $\boldsymbol{M}_{11}$, not the $\boldsymbol{M}_{21}$. The submatrix $\boldsymbol{M}_{21}$ is determined solely by the stability considerations.

Example 6.4. Given a minimal realization of

$$\boldsymbol{S} = \frac{\begin{bmatrix} z^{-1} & z^{-1}(1 - z^{-1})^2 \\ 0 & z^{-1}(1 - z^{-1})^2 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} & z^{-1}(1 - z^{-1}) \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 - z^{-1} & -1 + z^{-1} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix}$$

over the field $\mathfrak{R}$ valuated by (2.25), find all decoupling controllers.

We have

$$\operatorname{rank} B_1 = 2\,, \quad B_{11} = B_1\,,$$

$$\operatorname{adj} B_{11} = \begin{bmatrix} z^{-1}(1 - z^{-1}) & -z^{-1}(1 - z^{-1}) \\ 0 & z^{-1} \end{bmatrix}, \quad \det B_{11} = z^{-2}(1 - z^{-1})\,,$$

$$(6.40) \qquad b_{11} = z^{-1}(1 - z^{-1})\,, \quad b_{12} = z^{-1}\,,$$

$$b_{01} = z^{-1}\,, \qquad b_{02} = z^{-1}(1 - z^{-1})$$

and

$$\operatorname{adj} A_1 = \begin{bmatrix} 1 & 1 - z^{-1} \\ 0 & 1 - z^{-1} \end{bmatrix}, \quad \det A_1 = 1 - z^{-1}$$

$$(6.41) \qquad a_{11} = 1\,, \qquad a_{12} = 1 - z^{-1}\,,$$

$$a_{01} = 1 - z^{-1}\,, \quad a_{02} = 1\,.$$

Equation (6.33) becomes

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1-z^{-1}) \end{bmatrix} D_1 + D_2 \begin{bmatrix} 1-z^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and its general solution for diagonal D_1 and D_2 is

$$(6.42)\quad D_1 = \begin{bmatrix} 1 + (1 - z^{-1}t_1) & 0 \\ 0 & t_2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 - z^{-1}t_1 & 0 \\ 0 & 1 - z^{-1}(1 - z^{-1})\, t_2 \end{bmatrix}$$

for any $t_1, t_2 \in \Re^+\{z^{-1}\}$.

Since the S is a square matrix, the solution of the first equation (6.34) is obtained via (6.36) as

$$M_1 = \begin{bmatrix} 1 + (1 - z^{-1})\, t_1 & -(1 - z^{-1})\, t_2 \\ 0 & t_2 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1 - z^{-}t_1 & 1 - z^{-1} - z^{-1}(1 - z^{-1})\, t_1 \\ 0 & 1 - z^{-1}(1 - z^{-1})\, t_2 \end{bmatrix}.$$

The second equation (6.34) becomes

$$\begin{bmatrix} 1-z^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_2 + M_2 \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1-z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and yields

$$N_2 = \begin{bmatrix} 1 + z^{-1}v_{11} & z^{-1}v_{12} \\ z^{-1}v_{21} & 1 + z^{-1}(1 - z^{-1})\, v_{22} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 - (1 - z^{-1})\, v_{11} & -v_{12} \\ -\, v_{21} & -v_{22} \end{bmatrix}$$

for arbitrary $v_{ij} \in \Re^+\{z^{-1}\}$.

Mutual relations (6.35) then necessitate the choice

$$v_{11} = -t_1\,, \quad v_{12} = -(1 - z^{-1}) + (1 - z^{-1})^2\, (t_2 - t_1)\,,$$

$$v_{21} = 0\,, \qquad v_{22} = -t_2\,,$$

that is,

$$M_2 = \begin{bmatrix} 1 + (1 - z^{-1})\, t_1 & 1 - z^{-1} - (1 - z^{-1})^2\, (t_2 - t_1) \\ 0 & t_2 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 1 - z^{-1}t_1 & -z^{-1}(1 - z^{-1}) + z^{-1}(1 - z^{-1})^2\, (t_2 - t_1) \\ 0 & 1 - z^{-1})1 - z^{-1})\, t_2 \end{bmatrix}.$$

Our problem has a solution and all decoupling controllers are given by (6.39) as minimal realizations of

$$R = \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-1}(1 - z^{-1}) & -z^{-1}(1 - z^{-1}) \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} z^{-1}(1 - z^{-1}) & 0 \\ 0 & z^{-1} \end{bmatrix}^{-1} .$$

$$. \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 0 \end{bmatrix}^{-1} D_1 D_2^{-1} = \begin{bmatrix} 1 & -(1 - z^{-1})^2 \\ 0 & 1 \end{bmatrix} D_1 D_2^{-1} ,$$

where D_1 and D_2 are given in (6.42)

Then

$$K_1 = \begin{bmatrix} z^{-1} + z^{-1}(1 - z^{-1}) t_1 & 0 \\ 0 & z^{-1}(1 - z^{-1}) t_2 \end{bmatrix}$$

is indeed diagonal.

Example 6.5. It should be noted that it may be impossible, in some cases, to make the decoupled system stable. To demonstrate this phenomenon, consider the system over $\mathfrak{R}$ valuated by (2.25) that is a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} & z^{-1}(1 - z^{-1})^2 \\ -z^{-1} & z^{-1}(1 - z^{-1})^2 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} z^{-1} & z^{-1}(1 - z^{-1}) \\ -z^{-1} & z^{-1}(1 - z^{-1}) \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 0{\cdot}5(1 - z^{-1}) & -0{\cdot}5(1 - z^{-1}) \\ 0{\cdot}5 & 0{\cdot}0 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - z^{-1}) \end{bmatrix} .$$

We compute

$$B_{11} = \begin{bmatrix} z^{-1} & z^{-1}(1 - z^{-1}) \\ -z^{-1} & z^{-1}(1 - z^{-1}) \end{bmatrix} , \quad \operatorname{adj} B_{11} = \begin{bmatrix} z^{-1}(1 - z^{-1}) & -z^{-1}(1 - z^{-1}) \\ z^{-1} & z^{-1} \end{bmatrix} ,$$

$$\operatorname{rank} B_{11} = 2 , \quad \det B_{11} = 2z^{-2}(1 - z^{-1}) ,$$

$$b_{11} = z^{-1} , \qquad b_{12} = z^{-1} ,$$

$$b_{01} = 2z^{-1}(1 - z^{-1}) , \quad b_{02} = 2z^{-1}(1 - z^{-1})$$

and

$$A_1 = \begin{bmatrix} 0{\cdot}5(1 - z^{-1}) & -0{\cdot}5(1 - z^{-1}) \\ 0{\cdot}5 & 0{\cdot}5 \end{bmatrix} , \quad \operatorname{adj} A_1 = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}5(1 - z^{-1}) \\ -0{\cdot}5 & 0{\cdot}5(1 - z^{-1}) \end{bmatrix} ,$$

$$\det A_1 = 0{\cdot}5(1 - z^{-1}) ,$$

$$a_{11} = 0{\cdot}5 , \qquad a_{12} = 0{\cdot}5 ,$$

$$a_{01} = 1 - z^{-1} , \quad a_{02} = 1 - z^{-1} .$$

Since

$$(a_{01}, b_{01}) = 1 - z^{-1},$$
$$(a_{02}, b_{02}) = 1 - z^{-1}$$

are not units of $\mathfrak{R}^+\{z^{-1}\}$, equation (6.33) can have no solution. Therefore, the system cannot be stably decoupled.

6.3. Decoupling and optimal control

The ultimate purpose of decoupling a multivariable system is to simplify its control. It is often convenient in practice when an input component affects just the corresponding output component and no others. Such a system, in fact, acts as the direct sum of single-input single-output subsystems.

Given a system $\mathscr{S}$ which is a minimal realization of

$$S = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}.$$

Write

$$B_1 = [B_{11}\ 0]$$

and let rank $B_{11} = l$. Denote

$$b_{1j} = \text{greatest common divisor of the } j\text{-th column of adj } B_{11},$$
$$a_{1i} = \text{greatest common divisor of the } i\text{-th row of adj } A_1$$

and

$$b_{0j} = \frac{\det B_{11}}{b_{ij}}, \quad a_{0i} = \frac{\det A_1}{a_{1i}}.$$

Further let

$$D_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}, \quad D_2 \in \mathfrak{F}^+_{l,l}\{z^{-1}\}$$

be diagonal matrices and

$$M_1 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}, \quad N_1 \in \mathfrak{F}^+_{l,l}\{z^{-1}\},$$
$$M_2 \in \mathfrak{F}^+_{m,l}\{z^{-1}\}, \quad N_2 \in \mathfrak{F}^+_{m,m}\{z^{-1}\}$$

be matrices satisfying the equations

(6.43)
$$B_1 M_1 + N_1 A_1 = I_l,$$
$$A_2 N_2 + M_2 B_2 = I_m,$$
$$A_2 M_1 = M_2 A_2,$$
$$B_1 N_2 = N_1 B_2,$$

(6.44)
$$\text{diag}\{b_{01}, \ldots, b_{0l}\} D_1 + D_2 \,\text{diag}\{a_{01}, \ldots, a_{0l}\} = I_l$$

and

$$(6.45) \qquad M_1 = (\operatorname{adj} B_{11}) \operatorname{diag}\left\{\frac{1}{b_{11}}, \ldots, \frac{1}{b_{1l}}\right\} D_1 ,$$

$$N_1 = D_2 \operatorname{diag}\left\{\frac{1}{a_{11}}, \ldots, \frac{1}{a_{1l}}\right\} (\operatorname{adj} A_1) ,$$

where

$$M_1 = \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}.$$

All controllers $\mathscr{R}$ which stably decouple the closed-loop system are given by Theorem 6.5 as minimal realizations of

$$(6.46) \qquad R = M_2 N_1^{-1} = N_2^{-1} M_1 .$$

The degrees of freedom in the controllers $\mathscr{R}$ can be utilized for optimization. The problem is to find appropriate D_1 and D_2 so that an optimality criterion may be minimized. We denote

$$(6.47) \qquad D_1 D_2^{-1} = \operatorname{diag}\left\{\frac{s_1}{r_1}, \frac{s_2}{r_2}, \ldots, \frac{s_l}{r_l}\right\} ,$$

where s_i and r_i, $i = 1, 2, \ldots, l$, are polynomials coprime in $\mathfrak{F}[z^{-1}]$. Then

$$SR = B_1 A_2^{-1} M_2 N_1^{-1} = B_1 A_2^{-1} A_2 M_1 A_1^{-1} N_1^{-1} = B_{11} M_{11} A_1^{-1} N_1^{-1} =$$

$$= \operatorname{diag}\left\{\frac{b_{01}}{a_{01}} \frac{s_1}{r_1}, \ldots, \frac{b_{0l}}{a_{0l}} \frac{s_l}{r_l}\right\}$$

by (6.46), (6.43) and (6.45), (6.47).

We have seen that the a_{0i} and b_{0i} need not be coprime polynomials. Thus denote

$$\frac{b_{0i}}{a_{0i}} = \frac{b_i}{a_i}, \quad i = 1, 2, \ldots, l ,$$

after cancelling the common factors. In fact, only stable factors may cancel since otherwise the closed-loop system could not have been stably decoupled. Hence

$$SR = \operatorname{diag}\left\{\frac{b_1}{a_1} \frac{s_1}{r_1}, \ldots, \frac{b_l}{a_l} \frac{s_l}{r_l}\right\}$$

and

$$K_1 = SR(I_l + SR)^{-1} = \operatorname{diag}\left\{\frac{b_1 s_1}{a_1 r_1 + b_1 s_1}, \ldots, \frac{b_l s_l}{a_l r_l + b_l s_l}\right\}.$$

The above formulas can be interpreted as follows. The closed-loop system, as far as its input-output properties are concerned, acts as the direct sum of l single-input single-output closed-loop systems, each containing a virtual system $\mathscr{S}_i$ to be controlled given by

$$S_i = \frac{b_i}{a_i}, \quad i = 1, 2, \ldots, l,$$

and a virtual controller $\mathscr{R}_i$ given by

$$R_i = \frac{s_i}{r_i}, \quad i = 1, 2, \ldots, l.$$

Therefore, the system $\mathscr{S}$ itself can be viewed as the direct sum of the virtual systems $\mathscr{S}_i$ and the optimal control of the decoupled closed-loop system can be obtained by working separately on each $\mathscr{S}_i$. For this purpose the theory developed in [30; 31; 32; 33; 34] can be used.

It should be stressed that $\mathscr{S}$ can be viewed as the direct sum of the above virtual subsystems $\mathscr{S}_i$ only relative to the *external* properties of the closed-loop system. The *internal* behavior of the closed-loop system cannot be described by the virtual subsystems $\mathscr{S}_i$ and the methods developed in Chapter 4 for general multivariable systems have to be applied.

For example, the pseudocharacteristic polynomial c of the decoupled system is not equal to the product of the pseudocharacteristic polynomials c_i of the individual virtual closed loops. Example: Given an $\mathfrak{F}$ valuated by $\mathscr{V}$ and a minimal realization of

$$S = \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & z^{-1} & 0 \end{bmatrix}$$

over $\mathfrak{F}$, then a minimal realization of

$$R = \begin{bmatrix} t & 0 \\ 0 & 1 \\ 1 & t \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

stably decouples the closed-loop system for any stable (with respect to $\mathscr{V}$) polynomial $t \in \mathfrak{F}[z^{-1}]$. Indeed,

$$K_1 = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix}$$

and the decoupled system consists of two virtual systems with pseudocharacteristic polynomials $c_1 = 1$, $c_2 = 1$.

The pseudocharacteristic polynomial c, however, is given by (4.26) as

$$c = \det \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} = t\,.$$

When synthesizing the optimal controls we need not know the pseudocharacteristic polynomial as such, it is sufficient to know that it is stable.

The purpose of this section is to show that the decoupling imposes certain restrictions on the existence and attainable performance of the optimal controls and also to show how the optimal controller should be found.

Given a reference input sequence

$$\boldsymbol{W} = \frac{Q}{p} \in \mathfrak{F}_{l,1}\{z^{-1}\}\,,$$

for the decoupling purposes we shall partition the Q as

$$Q = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_l \end{bmatrix}$$

and let

$$\boldsymbol{W}_i = \frac{g_i}{p} = \frac{q_i}{p_i}\,, \quad i = 1, 2, \ldots, l\,,$$

after cancelling the common factors, i.e. $(p_i, q_i) = 1$

For convenience, let

$$(a_i, p_i) = d_i\,, \quad i = 1, 2, \ldots, l\,,$$

and write

$$a_i = a_{10} d_i\,,$$

$$p_i = \quad d_i p_{i0}\,.$$

The error sequence $\boldsymbol{E}$ can be conformably partitioned as

$$\boldsymbol{E} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_l \end{bmatrix}.$$