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Some Theorems on Geometric Measure of Distortion

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In this paper the relationship between the rate of transmission of information and the geometric measure of distortion is established both in discrete and continuous cases. The geometric rate-distortion function is defined as the infimum of the average mutual information between the sets of input and output symbols under the constraint that geometric distortion measure does not exceed a distortion limit. The slope of the geometric rate-distortion curve is evaluated and a lower bound is obtained. Finally geometric rate-distortion function is constructed for symmetric distortion measure.

I. INTRODUCTION

Consider an M-letter independent source with input symbols $\{0, 1, \ldots, M - 1\}$ which are used to communicate over channel whose set of output symbols is $\{0, 1, \ldots, \ldots, N - 1\}$. Let the channel matrix be $\{q_{j/i}\}$ where $q_{j/i}$ is the probability of receiving j when i is sent. If the input distribution is $\{p_i\}_{i=0}^{M-1}$ then the output distribution $\{q_j\}_{j=0}^{N-1}$ is determined by

$$q_j = \sum_i p_i q_{j/i} \quad \text{for all} \quad j . \tag{1.1}$$

Further let ϱ_{ij} denote the distortion when symbol i is received as j such that $\varrho_{ij} > > \alpha > 0$, $i \neq j$; $\varrho_{ii} = \alpha$. If we denote the geometric mean of single letter distortions ϱ_{ij} by ${}_\alpha D_G$, then

$${}_\alpha D_G = \prod_{i,j} \varrho_{ij}^{(p_i \cdot q_{j/i})} . \tag{1.2}$$

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Also, the average mutual information $I(X;Y)$ (or $R(\{q_{j/i}\})$) between the input and output, is given by

$$(1.3) \qquad I(X;Y) = R(\{q_{j/i}\}) = \sum_i \sum_j p_i q_{j/i} \log \frac{q_{j/i}}{\sum_l p_l q_{j/l}},$$

where logarithms are considered to the base 2.

On the lines of Shannon's rate-distortion function [4], Sharma, Mitter and Mathur have defined the geometric rate-distortion function $R({}_\alpha D_G^*)$ as

$$(1.4) \qquad R({}_\alpha D_G^*) = \min R(\{q_{j/i}\}),$$

where the minimization is done with respect to all those $\{q_{j/i}\}$ for which ${}_\alpha D_G \leqq {}_\alpha D_G^*$, ${}_\alpha D_G^*$ being a fixed quantity.

The measure defined in (1.2) has some advantages over the Shannon's measure of distortion given by

$$(1.5) \qquad D = \sum_i \sum_j p_i \cdot q_{j/i} \cdot \varrho_{ij}.$$

Some relations that it bears with entropies of the system and the rate of transmission, have been given in [5]. Bounds on $R({}_\alpha D_G^*)$ when measure of distortion is symmetric i.e., $\varrho_{ij} = \beta > \alpha \; \forall i, j;\; i \neq j$ were obtained by Sharma and the authors in [6].

In the present communication, we obtain some expressions for $R({}_\alpha D_G^*)$ and study the nature of the geometric rate-distortion curve. A lower bound on $R({}_\alpha D_G^*)$ when the measure $\varrho_{ij} = \varrho(x; y)$ (continuous case) depends on the difference of x and y and its value when the measure is symmetric, are obtained herein.

II. THEOREMS ON ${}_\alpha D_G$

Theorem 2.1. Let $R({}_\alpha D_G^*)$ be the geometric rate-distortion function of a discrete memoryless source with source probability $\{p_i\}$ and single letter distortion measure ϱ_{ij}, then $R({}_\alpha D_G^*)$ can be expressed as

$$(2.1) \qquad R({}_\alpha D_G^*) = v \cdot {}_\alpha D_G^* \log {}_\alpha D_G^* + \sum_i p_i \log \mu_i,$$

where

$$(2.2) \qquad \sum_i \mu_i p_i \varrho_{ij}^{v \cdot {}_\alpha D^*_G} = 1 \quad \text{for all} \quad j,$$

and

$$(2.3) \qquad \mu_i^{-1} = \sum_j q_j \varrho_{ij}^{v \cdot {}_\alpha D^*_G}, \quad \text{for all} \quad i.$$

Proof. Proceeding as in Gallager [2], it can be shown that by constructing a function φ as

$$\varphi = I(X;Y) - \nu \,.\, {}_{\alpha}D_G^* + \sum_i S_i \sum_j q_{j/i}\,,$$

where

$$\sum_j q_{j/i} = 1\,,$$

and differentiating it with respect to $q_{j/i}$, after setting $S_i = -p_i \log \mu_i$, the condition for $q_{j/i}$ to yield a stationary point for φ is that

$$q_{j/i} = \mu_i q_j \varrho_{ij}^{\nu \cdot {}_{\alpha}D^*_G}\,, \quad \text{for all} \quad i \quad \text{and} \quad j \tag{2.4}$$

satisfying

$$\sum_i \mu_i p_i \varrho_{ij}^{\nu \cdot {}_{\alpha}D^*_G} = 1 \quad \text{for all} \quad j\,.$$

Further setting (2.4) in (1.3), we get

$$R({}_{\alpha}D_G^*) = \sum_i \sum_j p_i \mu_i q_j \varrho_{ij}^{\nu \cdot {}_{\alpha}D^*_G} \log \left(\mu_i \varrho_{ij}^{\nu_{\alpha}D^*_G}\right) =$$

$$= \nu \,.\, {}_{\alpha}D_G^* \sum_i p_i q_j \mu_i \varrho_{ij}^{\nu_{\alpha}D^*_G} \log \varrho_{ij} + \sum_i \sum_j p_i q_j \mu_i \varrho_{ij}^{\nu_{\alpha}D^*_G} \log \mu_i =$$

$$= \nu \,.\, {}_{\alpha}D_G^* \log {}_{\alpha}D_G^* + \sum_i p_i \log \mu_i\,,$$

using (2.3). □

The result can be analogously extended to continuous case. With similar notations, we have

$$R({}_{\alpha}D_G^*) - \gamma \,.\, {}_{\alpha}D_G^* \log {}_{\alpha}D_G^* = \int p(x) \ln \mu(x)\, \mathrm{d}x \tag{2.6}$$

such that

$$\int \mu(x)\, p(x)\, \varrho^{\nu \cdot {}_{\alpha}D^*_G}(x;y)\, \mathrm{d}x = 1 \quad \text{for all} \quad y\,,$$

and

$$\mu^{-1}(x) = \int q(y)\, \varrho^{\nu \cdot {}_{\alpha}D^*_G}(x;y)\, \mathrm{d}y\,, \tag{2.6a}$$

where ln denotes logarithms to the base e.

The parameter ν admits of a geometrical interpretation which we now state below:

Theorem 2.2. The slope $R'({}_{\alpha}D_G^*)$ of the geometric rate-distortion curve at ${}_{\alpha}D_G^*$ is given by

$$R'({}_{\alpha}D_G^*) = \nu \,.\, \log (2 \,.\, {}_{\alpha}D_G^*)\,. \tag{2.7}$$

Proof. From (2.1), it follows that $R({}_\alpha D_G^*)$ is a function of v, ${}_\alpha D_G^*$ and μ_i ($i = 0, 1, \ldots$ $\ldots, M - 1$). Thus, we have

$$R'({}_\alpha D_G^*) = \frac{\mathrm{d}R({}_\alpha D_G^*)}{\mathrm{d}_\alpha D_G^*} = \frac{\partial R}{\partial_\alpha D_G^*} + \frac{\partial R}{\partial v}\left(\frac{\mathrm{d}v}{\mathrm{d}_\alpha D_G^*}\right) + \sum_i \frac{\partial R}{\partial \mu_i}\left(\frac{\mathrm{d}\mu_i}{\mathrm{d}_\alpha D_G^*}\right) =$$
$$= v + v \,.\, \log {}_\alpha D_G^* + \left[{}_\alpha D_G^* \log {}_\alpha D_G^* + \sum_i \frac{p_i}{\mu_i}\left(\frac{\mathrm{d}\mu_i}{\mathrm{d}v}\right)\right] \frac{\mathrm{d}v}{\mathrm{d}_\alpha D_G^*}\,.$$

As the transverse the $R({}_\alpha D_G^*)$ curve, the solution always satisfies (2.2), so that

$$\sum_i \left[\mu_i \, {}_\alpha D_G^* \log \varrho_{ij} + \frac{\mathrm{d}\mu_i}{\mathrm{d}v}\right] p_i \, \varrho_{ij}^{v \cdot {}_\alpha D*G} = 0\,.$$

Multiplying this by q_j and summing over j, we obtain

$${}_\alpha D_G^* \log {}_\alpha D_G^* + \sum_i \frac{p_i}{\mu_i}\left(\frac{\mathrm{d}\mu_i}{\mathrm{d}v}\right) = 0\,.$$

This, in turn gives

$$R'({}_\alpha D_G^*) = v + v \,.\, \log {}_\alpha D_G^*\,. \qquad \square$$

Theorem 2.3. For a reproducing probability distribution $q = (q_0, q_1, \ldots, q_{N-1})$ let $B_q = \{j : q_j = 0\}$ and $V_q = \{j : q_j > 0\}$ be the boundary and interior sets respectively. Then a conditional probability assignment $\{q_{j/i}\}$ such that

$$q_{j/i} = \mu_i q_j \varrho_{ij}^{v \cdot {}_\alpha D*G} \quad \text{for all} \quad i \quad \text{and} \quad j\,,$$

yields a point on the $R({}_\alpha D_G^*)$ curve if and only if

$$\sum_i \mu_i p_i \varrho_{ij}^{v \cdot {}_\alpha D*G} \leqq 1\,, \quad \text{for} \quad j \in B_q\,, \tag{2.8}$$

where μ_i and v satisfy (2.3).

Proof. Let a change of transition probabilities $\Delta q_{j/i}$ be such that

$$\Delta q_{j/i} \geqq 0\,, \quad \text{for} \quad j \in B_q\,, \tag{2.9}$$

$$\sum_j \Delta q_{j/i} = 0 \tag{2.10}$$

and

$$\Delta_\alpha D_G = \exp\left[\sum_i \sum_j p_i \, \Delta q_{j/i} \log \varrho_{ij}\right] = \exp(0) = 1\,. \tag{2.11}$$

 For V_q, the change in $I(X; Y)$ is given by

$$\Delta I_v = \sum_i \sum_{j \in v_q} p_i \, \Delta q_{j/i} \log \mu_i \varrho_{ij}^{v;{}_\alpha D*_G}$$

(from 2.4) and for B_q, the change is

$$\Delta I_B = \sum_i \sum_{j \in B_q} p_i \, \Delta q_{j/i} \log \frac{\Delta q_{j/i}}{\Delta q_j} .$$

Thus the total change is then given by

$$\Delta I = \sum_i \sum_{j \in V_q} p_i \, . \, \Delta_{j/i} \{\log \varrho_{ij}^{v;{}_\alpha D*_G} + \log \mu_i\} + \sum_i \sum_{j \in B_q} p_i \, . \, \Delta q_{j/i} \log \frac{\Delta q_{j/i}}{\Delta q_j} .$$

By adding and substracting the quantity

$$\sum_i \sum_{j \in B_q} p_i \, . \, \Delta q_{j/i} \{\log \varrho_{ij}^{v,{}_\alpha D*_G} + \log \mu_i\} ,$$

we obtain

$$\Delta I = \sum_i \sum_j p_i \, . \, \Delta q_{j/i} \{\log \varrho_{ij}^{v;{}_\alpha D*_G} + \log \mu_i\} +$$

$$+ \sum_i \sum_{j \in B_q} p_i \, . \, \Delta q_{j/i} \log \left(\frac{\Delta q_{j/i}}{\Delta q_j \, . \, \mu_i \varrho_{ij}^{v;{}_\alpha D*_G}} \right) . \tag{2.12}$$

Invoking the constraints (2.10), (2.11), the first expression on the right hand side of (2.12) vanishes. Again by applying the inequality

$$\log \, x \geqq 1 - \frac{1}{x} \quad \text{(with equality iff } x = 1\text{)} \tag{2.13}$$

to the second expression of right hand side of (2.12), we get

$$\Delta I \geqq 0$$

if (2.8) holds.

This shows that any change of transition probabilities can only increase $I(X; Y)$ if ${}_\alpha D_G$ is kept fixed when (2.8) holds, which implies that the above solution achieves the minimum of $I(X; Y)$. The second part of the theorem can be readily established by showing that the set of transition probabilities which does not satisfy (2.8), will decrease $I(X; Y)$, keeping ${}_\alpha D_G$ fixed. □

Theorem 2.4 (Another form of $R({}_\alpha D_G^*)$). Let $\hat{\mu}$ be the set $\{\mu\}$ where $\mu = (\mu_0, \mu_1, \ldots \ldots, \mu_{M-1})$ and $\mu_i > 0$ for each $i = 0, 1, \ldots, M - 1$ satisfying

$$\sum_i \mu_i \cdot p_i \cdot \varrho_{ij}^{v \cdot {}_\alpha D^*_G} \leqq 1 \quad \text{for all} \quad j , \tag{2.14}$$

then

$$R({}_\alpha D_G^*) = \max_{v,\mu \in \hat{\mu}} (v \cdot {}_\alpha D_G^* \log {}_\alpha D_G^* + \sum_i p_i \cdot \log \mu_i) \tag{2.15}$$

and a necessary and sufficient condition for μ to achieve maximum in (2.15) is that its components be given by

$$\mu_i^{-1} = \sum_j q_j \varrho_{ij}^{v \cdot {}_\alpha D^*_G} , \quad i = 0, 1, \ldots, M - 1 . \tag{2.16}$$

Proof. From the assumption ${}_\alpha D_G \leqq {}_\alpha D_G^*$ and making use of the inequality (2.14) and (2.13), we get

$$R({}_\alpha D_G^*) - \dot{v} \cdot {}_\alpha D_G^* \log {}_\alpha D_G^* - \sum_i p_i \cdot \log \mu_i \geqq$$

$$\geqq \sum_i \sum_j p_i q_{j/i} \left\{1 - \frac{q_j \cdot \mu_i \varrho_{ij}^{v \cdot {}_\alpha D^*_G}}{q_{j/i}}\right\} = 1 - \sum_j q_j \sum_i \mu_i p_i \varrho_{ij}^{v \cdot {}_\alpha D^*_G} \geqq 1 - \sum_j q_j = 0 .$$

Hence, for every set of conditional probabilities $\{q_{j/i}\}$ for which ${}_\alpha D_G \leqq {}_\alpha D_G^*$, $R({}_\alpha D_G^*)$ approaches the maximum on the right hand side of (2.15). Thus

$$R({}_\alpha D_G^*) \geqq \max_{v,\mu \in \hat{\mu}} (v \cdot {}_\alpha D_G^* \log {}_\alpha D_G^* + \sum_i p_i \log \mu_i) ; \tag{2.17}$$

we can easily see from Theorem 2.1, that

$$R({}_\alpha D_G^*) \leqq \max_{v,\mu \in \hat{\mu}} (v \cdot {}_\alpha D_G^* \log {}_\alpha D_G^* + \sum_i p_i \log \mu_i) . \tag{2.18}$$

Thus combining (2.17) and (2.18), we obtain (2.15). The necessary and sufficent condition for achieving the maximum in the statement of the theorem follows immediately from Theorem 2.3. □

In the next section, we shall come to a variational problem to find a lower bound of $R({}_\alpha D_G^*)$. For that we shall need the continuous analog of Theorem 2.4 which may be stated as follows:

If $\hat{\mu}$ is the set of all non-negative functions $\mu(x)$ satisfying

$$\int \mu(x)\, p(x)\, \varrho^{v \cdot {}_\alpha D^*_G}(x; y)\, \mathrm{d}x \leqq 1 \quad \text{for all} \quad y \tag{2.19}$$

then

$$R({}_\alpha D_G^*) = \operatorname*{Sup}_{v,\mu(x)\in\mu} \left[v \,.\, {}_\alpha D_G^* \log {}_\alpha D_G^* + \int p(x) \ln \mu(x)\, dx\right] \tag{2.20}$$

and a necessary and sufficient condition for $\mu(x)$ to achieve supremum in (2.20) is that there exists an output probability density function $q(y)$ satisfying (2.6a) for almost all y for which $q(y) < 0$. □

III. A LOWER BOUND WHEN $\varrho(x; y) = \varrho(x - y)$

When the distortion $\varrho(x; y)$ depends upon the difference of x and y, we call it as difference distortion measure.

Theorem 3.1. If R_L denotes the lower bound of $R({}_\alpha D_G^*)$ for difference distortion, then

$$R_L = H(x) - H(\psi(x)), \tag{3.1}$$

where $H(X)$ is input entropy, that is $- \int p(x) \log p(x)$,

$$\psi(x) = \frac{\varrho^{v \cdot {}_\alpha D^{**}G}(x)}{\int \varrho^{v \cdot {}_\alpha D^{**}G}(z)\, dz}, \tag{3.2}$$

and ${}_\alpha D_G^{**}$ is the value of ${}_\alpha D_G^*$ for $z = x - y$.

Proof. Let us suppose that

$$\mu(x) = \frac{S}{p(x)}, \tag{3.3}$$

where S is a constant.

If ${}_\alpha D_G^{**}$ denotes the value of ${}_\alpha D_G^*$ when $z = x - y$, then (2.19) gives

$$S \,.\, \int \varrho^{v \cdot {}_\alpha D^{**}G}(z)\, dz \leqq 1\,. \tag{3.4}$$

Choosing S such that (3.4) holds with equality, it follows from (2.20) that

$$R({}_\alpha D_G^{**}) \geqq v \,.\, {}_\alpha D_G^{**} \ln {}_\alpha D_G^{**} + H(x) - \ln \int \varrho^{v \cdot {}_\alpha D^{**}G}(z)\, dz = R_L \quad (\text{say})\,. \tag{3.5}$$

Therefore,

$$R_L' = {}_\alpha D_G^{**} \ln {}_\alpha D_G^{**} - {}_\alpha D_G^{**} \int (\ln \varrho(x))\, \psi(x)\, dx \tag{3.6}$$

and

(3.7) $$R_L'' = -\int\{{}_\alpha D_G^{**} \ln \varrho(x)\}^2 \cdot \psi(x)\,dx + (\int {}_\alpha D_G^{**} \ln \varrho(x)\,\psi(x)\,dx)^2\,.$$

From (3.7) it can be readily seen that $R_L'' \leqq 0$, therefore R_L is convex $\cap$ function of v. Hence, there exists the unique maximum at some v satisfying $R_L' = 0$, that is

(3.8) $${}_\alpha D_G^{**} \ln {}_\alpha D_G^{**} = {}_\alpha D_G^{**} \int \ln \varrho(x)\,\psi(x)\,dx\,.$$

Suppose that the value of ${}_\alpha D_G^{**} \ln {}_\alpha D_G^{**}$ for v obtained from (3.8) be denoted by D_v, then from (3.5) it follows that

$$R_L = vD_v + H(x) - \ln \int \varrho^{v \cdot {}_\alpha D^{**}G}(z)\,dz = H(x) + \int \psi(x) \ln \psi(x)\,dx\,. \quad \square$$

IV. CONSTRUCTION OF $R({}_\alpha D_G^*)$ FOR A SYMMETRIC MEASURE OF DISTORTION

If the number of input and output symbols are the same and if the cost of every correct transmission is α and that of any incorrect transmission is β (obviously $\alpha < \beta$) so that

(4.1) $$\varrho_{ij} = \begin{cases} \alpha & \text{if} \quad i = j\,, \\ \beta & \text{otherwise}\,, \end{cases}$$

then we may refer to this as symmetric measure of distortion. We shall give a theorem on the construction of $R({}_\alpha D_G^*)$ for the symmetric measure of distortion. We first prove two lemmas.

Lemma 1. Let $R({}_\alpha D_G^*)$ be defined for some source X with probability $P = \{p_0, p_1, \ldots, p_{M-1}\}$ and distortion matrix $\{\varrho_{ij}\}$ and suppose the new distortions are formed by multiplying each row of the distortion matrix by a constant i.e.,

$$\hat{\varrho}_{ij} = C_i \cdot \varrho_{ij}\,,$$

then

(4.2) $$\hat{R}({}_\alpha D_G^*) = R({}_\alpha D_G^*/C)\,,$$

where $C = 2^{\Sigma_i p_i \log C_i}$ and $\hat{R}$ is defined for the source X and distortion $\hat{\varrho}_{ij}$.

Proof. We know that

$$\hat{R}({}_\alpha D_G^*) = \min I(X; Y)$$

subject to the constraint

$$2^{\Sigma_i \Sigma_j p_i q_{j/i} \log \varrho_{ij}} \leqq {}_\alpha D_G^* \tag{4.3}$$

or

$$2^{\Sigma_i \Sigma_j p_i q_{j/i} \log \rho_{ij}} \leqq {}_\alpha D_G^*/C ,$$

which by definition is $R({}_\alpha D_G^*/C)$. □

Lemma 2. Let p_0 be the probability of the source letter corresponding to a row with all entries 1 in the distortion matrix. Then

$$R({}_\alpha D_G^*) = (1 - p_0)\, \hat{R}(({}_\alpha D_G^*)^{1/(1-p_0)}) , \tag{4.4}$$

where $\hat{R}$ is defined for the distortion matrix with row of 1's deleted, the source being $(1, 2, \ldots, M - 1)$ with input probability distribution

$$p^* = \left(\frac{p_1}{1 - p_0}, \frac{p_2}{1 - p_0}, \ldots, \frac{p_{M-1}}{1 - p_0}\right). \tag{4.5}$$

Proof. The geometric distortion ${}_\alpha D_G$ is not affected by omitting the distortion corresponding to the reproduction of source letter O. Thus to minimize $I(X; Y)$ we must choose $q_{j/i}$ so that $I(x_0, Y) = 0$. With this choice

$$\begin{aligned} R({}_\alpha D_G^*) &= \min I(X; Y) = \\ &= \min \left[p_0\, I(x_0; Y) + \sum_{i=1}^{M-1} p_i\, I(x_i; Y)\right] = \\ &= \min \left[(1 - p_0) \sum_{i=1}^{M-1} \frac{p_i}{1 - p_0} I(x_i; Y)\right] . \end{aligned}$$

The constraint is

$$\sum_{i=0}^{M-1} \sum_j p_i q_{j/i} \log \varrho_{ij} = \log {}_\alpha D_G^* .$$

But

$$\sum_j q_{j/0} \log \varrho_{0j} = 0 .$$

Therefore,

$$\sum_{i=1}^{M-1} \sum_j \frac{p_i}{1 - p_0} q_{j/i} \log \varrho_{ij} = \frac{1}{1 - p_0} \log {}_\alpha D_G^* = \log ({}_\alpha D_G^*)^{1/(1-p_0)} .$$

So by the definition of $R({}_\alpha D_G^*)$, we obtain the desired result. □

$$R({}_\alpha D_G^*) = (1 - \sigma_{K-1})\left[H_{M-K+1}(X) - \hat{H}(\Delta_{K-1}) - \Delta_{K-1}\log(M-K)\right] \tag{4.6}$$

for

$${}_\alpha D_G^{*(K-1)} < {}_\alpha D_G^* \leqq {}_\alpha D_G^{*(K)}, \quad \text{for} \quad 2 \leqq K < M + 1,$$

where

$$\sigma_K = \sum_{i=0}^{K-1} p_i, \quad \text{for} \quad K \geqq 1; \quad \sigma_0 \equiv 0,$$

$${}_\alpha D_G^{*(K)} = \beta^{\sigma_{K-1}}\left[\alpha \log \alpha + (\beta \log \beta - \alpha \log \alpha)(M-K)\frac{p_{K-1}}{1-\sigma_{K-1}}\right]^{1-\sigma_{K-1}} \tag{4.7}$$

and

$$\Delta_K = \left\{\frac{\left(\dfrac{{}_\alpha D_G^*}{\beta^{\sigma_K}}\right)^{1/(1-\sigma_K)} - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right\}, \tag{4.8}$$

$$\hat{H}(\Delta_K) = -\Delta_K \log \Delta_K - (1 - \Delta_K)\log(1 - \Delta_K);$$

also $H_{M-K}(X)$ is the entropy of the source $(p_K/(1-\sigma_K), \ldots, p_{M-1}/(1-\sigma_K))$, provided that $p_0 \leqq p_1 \leqq \cdots \leqq p_{M-1}$.

Proof. We have indicated in Theorem 2.1 that the set $\{q_{j/i}\}$ giving $R({}_\alpha D_G^*)$ is given by

$$q_{j/i} = q_j \mu_i \varrho_{ij}^{v;{}_\alpha D*_G} \quad \text{for all} \quad i \quad \text{and} \quad j,$$

where q_j's satisfy the constraint

$$\mu_i \sum_j q_j \varrho_{ij}^{v;{}_\alpha D*_G} = 1 \quad \text{for all} \quad i. \tag{4.9}$$

For symmetric measure of distortion, it has been shown in [6] that

$$R({}_\alpha D_G^*) \geqq H(X) - \hat{H}(\Delta) - \Delta \log(M-1), \tag{4.10}$$

where $H(X)$ is the source entropy,

$$\Delta = \frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}$$

 and

$$\hat{H}(\Delta) = -\Delta \log \Delta - (1 - \Delta) \log (1 - \Delta)$$

with equality in (4.10) if

$$(4.11) \qquad {}_\alpha D_G^* \leqq \alpha \log \alpha + (\beta \log \beta - \alpha \log \alpha)(M - 1) p_0 ,$$

p_0 is the minimum input probability and q_j in (4.9) under this measure is given by

$$(4.12) \qquad q_j = \frac{p_j[\beta^{\lambda\beta} + (M - 1) \alpha^{\lambda\alpha}] - \alpha^{\lambda\alpha}}{\beta^{\lambda\beta} - \alpha^{\lambda\alpha}} .$$

All q_j's will be non-negative if

$$(4.13) \qquad p_j \geqq \frac{1}{\beta^{\lambda\beta}\alpha^{-\lambda\alpha} + (M - 1)} .$$

Denote

$$(4.14) \qquad {}_\alpha D_G^* = \alpha \log \alpha + (\beta \log \beta - \alpha \log \alpha)(M - 1) p_0 = {}_\alpha D_G^{*(1)} .$$

Then for ${}_\alpha D_G^* > {}_\alpha D_G^{*(1)}$ [3], output zero will never be used and we can therefore remove it from the output alphabet and delete the corresponding column from the distortion matrix without affecting $R({}_\alpha D_G^*)$. Thus for ${}_\alpha D_G^* > {}_\alpha D_G^{*(1)}$ we have $M \times$ $\times (M - 1)$ distortion matrix $\{\varrho_{ij}\}$ with all β's in the first row. Dividing the first row by β and using Lemma 1, we have

$$(4.15) \qquad R({}_\alpha D_G^*) = R^{(1)}({}_\alpha D_G^* / \beta^{p_0}) ,$$

where $R^{(1)}$ corresponds to the matrix

$$(4.16) \qquad \{\varrho_{ij}^{(1)}\} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \alpha & \beta & \dots & \beta \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \beta & \beta & \dots & \alpha \end{bmatrix} .$$

Using Lemma 2, we get

$$(4.17) \qquad R^{(1)}({}_\alpha D_G^*) = (1 - p_0) R^{(2)}(({}_\alpha D_G^*)^{1/(1-p_0)}) .$$

From (4.15) and (4.17), we have

$$(4.18) \qquad R({}_\alpha D_G^*) = (1 - p_0)\, R^{(2)}\big(({}_\alpha D_G^*/\beta^{p_0})^{1/(1-p_0)}\big)$$

for ${}_\alpha D_G^* > {}_\alpha D_G^{*(1)}$ and $R^{(2)}$ corresponds to the $(M-1)\times(M-1)$ matrix and the source

$$p^{**} = \left(\frac{p_1}{1-p_0}, \frac{p_2}{1-p_0}, \ldots, \frac{p_{M-1}}{1-p_0}\right).$$

A lower bound of $R^2({}_\alpha D_G^*)$ can be obtained similarly which is valid for

$${}_\alpha D_G^* \leqq {}_\alpha \log \alpha + (\beta \log \beta - \alpha \log \alpha)(M-2)\frac{p_1}{1-p_0},$$

where p_1 is the second lowest probability. Thus the second break point occurs at ${}_\alpha D_G^{*(2)}$, where

$$({}_\alpha D_G^{*(2)}/\beta^{p_0})^{1/(1-p_0)} = \alpha \log \alpha + (\beta \log \beta - \alpha \log \alpha)(M-2)\frac{p_1}{1-p_0}.$$

Hence

$$R({}_\alpha D_G^*) = H(X) - \hat{H}\left(\frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right) - \left(\frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right)\log(M-1)$$

for

$${}_\alpha D_G^* \leqq \alpha \log \alpha + (\beta \log \beta - \alpha \log \alpha)(M-1)\, p_0$$

and

$$R({}_\alpha D_G^*) = (1-p_0)\left[H_{M-1}(X) - \hat{H}\left\{\frac{({}_\alpha D_G^*/\beta^{p_0})^{1/(1-p_0)} - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right\} - \right.$$
$$\left. - \left\{\frac{({}_\alpha D_G^*/\beta^{p_0})^{1/(1-p_0)} - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right\}\log(M-2)\right]$$

for

$${}_\alpha D_G^{*(1)} < {}_\alpha D_G^* \leqq {}_\alpha D_G^{*(2)},$$

where

$$H_{M-1}(X) = -\sum_{i=1}^{M-1} \frac{p_i}{1-p_0} \log \frac{p_i}{1-p_0}.$$

Continuing this way, we get the desired result. □

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