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# Existence of Optimal Solutions in General Discrete Systems

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The general existence theorem is presented for an abstract discrete optimal control problem in topological spaces. The admissible control region is assumed state-dependent and the number of stages finite and given.

To prove such theorem, some fundamental concepts from the theory of multivalued mappings are necessary, which are summerized for convenience. The assumptions made are general enough to be of practical interest. Thus the theorem includes and generalizes all known cases, which appear in the theory of discrete optimal control.

## 1. INTRODUCTION

The aim of this paper is to formulate and prove a theorem which guarantees the existence of an optimal solution for very general discrete optimal control problems with prescribed finite number of stages. The main generality lies in the fact that both, state and control, are supposed to be elements of certain topological spaces. Moreover, we assume state-dependent admissible control region.

In spite of this general and abstract setting, we were able to preserve reasonable and fairly mild assumptions so that our theorem applies to a large variety of discrete optimal control problems.

It is known that the existence problem for continuous systems is rather complicated and involved, e.g. see Olech [12]. On the other hand, in a discrete case the method for obtaining such theorem is more straightforward. Namely, if we consider a common discrete optimal control problem as given in the book of Canon et al. [6], it is not very difficult to realize that, in fact, we only seek for a minimum of certain real function over some constraining set in finite dimensional space. The existence conditions for such problem are well-known. So we want to find reasonable sufficient assumptions for the original optimal control problem to be able to apply the classical result.

This was also the approach used in some previous works devoted to this subject. To the author's knowledge, one of the first was paper of Propoj [13], but only simple control problem was considered there. Slightly more general formulation was given by Boltjanskij in his book [5]. His method is similar to that of Propoj. For discrete systems with the so called state-dependent admissible control region Boltjanskij in [5] only states (without proof) certain existence conditions, which can be easily derived from our general Existence Theorem. Moreover, our assumptions also in this special case are weaker than those used in [5].

Original subject of study was to find existence conditions in finite dimensional case. It has shown that to solve rigorously this question, some basic concepts and results from the theory of multivalued mappings will be necessary. This theory is now also successfully used for deriving necessary and sufficient conditions of optimality for continuous systems – see Blagodatskich [3; 4]. Realizing some fundamental facts about the topological spaces, it was noticed that our approach is valid generally without restricting the problem to finite dimensional one. As a result we obtained a general formulation of discrete optimal control problems in topological spaces.

The structure of the paper is the following. In the next section we give all necessary concepts and results concerning the theory of multivalued mappings. For the used topological concepts the reader is referred to some classical textbooks dealing with this subject. Section 3 is devoted to the precise formulation of an abstract discrete optimal control problem. We included also the classical formulation in finite dimensional spaces and a special case of explicitely given constraints.

The main result is given in Section 4, where the Existence Theorem is stated and proved. As a corollary we give the existence conditions for the explicite case. As an illustration, a simple example with state-dependent admissible control region is included. It is shown that our theorem assures the existence of an optimal solution.

## 2. ANALYTICAL FOUNDATIONS

To make the paper in certain sense self-contained, we devote this section to some basic results from the theory of the so called multivalued mappings. These results will be used later to prove general Existence Theorem for discrete optimal control problems. Because the theory of multivalued mappings is not widely known, we have tried to include also the proofs, which themselves could be of some interest and also instructive.

Throughout the paper we assume some fundamental knowledge about topological spaces. For all necessary information the reader can consult any of the standart textbooks; e.g. Dunford and Schwartz [9], Kolmogorov and Fomin [11] were mainly used by the author. Less experienced reader can without any substantial loss simply assume that all spaces in question are finite dimensional.

This section is partially based on Berge [2], who studied in detail multivalued mappings. Some of the given propositions can be also found in the paper of Davy [7]. 

Having in mind further applications, we can always assume that all considered topological spaces are Hausdorff (or $T_2$-spaces), i.e. any two distinct points of such topological spaces have disjoint neighbourhoods, although some particular results are valid in general case. The reason for this assumption is that topological spaces met in the analysis are almost exclusively Hausdorff. So this restriction is a formal one in order to simplify certain considerations performed later.

**Definition 1.** Let $X$ and $Y$ be topological spaces and $\Omega(Y)$ the set of all nonempty compact subsets of $Y$. The mapping $F$ from $X$ to $\Omega(Y)$ is called a (compact-valued) multivalued mapping.

**Definition 2.** Let $F : X \to \Omega(Y)$ and let $A$ be a subset of $X$. Then we define

$$F(A) = \bigcup_{x \in A} F(x) .$$

For convenience, if $A = \emptyset$, we define $F(\emptyset) = \emptyset$.

Very important role in the theory of multivalued mappings plays the so called upper semicontinuity. This concept is a direct generalization of the continuity concept of common, single-valued function.

**Definition 3.** We say that the multivalued mapping $F : X \to \Omega(Y)$ is upper semicontinuous at the point $x_0$ of $X$ if for all open sets $G \subseteq Y$ containing $F(x_0)$, there exists a neighbourhood $U(x_0)$ of $x_0$ such that $F(U(x_0)) \subseteq G$. We say that $F$ is upper semicontinuous if it is upper semicontinuous at every point of $X$.

In a similar way the so called lower semicontinuity of a multivalued mapping can be also defined – see Berge [2], but we do not need this concept here. The next results immediately follows from Definition 3.

**Proposition 1.** A single-valued function considered as a multivalued mapping is upper semicontinuous if and only if it is continuous.

**Definition 4.** The set

$$\Gamma(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}$$

is denoted as a graph of the multivalued mapping $F : X \to \Omega(Y)$.

Here by $X \times Y$ is denoted the topological product of $X$ and $Y$. We always assume that the topology of $X \times Y$ is the product topology (in Tichonov's sense).

**Proposition 2.** Let $F : X \to \Omega(Y)$ be upper semicontinuous. Then the set $\Gamma(F) \subseteq$ $\subseteq X \times Y$ is closed.

Proof. Suppose $(x, y) \notin \Gamma(F)$. From the compactness of $F(x)$ we can conclude the existence of an open set $G \subseteq Y$ containing $F(x)$ and a neighbourhood $V(y)$ of $y$ which are disjoint. By upper semicontinuity of $F$ there exists also a neighbourhood $U(x)$ such that $F(U(x)) \subseteq G$. Then $U(x) \times V(y)$ is a neighbourhood of $(x, y)$ and has no points in common with $\Gamma(F)$. Hence, the set $\Gamma(F)$ is closed.

Sometimes the upper semicontinuity of a multivalued mapping is defined by the closedness of its graph, e.g. see Joffe and Tichomirov [10]. But both definitions will be equivalent only under some additional restrictions, e.g. if the space $Y$ is compact. Now let us summarize some most important properties of upper semicontinuous multivalued mappings.

**Proposition 3.** A multivalued mapping $F : X \to \Omega(Y)$ is upper semicontinuous if and only the set

$$F^+(G) = \{x \in X \mid F(x) \subseteq G\}$$

is open for all open sets $G$ in $Y$.

Proof. Assume $F$ upper semicontinuous and consider a point $x_0 \in F^+(G)$ for some open $G$. There exists a neighbourhood $U(x_0)$ of $x_0$ such that $F(U(x_0)) \subseteq G$. Thus $U(x_0) \subseteq F^+(G)$ and $F^+(G)$ is therefore open.

Assume $G$ is open implies $F^+(G)$ is open. Let $x_0 \in X$ and let $G$ be an open set which contains $F(x_0)$. Then $F^+(G)$ is a neighbourhood of $x_0$ and $F(F^+(G))$ is contained in $G$. So $F$ is upper semicontinuous.

**Proposition 4.** Let $F : X \to \Omega(Y)$ be upper semicontinuous and $K$ compact subset of $X$. Then the set $F(K)$ is also compact.

Proof. Let $\{G_\alpha \mid \alpha \in A\}$ be an open covering of $F(K)$. If now $x \in K$, the compact set $F(x)$ is covered only by a finite number of $G_\alpha$. Denote their union as $G(x)$. Then $\{F^+(G(x)) \mid x \in K\}$ is an open covering of $K$ which has finite subcovering $\{F^+(G(x_1)), \ldots, F^+(G(x_n))\}$. So we have that the sets $G(x_1), \ldots, G(x_n)$ cover $F(K)$ and each $G(x_i)$, $i = 1, \ldots, n$, is the union of a finite number of $G_\alpha$. Therefore $F(K)$ is also covered by a finite number of $G_\alpha$ and thus compact.

**Proposition 5.** Let $F_1 : X \to \Omega(Y)$ and $F_2 : Y \to \Omega(Z)$ be upper semicontinuous. Define $F_2 \circ F_1(x) = F_2(F_1(x))$. Then the composed multivalued mapping $F_2 \circ F_1$ maps $X$ to $\Omega(Z)$ and is upper semicontinuous.

Proof. By Proposition 4, the set $F_2(F_1(x)$ is nonempty and compact, i.e. $F_2(F_1(x)) \in \Omega(Z)$. Now consider an open set $G$ in $Z$. Then 

$$(F_2 \circ F_1)^+ (G) = \{x \mid F_2 \circ F_1(x) \subseteqq G\} = \{x \mid F_1(x) \subseteqq F_2^+(G)\} = \\ = F_1^+[F_2^+(G)] .$$

The set $F_1^+[F_2^+(G)]$ is open in $X$ by Proposition 3. By the same proposition we also have that $F_2 \circ F_1$ is upper semicontinuous.

**Proposition 6.** Let $F : X \to \Omega(Y)$ be upper semicontinuous and let $K$ be a compact subset of $X$. Then the part of $\Gamma(F)$ considered only over set $K$, i.e. the set

$$\Gamma_K(F) = \{(x, y) \in X \times Y \mid y \in F(x) , \quad x \in K\} ,$$

is also compact.

Proof. By Proposition 4, the set $\tilde{\Gamma}_K(F) = K \times F(K)$ is compact. Clearly, the inclusion $\Gamma_K(F) \subseteqq \tilde{\Gamma}_K(F)$ holds. By Proposition 2, $\Gamma_K(F)$ is a closed subset of the compact set $\tilde{\Gamma}_K(F)$. Hence, $\Gamma_K(F)$ is compact.

We proved the last proposition directly, but it can be easily verified using the next result.

**Proposition 7.** Let $F_i : X \to \Omega(Y_i)$, $i = 1, 2$, be upper semicontinuous. Define $F_1 \times F_2(x) = F_1(x) \times F_2(x)$. Then the product mapping $F_1 \times F_2$ maps $X$ to $\Omega(Y_1 \times Y_2)$ and is upper semicontinuous.

Proof. The set $F_1(x) \times F_2(x) \subseteqq Y_1 \times Y_2$ is compact for all $x \in X$. Let $G \subseteqq Y_1 \times Y_2$ be an open set containing $F_1(x) \times F_2(x)$. From the definition of the topological product $Y_1 \times Y_2$ one can show the existence – see [2, p. 120], of open sets $G_1 \subseteqq Y_1$, $G_2 \subseteqq Y_2$ such that

$$F_1(x) \times F_2(x) \subseteqq G_1 \times G_2 \subseteqq G .$$

Then there exists a neighbourhood $U(x)$ of $x$ such that

$$F_1(U(x)) \times F_2(U(x)) \subseteqq G_1 \times G_2 \subseteqq G .$$

Thus the mapping $F_1 \times F_2$ is upper semicontinuous.

It is obvious, that Propositions 5 and 7 remain true if the composition or product are given by a finite number of upper semicontinuous multivalued mappings. Proposition 7 shows us also the way, how to define multivalued mappings of two and more variables.

**Definition 5.** Let $X$, $Y$, $Z$ be topological spaces. A multivalued mapping of two variables is defined as a mapping $F : X \times Y \rightarrow \Omega(Z)$.

We say that the mapping $F$ is upper semicontinuous if it is upper semicontinuous in the product topology of $X \times Y$.

Again, it would be possible to prove results analogous to those given by Propositions 1 – 7. But it is not necessary for our later construction. We shall need only the following corollary, which is implied by Proposition 7, or by realizing the obvious facts which result from Definition 5.

**Corollary 1.** Let $X$, $Y$, $Z$ be topological spaces, $f : X \times Y \rightarrow Z$ continuous function and $F : X \rightarrow \Omega(Y)$ upper semicontinuous multivalued mapping. Then the multivalued mapping $V : X \rightarrow \Omega(Z)$ defined by the relation

$$V(x) = f(x, F(x)) = \{z \in Z \mid z = f(x, y)\,, \quad y \in F(x)\}$$

is upper semicontinuous.

In the formulation of discrete optimal control problem also the concept of *lower semicontinuity* of a *real function* will be used. To avoid possible misunderstanding, we introduce this concept now.

**Definition 6.** We say that the real function $f$ defined on the topological space $X$ is lower semicontinuous at the point $x_0$ if $\liminf_{x \rightarrow x_0} f(x) \geqq f(x_0)$.

We say that $f$ is lower semicontinuous if it is lower semicontinuous at every point of $X$.

One can verify that the finite sum of lower semicontinuous real functions is again lower semicontinuous function. Further, if $f$ is lower semicontinuous and $X$ nonempty compact space, then the minimum of $f$ is attained.

## 3. PROBLEM FORMULATION

Now we shall given the precise formulation of an abstract control problem over finite number of stages. Our aim is to find sufficient conditions, under which the optimal solution exists. First we give a general formulation for which the Existence Theorem will be proved in the next section. Then we point out a practically interesting special case with explicitely given constraints.

Suppose that we are given by a system the behaviour of which is fully described by the recurrent (difference) equation

(1) $$x_{k+1} = f_k(x_k, u_k)\,, \quad k = 0, 1, \ldots, K - 1\,,$$

where a positive integer $K$ denotes the prescribed number of stages, $x_k \in X_k$, $k =$

$= 0, 1, \ldots, K$, is the state of the system at the stage $k$, $u_k \in U_k$, $k = 0, 1, \ldots, K - 1$, is the control (input) at the stage $k$, and $f_k : X_k \times U_k \to X_{k+1}$, $k = 0, 1, \ldots, K - 1$. The topological spaces $X_0, X_1, \ldots, X_K$; $U_0, U_1, \ldots, U_{K-1}$ are assumed to be Hausdorff. 

The aim is to choose a control sequence $\hat{u} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1})$ and a corresponding trajectory $\hat{x} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K)$, determined by (1) and subject to the constraints

$$(2) \qquad x_k \in \mathbf{A}_k \subseteqq X_k\,, \qquad k = 0, 1, \ldots, K\,,$$

$$(3) \qquad u_k \in \mathbf{U}_k(x_k) \subseteqq U_k\,,\ k = 0, 1, \ldots, K - 1\,,$$

which minimize the sum (cost functional)

$$(4) \qquad \mathbf{J} = \sum_{k=0}^{K-1} h_k(x_k, u_k)\,.$$

Here $h_k : X_k \times U_k \to \mathbf{E}^1$, $k = 0, 1, \ldots, K - 1$; $\mathbf{U}_k : X_k \to \Omega(U_k)$.

The pair $(\hat{x}, \hat{u})$ is then denoted as an optimal process in our system (1)–(4). The pair $(x, u)$ satisfying the system constraints (1)–(3) is said to be an admissible process.

In the next section we shall impose some further assumptions on just defined optimal control problem to be able to prove the existence of optimal controls. Now, let us give more concrete cases of discrete optimal control problem.

For this purpose we assume that

$$(5) \qquad X_k = \mathbf{E}^n,\ k = 0, 1, \ldots, K\,; \quad U_k = \mathbf{E}^m,\ k = 0, 1, \ldots, K - 1\,.$$

i.e. state and control spaces are finite dimensional. This is the formulation used by Boltjanskij [5] and by the author [8]. If we additionally assume that the admissible control regions $\mathbf{U}_k$, $k = 0, 1, \ldots, K - 1$, do not depend on $x$, we obtain the formulation given in [6].

From the practical point of view it is interesting if, in addition to (5), all sets appearing in (2) and (3) are given explicitely as a system of equalities and inequalities, namely,

$$(6) \qquad \mathbf{A}_k = \{x \in \mathbf{E}^n \mid S_k(x) = 0,\ s_k(x) \leqq 0\}\,, \qquad k = 0, 1, \ldots, K\,,$$

$$(7) \qquad \mathbf{U}_k(x) = \{u \in \mathbf{E}^m \mid Q_k(x, u) = 0\,,\ q_k(x, u) \leqq 0\}\,, \quad k = 0, 1, \ldots, K - 1\,,$$

where $S_k : \mathbf{E}^n \to \mathbf{E}^{\varrho_k}$, $s_k : \mathbf{E}^n \to \mathbf{E}^{\pi_k}$, $Q_k : \mathbf{E}^n \times \mathbf{E}^m \to \mathbf{E}^{\gamma_k}$ and $q_k : \mathbf{E}^n \times \mathbf{E}^m \to \mathbf{E}^{\varkappa_k}$, i.e. finite dimensional mappings. The inequality sign for vectors in (6) and (7) is to be taken componentwise.

Now we can formulate conditions which are sufficient to guarantee the existence for the abstract discrete optimal control problem stated in the last section. We repeat once again that all considered topological spaces are Hausdorff. Otherwise, certain conclusions made in the course of the proof would need either more detailed discussion or would be simply incorrect.

**Existence theorem.** Suppose that an abstract discrete system (1)–(4) is given and that the following assumptions hold:

(a) the functions $f_k$ are continuous and functions $h_k$ lower semicontinuous in $X_k \times U_k$, $k = 0, 1, \ldots, K - 1$;

(b) the initial set $\mathbf{A}_0$ is compact and the sets $\mathbf{A}_k$, $k = 1, \ldots, K$, are closed;

(c) the compact-valued multivalued mappings $\mathbf{U}_k$, $k = 0, 1, \ldots, K - 1$, are upper semicontinuous;

(d) there exists at least one admissible process in the given system.

Then the discrete optimal control problem has a solution, i.e. there exists an optimal process.

P r o o f. Consider the following multivalued mappings $V_k : X_k \to \Omega(X_{k+1})$, $k = 0, 1, \ldots, K - 1$, given by the formula

$$V_k(x_k) = f_k(x_k, \mathbf{U}_k(x_k)) = \{v \in X_{k+1} \mid v = f_k(x_k, u_k), \quad u_k \in \mathbf{U}_k(x_k)\}, \quad k = 0, 1, \ldots, K - 1 .$$

Further define recurrently the sets

$$\mathbf{P}_{k+1} = V_k(\mathbf{A}_k \cap \mathbf{P}_k), \quad \mathbf{P}_{k+1} \subseteq X_{k+1}, \quad k = 0, 1, \ldots, K - 1,$$

where we define $\mathbf{P}_0 = X_0$ for convenience. Thus for any admissible process $(x, u)$ we have

$$x_k \in \mathbf{A}_k \cap \mathbf{P}_k, \quad k = 0, 1, \ldots, K .$$

From Assumptions (a) and (c) we immediately obtain, using Corollary 1, that the multivalued mappings $V_k$, $k = 0, 1, \ldots, K - 1$, are upper semicontinuous.

The set $\mathbf{A}_0$ is assumed compact. By Proposition 5, the set $\mathbf{P}_1$ is also compact and, therefore, closed. Thus $\mathbf{A}_1 \cap \mathbf{P}_1$ must be closed and, as a subset of compact set $\mathbf{P}_1$ necessarily compact. Then $\mathbf{P}_2$ will be compact. Continuing in an obvious manner this procedure further, we see that the sets $\mathbf{A}_k \cap \mathbf{P}_k$, $k = 0, 1, \ldots, K$, are compact.

Define now as a topological product the space

$$Z = X_0 \times X_1 \times \ldots \times X_K \times U_0 \times U_1 \times \ldots \times U_{K-1},$$

which is also Hausdorff space. The collection of all admissible processes in the system (1)–(4) can be interpreted as a set of those points $z \in Z$, components of which satisfy the conditions:

1) system equations (1);
2) $x_k \in \mathbf{A}_k \cap \mathbf{P}_k$, $k = 0, 1, \ldots, K$;
3) $u_k \in \mathbf{U}_k(x_k)$, $k = 0, 1, \ldots, K - 1$.

Conditions 2) and 3) imply that any admissible $z$ belongs to the compact set

$$\Phi = (\mathbf{A}_0 \cap \mathbf{P}_0) \times (\mathbf{A}_1 \cap \mathbf{P}_1) \times \ldots \times (\mathbf{A}_K \cap \mathbf{P}_K) \times \mathbf{U}_0(\mathbf{A}_0 \cap \mathbf{P}_0) \times \mathbf{U}_1(\mathbf{A}_1 \cap \mathbf{P}_1) \times \\ \times \ldots \times \mathbf{U}_{K-1}(\mathbf{A}_{K-1} \cap \mathbf{P}_{K-1}).$$

By Proposition 2, the conditions 1) and 3) always define in $Z$ certain system of closed subsets. Namely, from the system equations (1) we obtain that the admissible processes must lie in the closed sets (see Proposition 2)

$$\Psi_k = \{z \in Z \mid x_{k+1} = f_k(x_k, u_k)\}, \quad k = 0, 1, \ldots, K - 1.$$

Similarly, from the condition 3) we have that the admissible processes must also lie in the closed sets

$$\Gamma_k = \{z \in Z \mid u_k \in \mathbf{U}_k(x_k)\}, \quad k = 0, 1, \ldots, K - 1.$$

Thus, we finally obtain that the set

$$\Delta = \Phi \cap \Big(\bigcap_{k=0}^{K-1} \Psi_k\Big) \cap \Big(\bigcap_{k=0}^{K-1} \Gamma_k\Big)$$

is a compact subset of $Z$ and represents the above mentioned collection of all admissible processes in our system.

The cost functional (4) can be considered as a real function on $Z$, i.e.

$$H(z) = \sum_{k=0}^{K-1} h_k(x_k, u_k), \quad H : Z \to \mathbf{E}^1,$$

which is evidently lower semicontinuous there.

To conclude the proof we only realize that the original discrete optimal control problem was reduced to that of finding a point $z_0 \in \Delta$ which minimizes the function $H$. Assumption (d) assures that the compact set $\Delta$ is nonempty. As we know, the minimum is reached in this case, i.e. an optimal process exists Q.E.D.

Now we shall show, how to apply this theorem to some more concrete cases. For this purpose consider the explicitely given discrete optimal control problem (1), (4)–(7). Suppose that all functions appearing in these relations are continuous. Further let $\mathbf{A}_0$ in (6) and $\mathbf{U}_k(x)$, $k = 0, 1, \ldots, K - 1$, in (7) be bounded.

**Corollary 2.** Under just stated assumptions the explicitely given optimal control problem (1), (4)–(7), has a solution provided that at least one admissible process exists.

Similar existence theorem was stated in [5] for the finite dimensional case and continuously varying admissible control regions, i.e. the multivalued mappings $\mathbf{U}_k$, $k = 0, 1, \ldots, K - 1$ were assumed both, upper and lower semicontinuous. In our approach only the upper semicontinuity assumption was necessary, so that the obtained Existence Theorem applies to a considerably broader class of discrete optimal control problems which need not be finite dimensional. This fact is illustrated by the next very simple example.

**Example 1.** Let us consider the following discrete optimal control problem with the state-dependent control regions ($x$, $u$ are in $\mathbf{E}^1$).

$$x_{k+1} = x_k + u_k\,, \quad k = 0, 1, \ldots, K - 1\,, \quad x_0 \quad \text{given}\,,$$

$$\mathrm{J} = \sum_{k=0}^{K-1} x_k^2\,,$$

$$\mathbf{U}_k(x) = \mathbf{U}(x) = \left.\begin{array}{l} [-1,\ 1],\ x < x^*, \\ [-2,\ 2],\ x \geqq x^*, \end{array}\right\} \quad k = 0, 1, \ldots, K - 1\,.$$

It is not very hard to see that the multivalued mapping $\mathbf{U}(x)$ is upper semicontinuous in $\mathbf{E}^1$ (but not continuous at $x^*$). Then Existence Theorem can be applied and we see that an optimal solution exists in this case. Of course, such conclusion is not possible if we try to use the results from [5].

Also the second example is only illustrative, but in a certain sense more concrete.

**Example 2.** Consider the following well-known example, e.g. see Bellman [1]. Find the numbers $\alpha_1, \ldots, \alpha_K$ such that

(i) $$\sum_{i=1}^{K} \alpha_i \leqq a\,, \quad a > 0\,;$$

(ii) $$\alpha_i \geqq 0\,, \quad i = 1, \ldots, K\,;$$

(iii) $$\prod_{i=1}^{K} \alpha_i = \max.$$

We can easily transcribe this problem to the form of discrete optimal control one, which has state-dependent admissible control region (all variables are in $\mathbf{E}^1$).

Minimize the cost functional

$$\mathrm{J} = -x_{K-1}^{(2)} u_{K-1}$$

subject to 

$$\text{(a)} \quad \left.\begin{aligned} x_{k+1}^{(1)} &= x_k^{(1)} + u_k, \\ x_{k+1}^{(2)} &= x_k^{(2)} u_k, \end{aligned}\right\} \quad k = 0, 1, \ldots, K - 1 ;$$

$$\text{(b)} \quad x_0^{(1)} = 0 , \quad x_0^{(2)} = 1 ;$$

$$\text{(c)} \quad 0 \leqq u_k \leqq a - x_k^{(1)} , \quad k = 0, 1, \ldots, K - 1 .$$

It is easy to see that both just stated problems are equivalent and that the minimizing control sequence $(\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1})$ for the discrete optimal control problem will be also a solution of the original problem if we set $\alpha_i = \hat{u}_{i-1}$, $i = 1, \ldots, K$. Further, all assumptions of Corollary 2 are obviously satisfied (e.g. the control sequence $(a, 0, \ldots, 0)$ generates an admissible process). So we can assert that this control problem and, therefore, also the original one, has a solution.

## 5. CONCLUDING REMARKS

The existence theorem for a general case of discrete optimal control problems in topological spaces was proved. This general formulation includes all known and studied cases of this type.

The assumptions made in the theorem are general enough to be of practical interest, as was shown by a simple example. Nevertheless, it was necessary to introduce certain basic concepts concerning the so called multivalued mappings in order to prove the theorem. For the special case of discrete optimal control problem with explicitely given constraints the existence conditions were stated separately.

For the finite dimensional case given by the relation (5) Boltjanskij in [5] formulates without proof some conditions sufficient for the existence of optimal solutions. However, our conditions also in this special case are more general, because only upper semicontinuity of $\boldsymbol{U}_k$ and lower semicontinuity of $h_k$, $k = 0, 1, \ldots, K - 1$, are assumed, while in [5] the continuity is always used.

It is felt that the general formulation in topological spaces can be usefull also for some other problems, which are not necessarily finite dimensional and which can be brought to the form considered here, e.g. stochastic discrete optimal control problems.

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