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A Viewpoint to the Minimum Coloring Problem of Hypergraphs

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A pseudo-Boolean programming scheme is constructed for solving the minimum coloring problem of hypergraphs. The scheme is linearized for small values of the number of vertices in a hypergraph and an elementary example is given.

1. INTRODUCTION AND BASIC CONCEPTS

The solving of the minimum coloring problem of hypergraphs is of interest to graph theory and applications. In this paper we shall construct a pseudo-Boolean programming scheme for solving this problem. As well known, all present high speed computer programs for solving pseudo-Boolean programming schemes are somewhat inefficient, and hence this paper offers only a viewpoint to this problem. The computer solution algorithms are best in case of linear programming schemes, and hence we shall look for a linearization of the scheme; a linear scheme can be constructed for hypergraphs with small number of vertices.

A hypergraph $H = (X; E_1, E_2, \ldots, E_m)$ is given by a finite set $X = \{x_1, \ldots, x_n\}$, whose elements are the vertices of H, and by subsets $E_1, \ldots, E_m$ of X called the edges of H. If $|E_j| \leqq 2$ for all j, H is an undirected graph. We shall denote the family $\{E_1, \ldots, E_m\}$ briefly by $E(H)$, and thus $H = (X; E(H))$.

The chromatic number $\chi(H)$ is defined as the minimum number of colors for which the vertices of H can be colored such that for any edge E_j, $|E_j| > 1$, the vertices in E_j are colored by at least two colors. A coloring of H with $\chi(H)$ colors is called a minimum coloring of H.

By a pseudo-Boolean function (for more details, see the monography of Hammer and Rudeanu [2]) $f(z_1, \ldots, z_p)$ we shall mean a function of p variables z_r of values zero and one mapping p-tuples of zeros and ones into the real field.

In what follows, we shall consider only hypergraphs for which $|E_j| > 1$ for any value of j, $j = 1, \ldots, m$.

Let us form an undirected graph G which offers the base to the considerations here. Let $H = (X; E(H))$ be a given hypergraph; we define the graph $G = (V(G), E(G))$ as follows: The set $V(G)$ of vertices of G consists of three disjoint sets of vertices, $V_{E(H)} = \{v_1, \ldots, v_m\}$, $V_X = \{x_1, \ldots, x_n\}$ and $V_C = \{u_1, \ldots, u_g\}$. The vertices in $V_{E(H)}$, the set vertices, correspond to the edges in H, the vertices in V_X, called vertex vertices, to the vertices in H, and those in V_C, called color vertices, corresponds to the colors used in the coloring process, where g is an upper bound for the chromatic number of H. There is in G an edge joining any two vertices u_q and x_i, $q = 1, \ldots, g$ and $i = 1, \ldots, n$, and an edge (x_i, v_j) joins two vertices x_i and v_j of G if and only if $x_i \in E_j$ in H, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. There are no other edges in G.

Clearly the minimum coloring problem of H is equivalent to the following problem in the graph G: Find a subgraph G' of G

(i) with a minimum number of vertices V_C' from the set V_C such that
(ii) any vertex vertex (i.e. a vertex of V_X) is joined by an edge to one and only one vertex of the subset V_C' of G and
(iii) any set vertex of G is joined in G' by a path of length two to at least two vertices of the subset V_C' of G.

In the following we shall translate the statements (i), (ii) and (iii) into the language of pseudo-Boolean functions.

We shall describe the desired subgraph $G' = (V(G'), E(G'))$, where $V(G') = V_{E(H)} \cup V_X \cup V_C'$, and $E(G') \subset E(G)$, as follows: A bivalent variable c_{qi} describes the edges between the vertex sets V_X and V_C' in G' such that $c_{qi} = 1$, if the edge (u_q, x_i) belongs to the edge set $E(G')$, and $c_{qi} = 0$ in other cases. Further, the paths of length two from a set vertex v_j to a color vertex u_q in V_C' via a vertex vertex v_i are characterized by the expression $a_{ji}c_{qi}$, where $a_{ji} = 1$ if and only if $x_i \in E_j$ in H and in other cases $a_{ji} = 0$. Thus $a_{ji}c_{qi} = 1$ if and only if there is in G' a path of length two from v_j to u_q via a vertex x_i. The statements (ii) and (iii) can now be expressed as follows:

(1) $$\sum_q c_{qi} = 1 \text{ for any fixed value of } i,\ i = 1, \ldots, n\,.$$

(2) $$\sum_i a_{ji}c_{qi} < |E_j| \text{ for any fixed values of } q \text{ and } j,$$
$$j = 1, \ldots, m \text{ and } q = 1, \ldots, m\,.$$

The equivalence of (1) with (ii) is obvious, and (2) expresses that not every path of length two from V_C' to a set vertex v_j is initiated from a single color vertex $u_q \in V_C'$. Hence (2) is equivalent to (iii) above. Now we must formulate (i) in a pseudo-Boolean form.

As any vertex of H is colored by a color in a minimum coloring of H, the number of edges between the vertices of the sets V_X and V'_C in G' equals the number of vertices in H (i.e. the number of vertex vertices in G and G'). Hence

$$\sum_q y_q(\sum_i c_{qi}) = n = |X| = |V_X|, \tag{3}$$

where the bivalent variable y_q has value 1 if color q is used, i.e. there is at least one edge incident to the vertex u_q in G', and in other cases $y_q = 0$, i.e. if color q is not used. Thus (i) is equivalent to the conditions (3) and (4), where (4) is

$$\text{minimize } y_1 + y_2 + \ldots + y_q\,. \tag{4}$$

As the arguments above show, the programming scheme of pseudo-Boolean expressions in (1), (2), (3) and (4) characterizes completely the minimum coloring problem of a hypergraph H. Hence any absolutely minimizing point of (4) satisfying also (1), (2) and (3) together with the values of variables c_{qi} determines a minimum coloring of H.

Unfortunately, the expression in (3) is nonlinear and hence the scheme is laborious to solve. Furthermore, as the edges incident to color vertices in G show, a color q_1 in a minimum coloring of H can be substituted by an arbitrary color $q_2 \neq q_1$, and thus most of the solutions of the scheme are not essentially new. On the other hand, obviously any minimum coloring of H is found by solving the programming scheme of (1), (2), (3) and (4). In the next section we shall consider a way of avoiding both of the difficulties mentioned above.

3. A LINEAR SCHEME

In this section we consider a linear scheme, where, after finding a minimum coloring minimizing absolutely the object function, all other minimum colorings of H can be determined by means of a modified linear object function. Unfortunately, this way applies to low values of $|X|$ only.

We substitute first the expressions (3) and (4) by the following object function

$$\text{minimize } (n^0 \sum_i c_{1i} + n^1 \sum_i c_{2i} + \ldots + n^{g-1} \sum_i c_{gi})\,. \tag{5}$$

The absolutely minimizing point of (5) satisfying also (1) and (2) determines a minimum coloring of H. Indeed, according to (1) and (2), the solution to (1), (2) and (5) determines a coloring of H. Assume that there would be a coloring of H with fewer, $k - 1$, colors than in the coloring of k colors determined by the absolutely minimizing point for (1), (2) and (5). According to the symmetry of G, the first k and $k - 1$ colors $1, \ldots, k$ and $1, \ldots, k - 1$, respectively, can be choosen as the colors of the colorings under interest. As $n^{k-2} + (n - 1)\, n^{k-1} < 1 \,.\, n^{k-1} + 1 \,.\, n^k$, the coloring of $k - 1$ colors determines always a point for which the value of (5) is smaller than the value of the absolutely minimizing point determining the k-coloring of H. This is a contradiction, and hence the absolutely minimizing point of (5) satisfying (1) and (2) determines a minimum coloring of H.

Assume that the absolutely minimizing point of (1), (2) and (5) determines a k-coloring of H, i.e. $k = \chi(H)$. As in every coloring of $k + 1$ colors there are at least two vertices, one colored by color k and one by color $k + 1$, and as the value of (5) is in case of the minimizing point smaller than $n^{k-1} + (n - 1)\, n^k < n^k + n^{k+1}$, any other minimum coloring of H can now be found by solving the pseudo-Boolean expressions in (1), (2) and (6), where

$$(6) \qquad n^0 \sum_i c_{1i} + n^1 \sum_i c_{2i} + \ldots + n^{g-1} \sum_i c_{gi} < n^{k-1} + (n - 1)\, n^k \,.$$

The different weights of the colors in (5) and (6) imply that two colors of a coloring cannot be changed, and hence any two solutions give in general two different minimum colorings of H. As the value of n^{g-1} increases very rapidly, the schemes of this section can be applied to hypergraphs with low values of $|X|$ only.

4. AN EXAMPLE

Let us consider hypergraph $H = (X; E(H))$, where $X = \{a, b, c, d, e, f, g\}$ and $E_1 = \{a, e, f\}$, $E_2 = \{a, d, g\}$, $E_3 = \{a, b, c\}$, $E_4 = \{f, d, b\}$, $E_5 = \{f, g, c\}$, $E_6 = \{c, e, d\}$ and $E_7 = \{b, e, g\}$ (see Berge [1, p. 410]). By using the Tomescu method for evaluating g (see Berge [1, p. 412]), we obtain $g = 4$, and so the programming scheme determined by (5), (1) and (2) is ($a = x_1, \ldots, g = x_7$):

Minimize $c_{11} + c_{12} + c_{13} + c_{14} + c_{15} + c_{16} + c_{17} + 7(c_{21} + c_{22} + c_{23} + c_{24} + c_{25} + c_{26} + c_{27}) + 49(c_{31} + c_{32} + c_{33} + c_{34} + c_{35} + c_{36} + c_{37}) + 343(c_{41} + c_{42} + c_{43} + c_{44} + c_{45} + c_{46} + c_{47})$

with subject to

$c_{11} + c_{21} + c_{31} + c_{41} = 1$ ($x_1 = a$ and colors 1, 2, 3 and 4)
$c_{12} + c_{22} + c_{32} + c_{42} = 1$ ($x_2 = b$ and colors 1, 2, 3 and 4)
$\vdots$
$c_{17} + c_{27} + c_{37} + c_{47} = 1$ ($x_7 = g$ and colors 1,2, 3 and 4)

and

$c_{11} + c_{15} + c_{16} < 3$ ($E_1 = \{a, e, f\}$ and color 1)
$c_{21} + c_{25} + c_{26} < 3$ (E_1 and color 2)
$\vdots$
$c_{11} + c_{14} + c_{17} < 3$ ($E_2 = \{a, d, g\}$ and color 1)
$\vdots$
$c_{42} + c_{45} + c_{47} < 3$ ($E_7 = \{b, e, g\}$ and color 4)

The absolute minimum of the object function is 67 and it is obtained when $c_{11} = c_{13} = c_{14} = c_{16} = c_2 = c_{25} = c_{37} = 1$ and the other variables have zero value. Thus $\{a, c, d, f\}$, $\{b, e\}$, $\{g\}$ is a minimum coloring of H. In order to obtain the other minimum colorings of H, if such exist, the object function is substituted by the function

$$c_{11} + \ldots + c_{17} + 7(c_{21} \ldots + c_{27}) + 49(c_{31} + \ldots + c_{37}) + \\ + 343(c_{41} + \ldots + c_{47}) < 301 .$$

$\{a, b, e\}$, $\{c, d, f\}$, $\{g\}$ is one of the other minimum colorings of H, for which the object function has the value 73.

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REFERENCES

[1] C. Berge: Graphes et hypergraphes. Dunod, Paris 1970.

[2] P. L. Hammer and S. Rudeanu: Boolean methods in operations research and related areas. Springer-Verlag, Berlin—Heidelberg—New York 1968.

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