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# Automatic Listing of Important Observational Statements III

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The ALIOS theory (of Automatic Listing of Important Observational Statements) was introduced and developed in the previous papers of this series. In the present paper, which is a free continuation, we study an important particular class of observational functor calculi and develop its logic.

## Part III — Nominal calculi with incomplete information

### INTRODUCTION

In [2] we introduced the notions of a semantic system, a problem and of a solution of a problem in a model; in [3] we studied functor calculi and corresponding semantic system, in particular, observational monadic functor calculi (OMFC's, see also below). In [4] we called the problematics of the ALIOS theory the problematic of *Automated Research*; this seems to be an acceptable term (first of all since it is short). It can be said that the logic of Automated Research is the *logic of observational functor calculi*, especially (at least at the present stage), of OMFC's. In the present part we are going to study OMFC's *with nominal values and incomplete information*, briefly, $\times$-*nominal calculi* (see below); they form an important particular class of OMFC's including the classical observational monadic predicate calculus. The generalization of the classical calculus consists in generalizing (i) truth values to more general values, (ii) connectives to some junctors and (iii) quantifiers to some operators. Values are generalized only slightly and in two directions: we allow values $0, 1, \ldots, h$ corresponding to some nominal quantities and also the values $\times$ – "unknown". (Cf. [3] 8.10 – 11 and [5].)*

* I promised in [3] a paper denoted there by [12] on nominal quantities; what I planned to write in that paper is included here. I also present here some results of [4].

The main attention is paid to the study of some particular classes of operators. The value $\times$ is treated in Kleene-Körner's style as it was in [5]. (Cf. [6]; note that Kleene-Körner calculus is studied in the recent paper by Cleave [1].) The reader interested only in the logical aspects of the present paper but not in Automated Research should consult § 1 of [2] and § 7 of [3] for terminology.

## 12. BASIC DEFINITIONS; OPEN FORMULAS

OMFC's are functor calculi whose all models are finite, all associated functions are calculable (say, recursive), all functors are unary, there is only one variable (omitted at all occurences) and (an added condition) all operators are of a type $\langle 1, \ldots, 1\rangle$ (i.e. the operator binds the variable in each of the joined formulas).

**12.1. Definition.** Let $h \geqq 1$. Numbers $0, 1, \ldots, h$ are called *regular values* ($V_{reg} = \{0, \ldots, h\}$); the symbol $\times$ is the *singular value*. So $V = V_{reg} \cap \{\times\}$. We put $0 < \times < 1$ (one might identify $\times$ e.g. with $\frac{1}{2}$). Fix a type $\underbrace{\langle 1, \ldots, 1\rangle}_{n \text{ times}}$ (say, $1^n$). A $V$-structure $\langle M, f_1, \ldots, f_n\rangle = \boldsymbol{M}$ of this type is *quantitative* if the ranges of all $f$'s are included in $\{0, \times, 1\}$; $\boldsymbol{M}$ is a structure *with complete information* if all $f$'s take only regular values. A structure $\boldsymbol{M}' = \langle M, g_1, \ldots, g_n\rangle$ with complete information is a *completion* of $\boldsymbol{M} = \langle M, f_1, \ldots, f_n\rangle$ if, for each $a$, $i$, $f_i(a) \in V_{reg}$ implies $f_i(a) = g_i(a)$.

**12.2. Remarks.** (1) A *partial $V_{reg}$-structure* is a tuple $\langle M, h_1, \ldots, h_n\rangle$ where each $h_i$ is a mapping whose domain is a subset of $M$ and whose range is a subset of $V_{reg}$. There is an obvious one-one correspondence between $V$-structures and partial $V_{reg}$-structures ($\langle M, f_1, \ldots, f_n\rangle$ corresponds to $\langle M, h_1, \ldots, h_n\rangle$ iff, for each $i$, $f_i$ extends $h_i$ and $f_i$ takes the value $\times$ on $Dom(f) - Dom(h)$).

(2) A *partition* in $M$ is a set $p$ of pairwise disjoint nonempty subsets of $M$. A *partitional structure* is a tuple $\langle M, p_1, \ldots, p_n\rangle = \boldsymbol{M}$ where $p_i$'s are partitions in $M$; we suppose that the cardinality of each $p_i$ is $\leqq h + 1$. Let, for each $i$, $e_i$ be a one-one mapping of $p_i$ into $\{0, \ldots, h\}$ (*enumeration*). Then $\langle \boldsymbol{M}, \boldsymbol{e}\rangle = \langle M, p_1, e_1, \ldots, p_n, e_n\rangle$ is an *enumerated partitional structure* called an *enumeration of $\boldsymbol{M}$.* There is an obvious one-one correspondence between all enumerated partitional structures and all $V$-structures ($\langle M, f_1, \ldots, f_n\rangle$ corresponds to $\langle M, p_1, e_1, \ldots, p_n, e_n\rangle$ iff $f_i(a) = e_i(A)$ for $a \in A \in p_i$ and $f_i(a) = \times$ for $a \in M - \bigcup p_i$).

(3) There is another obvious one-one correspondence between all $V$-structures and all qualitative structures $\langle M, q_1, \ldots, q_{n.h}\rangle$ of the type $1^{n.h}$ such that the following holds for each $i$ and each $a \in M$: if one of $q_{i.h+1}(a), \ldots, q_{i(h+1)}(a)$ is $\times$ then all of them; if no member of the above sequence is $\times$ then exactly one member is 1. (Call such a structure a *disjointed structure* of the type $\langle n; h\rangle$.) .

**12.3. Discussion.** It follows from the intuitive notion of a nominal quantity (as a quantity whose values merely enumerate factors of an equivalence relation) that speaking about a $\times$-nominal $V$-structure $\boldsymbol{M}$ we in fact want to speak about a partional structure $\boldsymbol{M}_0$ such that $M$ corresponds to an enumeration of $\boldsymbol{M}_0$. This will influence the choice of our language. Note that thereoretically we could deal only with qualitative structures, namely with disjointed structures of the type $\langle n; h\rangle$ but practically we prefer $V$-structures e.g. for saving place in the computer's memory, (See also 12.7.)

**12.4. Definition.** (1) Take functors $F_1, \ldots, F_n$. We introduce a unary junctor $(X)$ (called a *coefficient*) for each $X \subseteq V_{reg}$ putting $Asf_{(X)}(u) = 1$ iff $u \in X$, $Asf_{(X)}(u) = 0$ iff $u \in V_{reg} - X$ and $Asf_{(X)}(u) = \times$ iff $u = \times$.

(2) A function $\alpha : V^j \to V$ is *qualitative* if there is a function $\alpha_0 : \{0, \times, 1\}^j \to$ $\to \{0, \times, 1\}$ such that $\alpha(u_1, \ldots, u_j) = \alpha_0(\bar{u}_1, \ldots, \bar{u}_j)$, where $\bar{0} = 0$, $\bar{\times} = \times$ and $\bar{u} = 1$ for $u \geqq 1$. We say that $\alpha$ *corresponds* to $\alpha_0$. (Evidently, for each $\alpha_0 : \{0, \times, 1\}^j \to$ $\to \{0, \times, 1\}$ the corresponding operation on $V$ is determined uniquely.)

(3) We introduce junctors $\neg$, $\vee$, $\&$ by associated functions corresponding to the respective three-valued assodiated functions (see [5], [6] or [1]), e.g. $Asf_{\&}(u, v) = 1$ iff $u \geqq 1$ and $v \geqq 1$, $Asf_{\&}(u, v) = 0$ iff $u = 0$ or $v = 0$ and $Asf_{\&}(u, v) = \times$ otherwise.

(4) (Auxiliary.) For a while, call any OMFC whose language has the functors $F_1, \ldots, F_n$ and the junctors just defined an *openly* $\times$*-nominal* OMFC. (Nothing is assumed on operators; we study open formulas in such OMFC's.)

(5) Each formula of the form $(X)\,F$ where $F$ is a functor and $0 \subset X \subset V$ is called a *literal*. An *elementary disjunction* (ED) is a disjunction of literals in which each functor occurs at most once; similary we define *elementary conjunctions* (EC).

(6) Two open formulas $\varphi$, $\psi$ are *strongly equivalent* in $\boldsymbol{M}$ if $\|\varphi\|_M(a) = \|\psi\|_M(a)$ for each $a \in M$. (Denotation: $\varphi \cong_{\boldsymbol{M}} \psi$.)

(7) A formula is *qualitative* if $(\forall \boldsymbol{M})\,(\forall a \in M)\,(\|\varphi\|_{\boldsymbol{M}}(a) \in \{0, \times, 1\})$.

(8) Let $\varphi$, $\psi$ be qualitative open formulas. $\varphi$ *strongly implies* $\psi$ in $\boldsymbol{M}$ if $\|\varphi\|_{\boldsymbol{M}}(a) \leqq$ $\leqq \|\psi\|_{\boldsymbol{M}}(a)$ for each $a \in M$. (Denotation: $\varphi \gtrsim_{\boldsymbol{M}} \psi$, cf. [1].)

**12.5. Remark.** (1) Each non-atomic open formula is qualitative. (2) $\|F\|_{\boldsymbol{M}}(a) = \times$ iff $\|(X)\,F\|_{\boldsymbol{M}}(a) = \times$ for each coefficient $(X)$.

(3) If $\varphi$, $\psi$ are qualitative open then $\varphi \cong_{\boldsymbol{M}} \psi$ iff ($\varphi \gtrsim_{\boldsymbol{M}} \psi$ and $\psi \gtrsim_{\boldsymbol{M}} \varphi$).

**12.6. Lemma.** (1) $\neg(X)\,\varphi \cong_{\boldsymbol{M}} (V_{reg} - X)\,\varphi$,

(2) $(X)\,\varphi \cong_{\boldsymbol{M}} \bigvee_{k \in X} (\{k\})\,\varphi$ for $X \neq \emptyset$,

(3) $(X)\,\varphi \cong_{\boldsymbol{M}} \bigvee_{k \notin X} \neg(\{k\})\,\varphi$ for $X \neq V_{reg}$,

(4) $(X)\,\varphi \vee (Y)\,\varphi \cong_{\boldsymbol{M}} (X \cup Y)\,\varphi$,

(5) $(X)\,\varphi \,\&\, (Y)\,\varphi \cong_{\boldsymbol{M}} (X \cap Y)\,\varphi$.

Easy proofs are left to the reader.

**12.7. Discussion.** Call a mapping of $M$ into $\{0, \times, 1\}$ a *partial subset* of $M$. Let $\boldsymbol{M}$ be a $V$-structure and let $\boldsymbol{N}$ be the corresponding disjointed structure of the type $\langle n; h\rangle$. Fet $\mathfrak{F}$ be an openly $\times$-nominal OMFC and let $\mathfrak{P}$ be the three-valued observational monadic predicate calculus with $n \,.\, h$ predicates. One easily shows that a partial subset of $M$ is definable in $\boldsymbol{M}$ by an open qualitative formula of $\mathfrak{F}$ iff it is definable in $\boldsymbol{N}$ by an open formula of $\mathfrak{P}$. So theoretically one could deal with $\mathfrak{P}$; but practically we prefer (openly) $\times$-nominal calculi not only because of the reasons of 12.3 but also since we have some reasonable notions of complexity of formulas of $\mathfrak{F}$ not preserved by the transition to corresponding formulas of $\mathfrak{P}$; in particular, ED's and EC's seem to be very natural open formulas (one can construct "normal forms" for non-atomic open formulas in the obvious way) and in 12.8 (4) we introduce a natural "simpler than" relation for EC's and ED's.

**12.8. Definition.** (1) An EC $\kappa = \bigwedge_{i\in I} (X_i)\, F_i$ is *poorer than* $\lambda = \bigwedge_{i\in I} (Y_i)\, F_i$ if for each $i \in I$ we have $X_i \subseteq Y_i$. $\kappa \subseteq \lambda$ means that $\kappa$ is a subconjunction of $\lambda$.

(2) An EC $\kappa$ is *incompressible* in $\boldsymbol{M}$ (or $\boldsymbol{M}$-incompressible) if $\kappa$ is in $\boldsymbol{M}$ not strongly equivalent to any $\kappa_0$ strictly poorer than $\kappa$. $\kappa$ is *incancellable* in $\boldsymbol{M}$ if $\kappa$ is in $\boldsymbol{M}$ not strongly equivalent to any $\kappa_0$ which is a proper subconjunction of $\kappa$, $\kappa$ is *prime* in $\boldsymbol{M}$ if it is both incompressible and incancellable in $\boldsymbol{M}$.

(3) If $\kappa$ is $\bigwedge_{i\in I} (X_i)\, F_i$ then $neg(\kappa)$ is $\bigvee_{i\in I} (V_{reg} - X_i)\, F_i$.

(4) Let $\kappa = \bigwedge_{i\in I} (X_i) F_i$ and $\lambda = \bigwedge_{i\in J} (Y_i)\, F_i$. We put $\kappa \lhd \lambda$ ($\kappa$ is simpler than $\lambda$) if $I \subseteq J$ and for each $i \in I$ we have $X_i \subseteq Y_i$.

(5) We put $\kappa \lessdot \lambda$ if $I \subseteq J$ and for each $i \in I$ we have $Y_i \subseteq X_i$.

(6) All the above definitions are made analogously for ED's.

**12.9. Remark.** If $\kappa$, $\lambda$ are EC's if $\boldsymbol{M}$ is a model and $\kappa$ is poorer than $\lambda$ then $\kappa \gtrsim_{\boldsymbol{M}} \lambda$; if $\kappa \subseteq \lambda$ then $\lambda \gtrsim_{\boldsymbol{M}} \kappa$. Hence if $\kappa \lessdot \lambda$ then $\lambda \gtrsim_{\boldsymbol{M}} \kappa$; analogously, if $\gamma$, $\delta$ are ED's and $\gamma \lhd \delta$ then $\gamma \gtrsim_{\boldsymbol{M}} \delta$. If $\kappa \lhd \lambda$ ($\kappa$, $\lambda$ EC's) then in general neither $\kappa \gtrsim_{\boldsymbol{M}} \lambda$ nor $\lambda \gtrsim_{\boldsymbol{M}} \kappa$. Nevertheless, we have the following

**12.10. Lemma.** If $\kappa_0 \cong_{\boldsymbol{M}} \kappa$ and $\kappa_0 \lhd \kappa$ then there is a $\kappa_2$ such that $\kappa_0 \subseteq \kappa_2$, $\kappa_2$ is poorer than $\kappa$ and $\kappa \cong_{\boldsymbol{M}} \kappa_2$.

Proof. Let $\kappa_0$ be poorer than $\kappa_1 \subseteq \kappa$, $\kappa = \bigwedge_I (X_i)\, F_i$, $\kappa_1 = \bigwedge_{I_0} (X_i)\, F_i$, $\kappa_0 = \bigwedge_{I_0} (X_i^0)\, F_i\, (I_0 \subseteq I, X_i^0 \subseteq X_i)$. Put $X_i^0 = X_i$ for $i \in I - I_0$ and put $\kappa_2 = \bigwedge_I (X_i^0)\, F_i$. Then $\kappa_0 \subseteq \kappa_2$, $\kappa_2$ is poorer than $\kappa$; hence if $\|\kappa_2\|_{\boldsymbol{M}}(a) = 1$ then $\|\kappa\|_{\boldsymbol{M}}(a) = 1$. If $\|\kappa\|_{\boldsymbol{M}}(a) = 1$ then $\|\kappa\|_{\boldsymbol{M}}(a) = 1$ and $\|\bigwedge_{I-I_0} (X_i)\, F_i\|_{\boldsymbol{M}}(a) = 1$, hence $\|\kappa_2\|_{\boldsymbol{M}}(a) = 1$. If $\|\kappa_2\|_{\boldsymbol{M}}(a) = \times$ then $\|\kappa\|_{\boldsymbol{M}}(a) \in \{\times, 1\}$; conversely, if $\|\kappa\|_{\boldsymbol{M}}(a) = \times$ then $\|\kappa_0\|_{\boldsymbol{M}}(a) = \times$ and $\|\bigwedge_{I-I_0} (X_i)\, F_i\|_{\boldsymbol{M}}(a) \in \{\times, 1\}$, so $\|\kappa_2\|_{\boldsymbol{M}}(a) = \times$.

**12.11. Corollary.** For each $\boldsymbol{M}$ and each $\kappa$, $\kappa$ is $\boldsymbol{M}$-prime iff there is no $\kappa_0 \neq \kappa$ such that $\kappa_0 \lhd \kappa$ and $\kappa_0 \cong_{\boldsymbol{M}} \kappa$.

**12.12. Lemma.** (1) Each subconjuction of an $\boldsymbol{M}$-incompressible EC is $M$-incompressible. (2) Each $\boldsymbol{M}$-incompressible EC $\kappa$ contains an $\boldsymbol{M}$-prime subconjuction strongly equivalent to $\kappa$.

Proof. (1) follows immediately from 12.10; (2) is then evident.

## 13. OPERATORS; IMPROVEMENT OPERATORS

**13.1. Definition.** (1) Let $\mathfrak{N}$ be a set of $V$-structures. A mapping $\alpha : \mathfrak{N} \to \{0, \times, 1\}$ is *qualitative* if for each $\boldsymbol{M} = \langle M, f_1, \ldots, f_n \rangle$ we have $\alpha(\boldsymbol{M}) = \alpha(\langle M, \bar{f}_1, \ldots, \bar{f}_n \rangle)$ where $\overline{f_i(a)} = \bar{f}_i(a)$ for each $a \in M$ (cf. 12.4).

(2) Let $\mathfrak{F}$ be an openly $\times$-nominal OMFC and let $Q$ be an operator of $\mathfrak{F}$. $Q$ is *qualitative* if its associated function is qualitative. (Evidently, such a function is determined by its values on qualitative models.)

**13.2. Lemma.** $Q$ is qualitative iff, for each $\boldsymbol{M}$, $\|Q(F_1, \ldots, F_n)\|_{\boldsymbol{M}} = \|Q((X)\, F_1, \ldots$ $\ldots, (X)\, F_n\|_{\boldsymbol{M}}$ where $X = V_{reg} - \{0\}$. (Since $\langle M, \|(X)\, F_1\|_{\boldsymbol{M}}, \ldots, \|(X)\, F_n\|_{\boldsymbol{M}} \rangle$ is $\langle M, \bar{f}_1, \ldots, \bar{f}_n \rangle$).

**13.3. Definition and Remark.** (1) Introduce a qualitative operator $\hat{=}$ of the type $1^2$ (*strong equivalence*) whose associated function takes for a qualitative model $\boldsymbol{M} = \langle M, f_1, f_2 \rangle$ value 1 if $f_1 = f_2$, otherwise $Asf_{\hat{=}}(M) = 0$. Then, for qualitative open formulas $\varphi$, $\psi$ we have: $\varphi \cong_{\boldsymbol{M}} \psi$ iff $\|\varphi \hat{=} \psi\|_{\boldsymbol{M}} = 1$.

(2) We further introduce a qualitative operator $\simeq$ of the type $1^2$ (*weak equivalence*) whose associated function takes for a qualitative model $\boldsymbol{M} = \langle M_1 f_1, f_2 \rangle$ value 1 if $(\forall a \in M)\, (f_1(a) = 1$ iff $f_2(a) = 1)$ and otherwise $Asf_{\simeq}(\boldsymbol{M}) = 0$.

(3) Evidently, an EC $\kappa$ is $\boldsymbol{M}$-compressible iff there is a $\kappa_0 \neq \kappa$, $\kappa_0$ poorer that $\kappa$ such that $\|\kappa_0 \hat{=} \kappa\|_{\boldsymbol{M}} = 1$. Analogously we define: $\kappa$ is *weakly $\boldsymbol{M}$-compressible* if there is a $\kappa_0 \neq \kappa$, $\kappa_0$ poorer than $\kappa$ such that $\|\kappa_0 \simeq \kappa\|_{\boldsymbol{M}} = 1$. Caution: the negation of "weakly $\boldsymbol{M}$-compressible" is "*strongly $\boldsymbol{M}$-incompressible*".

**13.4. Definition.** An openly $\times$-nominal OMFC $\mathfrak{F}$ is a *$\times$-nominal* OMFC (or an OMFC *with nominal values and incomplete information*) if all operators of $\mathfrak{F}$ are qualitative.

**13.5. Remark.** This definition corresponds to our interest in qualitative formulas; Lemma 13.2 shows that every non-atomic formula (not necessarily open) is strongly equivalent to a qualitative formula.

**13.6. Definition.** Let $\mathfrak{F}$ be a $\times$-nominal OMFC. An operator $Q$ is *secured* if for each model $\boldsymbol{M}$ of the type of $Q$ we have: $Asf_Q(\boldsymbol{M}) = 1$ iff for each completion $\boldsymbol{M}'$ of $\boldsymbol{M}$ $Asf_Q(\boldsymbol{M}') = 1$, $Asf_Q(\boldsymbol{M}) = 0$ iff for each completion $\boldsymbol{M}'$ of $\boldsymbol{M}$ $Asf_Q(\boldsymbol{M}') = 0$, $Asf_Q(\boldsymbol{M}) = \times$ otherwise.

**13.7. Remark.** (1) Evidently, it suffices when the above conditions are satisfied for all qualitative models (since $Q$ is qualitative, $\mathfrak{F}$ being $\times$-nominal). Hence, the semantics of a secured operator is determined by its semantics on qualitative models with complete information.

(2) One often defines an operator using a real function $S(\boldsymbol{M})$ (e.g. a statistic) putting for $\{0, 1\}$ – models $Asf_Q(\boldsymbol{M}) = 1$ if $S(M) \leqq \alpha$ and $= 0$ otherwise. If one uses such a definition to define a secured operator, it is advantageous to have an effective procedure which, given a qualitative model $\boldsymbol{M}$ produces its completion for which the value $S(\boldsymbol{M}_{\max})$ is maximal among all $S(\boldsymbol{M}')$ for $\boldsymbol{M}'$ being a completion of $\boldsymbol{M}$.

**13.8. Discussion.** The definition of a secured operator is inspired by the idea of a "heavenly model" (cf. [5]): we treat $\times$ as a sign for missing information; if $\|\varphi\|_M(a) = \times$ then in fact the object $a$ either has the property $\varphi$ of has not; we only do not know what case occurs. Since the right "heavenly" completion is not available, wanting to be sure that a formula $Q(\varphi_1, \ldots, \varphi_n)$ is true in it we must assure that it is true in all the completions. A philosophical investigation of the value $\times$ in connection with "inexact classes" is contained in [6].

Non-secured operators are auxiliary from our point of view; but they can be quite helpful. For example, $\triangleq$ is not secured. In this paper we shall study certain non-secured operatos called improvement operators. We find important particular improvement operators in Sections 15 and 16.

In the rest of the present section we consider an arbitrary OMFC $\mathfrak{F}$ whose set of values contains 0, 1. "Tautology" means $\{1\}$-tautology etc.

**13.9. Definition.** Let $H$ be an operator of the type $1^n$, let $Of$ be a (finite) set of $n$-tuples of open formulas, let $\subseteq$ be an ordering of $Of$ such that the supremum of any two elements exists; finally, let $\preccurlyeq$ be an operator of the type $1^{2n}$; $\Phi$, $\Psi$, $\Omega$ denote elements of $Of$.

(1) $\preccurlyeq$ is a *closure operator* for $Of$ and $\subseteq$ if $\Phi \preccurlyeq \Phi$ is a tautology for each $\Phi$ and if the following rules are sound:

$$\left.\begin{array}{ll}\text{(I)} & \dfrac{\Phi \preccurlyeq \Psi,\ \Psi \preccurlyeq \Omega}{\Phi \preccurlyeq \Omega} \\[2ex] \text{(II)} & \dfrac{\Phi \preccurlyeq \Omega}{\Phi \preccurlyeq \Psi,\ \Psi \preccurlyeq \Omega}\end{array}\right\} \quad \text{for} \quad \Phi \subseteq \Psi \subseteq \Omega,$$

(III) $$\frac{\Phi \leqslant \Psi_1, \Phi \leqslant \Psi_2}{\Phi \leqslant \Omega} \quad \text{for} \quad \Omega = sup_{\subseteq}(\Psi_1, \Psi_2)\,.$$

(2) $\leqslant$ *satisfies modus ponens* w.r.t. $H$, $Of$, $\subseteq$ if the following rule is sound:

$$\frac{H(\Phi), \Phi \leqslant \Psi}{H(\Psi)} \quad \text{for} \quad \Phi \subseteq \Psi\,.$$

(3) $\leqslant$ is an *improvement operator* for $H$, $Of$, $\subseteq$ if it is a closure operator $Of$, $\subseteq$ and satisfies modus ponens w.r.t. $H$, $Of$, $\subseteq$.

**13.10. Lemma and Definition.** (1) Let $\leqslant$ be a closure operator for $Of$, $\subseteq$. Then for each $\Phi \in Of$ and each $\boldsymbol{M}$ there is a uniquely determined $\Psi \supseteq \Phi$ which is $\subseteq$-maximal such that $\|\Phi \leqslant \Psi\|_{\boldsymbol{M}} = 1$; put $\Psi = Reg \leqslant_{\boldsymbol{M}}(\Phi)$. $Reg \leqslant_{\boldsymbol{M}}(\Phi)$ is the supremum of all $\Psi \subseteq \Phi$ such that $\|\Phi \leqslant \Psi\|_{\boldsymbol{M}} = 1$.

(2) The following holds for each $\Phi$, $\Psi$:

$$\Phi \subseteq Reg \leqslant_{\boldsymbol{M}}(\Phi);\ \Phi \subseteq \Psi \subseteq Reg \leqslant_{\boldsymbol{M}}(\Phi) \rightarrow Reg \leqslant_{\boldsymbol{M}}(\Psi) = Reg \leqslant_{\boldsymbol{M}}(\Phi)\,.$$

Proofs are easy from the definition.

**13.11. Remark.** Suppose that $\langle Of, \subseteq \rangle$ has the following algebraic property: whenever $\Phi \subset \Psi \subset \Omega$ then there is a $\overline{\Psi}$ such that $\Phi \subset \overline{\Psi} \subset \Omega$, $\overline{\Psi}$ is an immediate successor of $\Phi$ and $\Psi$, $\overline{\Psi}$ are incomparable. Then for $\Phi \subset \Omega$, $\Omega$ is the supremum of all immediate successors $\Psi$ of $\Phi$ such that $\Psi \subseteq \Omega$. In particular, $Reg \leqslant_{\boldsymbol{M}}(\Phi)$ is the supremum of all the immediate successors $\Psi$ of $\Phi$ such that $\|\Phi \leqslant \Psi\|_{\boldsymbol{M}} = 1$ provided there are any; if not, $Reg \leqslant_{\boldsymbol{M}}(\Phi) = \Phi$.

**13.12. Theorem.** Let $\leqslant$ be an improvement operator for $H$, $Of$, $\subseteq$; put $\boldsymbol{P} = \langle \boldsymbol{F}, \{1\}, IC \rangle$ where $\boldsymbol{F} = \{H(\Phi); \Phi \in Of\}$ and $IC$ is the following rule:

$$\frac{H(\Phi)\ \&\ \Phi \leqslant \Omega}{H(\Psi)} \quad \text{for} \quad \Phi \subseteq \Psi \subseteq \Omega\,.$$

Then $IC$ is sound and hence $\boldsymbol{P}$ is a problem; for each model $\boldsymbol{M}$, call $H(\Phi)$ $\boldsymbol{M}$-prime if $\|H(\Phi)\|_{\boldsymbol{M}} = 1$ and there is no $\Psi \subset \Phi$ such that $\|H(\Psi)\|_{\boldsymbol{M}} = 1$ and $\|\Psi \leqslant \Phi\|_{\boldsymbol{M}} = 1$. Put $\boldsymbol{X} = \{H(\Phi)\ \&\ \Phi \leqslant \Psi;\ H(\Phi)\ \boldsymbol{M}\text{-prime and } \Psi = Reg \leqslant_{\boldsymbol{M}}(\Phi)\}$. Then $\boldsymbol{X}$ is a $\subseteq$-minimal solution of $\boldsymbol{P}$ in $\boldsymbol{M}$.

Proof. $\boldsymbol{X} \subseteq Tr(\boldsymbol{M})$ is trivial; if $\|H(\Omega)\|_{\boldsymbol{M}} = 1$ then consider the set $Pr_{\Omega}$ of all $\Phi \subseteq \Omega$ such that $\|H(\Phi)\ \&\ \Phi \leqslant \Omega\|_{\boldsymbol{M}} = 1$. Let $\Phi$ be a $\subseteq$-minimal element of $Pr_{\Omega}$. Then $H(\Phi)$ is $\boldsymbol{M}$-prime and $\Phi \subseteq \Omega \subseteq Reg \leqslant_{\boldsymbol{M}}(\Phi)$. Hence if $Reg \leqslant_{\boldsymbol{M}}(\Phi) = \Psi$ then $(H(\Phi)\ \&\ \Phi \leqslant \Psi)$ is in $\boldsymbol{X}$ and $H(\Omega)$ is an immediate consequence of $H(\Phi)\ \&\ \Phi \leqslant \Psi$. If $(H(\Phi)\ \&\ \Phi \leqslant \Psi) \in \boldsymbol{X}$ then this formula is the only one from which $H(\Phi)$ immediately follows; so $\boldsymbol{X}$ is $\subseteq$-minimal.

**13.13. Remarks.** (1) Compare theorem 13.12 with [3] 9.3–9.9. One can form a hierarchicity statement as in 9.4: Let

$$cf(H(\Phi)) = \Phi\,, \quad cf(H(\Phi)\,\&\,\Phi \ll \Psi) = \Phi\,,$$
$$Sent = \{H(\Phi)\,;\ \Phi\} \cup \{H(\Phi)\,\&\,\Phi \ll \Psi\,;\quad \Phi \subseteq \Psi\}\,.$$

Let, for $\Sigma_1, \Sigma_2 \in Sent$, $\Sigma_1 \leqq_R \Sigma_2$ iff $cf(\Sigma_1) \subseteq cf(\Sigma_2)$. Let $\boldsymbol{H}$ be a hierarchy on *Sent* such that $R \subseteq R_H$. Then $\{X \cap h;\, h \in \boldsymbol{H}\}$ is a solution of the hierarchical problem $\langle \boldsymbol{P}, \boldsymbol{H}\rangle$. (The proof is routine; cf. [2] 5.4 and 5.9.)

(2) Let $\ll_1$ and $\ll_2$ be two improvement operators for $H$, *Of*, $\subseteq$; define problems $\boldsymbol{P}_1, \boldsymbol{P}_2$ and solutions $\boldsymbol{X}_1, \boldsymbol{X}_2$ as above. $\ll_1$ is said to be stronger than $\ll_2$ if $(\Phi \ll_2 \Psi) : : (\Phi \ll_1 \Psi)$ is a sound rule. Evidently, if $\ll_1$ is stronger than $\ll_2$ then $card(\boldsymbol{X}_1) \leqq \leqq card(\boldsymbol{X}_2)$ so that $\boldsymbol{X}$ is better.

(3) Improvement operators can be used also to improve solutions of problems having already a non-trivial relation of immediate consequence; one strengthens the relation using properties of improvement operators and diminishes the cardinality of solutions. An example will be considered in Section 16.

(4) Let us mention here that various notions of minimality of a solution do not formalize all intuivite criteria for a solution to be good but only some of them. We want the solution (as the machine output e.g. in printed form) to be both as small as possible and as transparent as possible. Preference of a certain (sort of a) solution can depend also on one's taste.

## 14. CLOSURE OPERATORS

Closure operators were defined in 13.9; we shall now study some particular cases

**14.1. Definition.** A qualitative operator $Q$ of the type $1^n$ is *universally definable* if there is a set $U \subseteq \{0, \times, 1\}^n$ such that, for each qualitative model $\boldsymbol{M} = = \langle M, f_1, \ldots, f_n\rangle$ of the type $1^n$ we have the following:

$$Asf_Q(\boldsymbol{M}) = 1 \quad \text{iff} \quad (\forall a \in M)\,(\langle f_1(a), \ldots, f_n(a)\rangle \in U)\,,$$
$$Asf_Q(\boldsymbol{M}) = 0 \quad \text{otherwise.}$$

(In words: $Asf_Q(\boldsymbol{M}) = 1$ iff ale the cards in $\boldsymbol{M}$ belong to $U$.)

**14.2. Remark.** The set $U$ defines a junctor $\iota\,(Asf_\iota(u_1, \ldots, u_n) = 1$ iff $\langle u_1, \ldots, u_n\rangle \in U$, $=0$ otherwise) such that, for each $\boldsymbol{M}$, $Q(\varphi_1, \ldots, \varphi_n) \cong_{\boldsymbol{M}} \forall_\iota(\varphi_1, \ldots, \varphi_n)$ ($\forall$ is the universal quantifier). We restrict ourselves to universally definable closure operators.

**14.3. Definition.** If $\ll$ is a closure operator (for *Of*, $\subseteq$) and if $\ll$ is universally defined by $U$ then $U$ is called a *closure set* (for *Of*, $\subseteq$).

**14.4. Definition.** (1) A *pseudoliteral* is a formula of the form $(X)\,F$ where $X \subset V_{reg}(X = \emptyset$ is not excluded). Pseudoelementary conjunctions and disjunctions (psEC, psED) are defined from pseudoliterals as EC's and ED's from literals.

(2) The orderings $\lhd$, $\lessdot$ are extended in the obvious manner to psEC's and psED's.

**14.5. Lemma.** If $\kappa = \bigwedge_I (X_i)\,F_i$ and $\lambda = \bigwedge_J (Y_i)\,F_i$ are psEC's then, in the sense of $\lessdot$, the supremum of $\kappa, \lambda$ is $\bigwedge_{I\cup J} (Z_i)\,F_i$ where $Z_i = X_i \cap Y_i$ for $i \in I \cap J$, $Z_i = X_i$ for $i \in I - J$ and $Z_i = Y_i$. for $i \in J - I$; for each $M$, $\sup(\kappa, \lambda) \cong_M \kappa \,\&\, \lambda$.

**14.6. Theorem.** (1) The strong equivalence operator $\triangleq$ is a universally definable closure operator for psAC's and $\lessdot$.

(2) If $\kappa$ is a psEC then $Reg{\triangleq_M}(\kappa)$ is $M$-incompressible.

(3) If $\kappa$ is an EC satisfiable in $M$ (i.e. $(\exists a \in M)(\|\kappa\|_M(a) = 1)$ then $Reg{\triangleq_M}(\kappa)$ is an EC (no coefficient is empty).

(4) The weak equivalence operator $\doteq$ is a universally definable closure operator for psEC's and $\lessdot$.

(5) If $\kappa$ is a psEC then $Reg{\doteq_M}(\kappa)$ is strongly $M$-incompressible (i.e. not weakly $M$-compressible).

(6) If $\kappa$ is an EC satisfiable in $M$ then $Reg{\doteq_M}(\kappa)$ is an EC.

Proof. (1) It follows from the definition using 14.5 that is a closure operator for psEC's and $\lessdot$ (observe that $\kappa \lessdot \lambda$ implies $\|\kappa\|_M(a) \geqq \|\lambda\|_M(a)$ for each $M$ and each $a \in M$. Clearly, $\triangleq$ is universally defined by $U = \{\langle 0, 0\rangle, \langle \times, \times\rangle, \langle 1, 1\rangle\}$.

(2) Let $Reg{\triangleq_M}(\kappa)$ be $\kappa_1 \,\&\, (X)\,F$ and suppose $\|\kappa_1 \,\&\, (X)\,F \triangleq \kappa_1 \,\&\, (X_0)\,F\|_M = 1$ for some $X_0 \subseteq X$. Then, since $\|\kappa \triangleq \kappa_1 \,\&\, (X)\,F\|_M = 1$, we have $\|\kappa \triangleq \kappa_1 \,\&\, (X_0)\,F\|_M = 1$, hence $\kappa_1 \,\&\, (X_0)\,F \lessdot \kappa_1 \,\&\, (X)\,F$ by the properties of $Reg$. Consequently, $X \subseteq X_0$.

(3) Suppose $Reg{\triangleq_M}(\kappa)$ to be $\kappa_1 \,\&\, (\emptyset)\,F$ and let $\|\kappa\|_M(a) = 1$. Then either $\|F\|_M(a) \in \{0, 1\}$, $\|(\emptyset)\,F\|_M(a) = 0$ and $\|\kappa_1 \,\&\, (\emptyset)\,F\|_M(a) = 0$ or $\|F\|_M(a) = \times$ and $\|\kappa_1 \,\&\, (\emptyset)\,F\|_M(a) = \times$; in either case we arrive at a contradiction with $\kappa \cong M \cong_M Reg{\triangleq_M}(\kappa)$.

Proofs of (4)–(6) are similar.

**14.7. Corollary.** For each psEC $\kappa$ and each $M$, (1) there is a uniquely determined psEC $\kappa_0$ poorer than $\kappa$, $M$-incompressible and such that $\kappa_0 \cong_M \kappa$;

(2) there is a uniquely determined psEC $\kappa_1$ poorer than $\kappa$, strongly $M$-incompressible and such that $\|\kappa_0 \doteq \kappa\|_M = 1$. (The last fact could be denoted by $\kappa_0 \simeq_M \kappa$ if you like.)

Proof. (1) Take the subconjuction $\kappa_0$ of $Reg \triangleq_M(\kappa)$ containing exactly the same functors as $\kappa$; then $\kappa_0$ is poorer than $\kappa$ (since $\kappa \lessdot Reg{\triangleq_M}(\kappa)$), $\kappa_0$ is $M$-incompressible by 12.12 and $\kappa \cong_M Reg{\doteq_M}(\kappa) \gtrsim_M \kappa_0 \gtrsim_M \kappa$, hence $\kappa \cong_M \kappa_0$. The uniquennes is obvious (e.g. by 14.5).

**14.8. Remark and Definition.** In the sequel, we shall be interested in closure operators for sets of pairs of open formulas, in particular, in the set $CC$ of pairs $\langle \kappa, \lambda \rangle$ of psEC's and in the set $CD$ of pairs $\langle \kappa, \delta \rangle$ where $\kappa$ is a psEC and $\delta$ is a psED. (We could consider also $DC$ and $DD$ in the obvious meaning and way.) We define $\prec$ for elements of $CC$ as the "product" ordering: $\langle \kappa, \lambda \rangle \prec \langle \bar{\kappa}, \bar{\lambda} \rangle$ if $\kappa \prec \bar{\kappa}$ and $\lambda \prec \bar{\lambda}$. Similarly, $\langle \kappa, \delta \rangle \,(\prec \lhd)\, \langle \bar{\kappa}, \bar{\delta} \rangle$ if $\kappa \prec \bar{\kappa}$ and $\delta \lhd \bar{\delta}$.

By a CC-*closure set* we mean a closure set $U$ for $CC$ and $\prec$ satisfying the following additional condition: $\langle u, v, \bar{u}, \bar{v} \rangle \in U$ implies $u \geqq \bar{u}, v \geqq \bar{v}$ for each $u, v, \bar{u}, \bar{v}$. A CC-*closure operator* is an operator defined universally by a CC-closure set. (The condition is natural since having a closure operator $\lll$ for $CC$ and $\prec$ we are interested only in values $\|(\kappa, \lambda) \lll (\bar{\kappa}, \bar{\lambda})\|_M$ for $\langle \kappa, \lambda \rangle \prec \langle \bar{\kappa}, \bar{\lambda} \rangle$; but then, for each $a \in M$, we have $\|\kappa\|_M(a) \geqq \|\bar{\kappa}\|_M(a)$ and $\|\lambda\|_M(a) \geqq \|\bar{\lambda}\|_M(a)$. Similarly for a CD-closure set (operator). The following lemma shows that the theory of CD-closure sets reduces completely to the theory of CC-closure sets (cf. 14.10):

**14.9. Lemma** Let $\lll, \lll^*$ be two operators of the type $1^n$ and let $[(\kappa, \delta) \lll^* (\bar{\kappa}, \bar{\lambda})] \cong_M$ $\cong_M [(\kappa, neg(\delta)) \lll (\bar{\kappa}, neg(\bar{\lambda}))]$ for each model $M$ and any EC's $\kappa \prec \bar{\kappa}$ and ED's $\delta \lhd \bar{\delta}$. Then $\lll$ is a CC-closure operator iff $\lll^*$ is a CD-closure operator.

Evident; note that $\delta \lhd \bar{\delta}$ iff $neg(\delta) \prec neg(\bar{\delta})$.

**14.10. Corollary.** For each $\langle u, v, \bar{u}, \bar{v} \rangle \in \{0, \times, 1\}^4$, let $N(\langle u, v, u, v \rangle) =$ $= \langle u, \neg v, \bar{u}, \neg \bar{v} \rangle$ (where, of course, $\neg 1 = 0$, $\neg \times = \times$, $\neg 0 = 1$). Let $U \subseteq$ $\subseteq \{0, \times, 1\}^4$. Then $U$ is a $CC$-closure set iff $N''U$ is a CD-closure set.

**14.11. Remark and Definition.** (1) Note that if $\sim$ is an operator of the type $1^2$, it is not true that improvement operators for $\sim$, $CC$ and $\prec$ reduce to improvement operators for $\sim$, $CD$ and $(\prec \lhd)$! See Section 16.

(2) We shall find reasonable necessary and sufficient conditions for a set $U \subseteq$ $\subseteq \{0, \times, 1\}^4$ to be a CC-closure set. We shall deal with $U$ as with a binary relation on $\{0, \times, 1\}^2$.

(3) $\geqq_{CC}$ is the following ordering of $\{0, \times, 1\}^2$. $\langle u, v \rangle \leqq_{CC} \langle \bar{u}, \bar{v} \rangle$ iff $u \leqq \bar{u}$ and $v \leqq \bar{v}$.

(4) If $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$ then $u \,\&\, v = \langle u_1 \,\&\, v_1, u_2 \,\&\, v_2 \rangle$.

**14.12. Lemma.** A set $U \subseteq \{0, \times, 1\}$ is a closure set iff the following holds for each $u, v, w \in \{0, \times, 1\}^2$:

(I) $uUu$;
(II) $uUv$ implies $u \geqq_{CC} v$;
(III) $uUv$ and $u\,U\,w$ implies $uUw$;
(III) $uUv$ and $uUw$ implies $uUw$;
(IV) $uUw$ and $u \geqq_{CC} v \geqq_{CC} w$ implies $uUv$, $vUw$;
(V) $uUv$ and $uUw$ implies $uUv \,\&\, w$.

Obvious from the definition 13.9 and 14.9 (of a closure operator and a closure set).

**14.13. Theorem.** A set $U \subseteq \{0, \times, 1\}^4$ is a CC-closure set iff there is an equivalence $E$ on $\{0, \times, 1\}^2$ such that

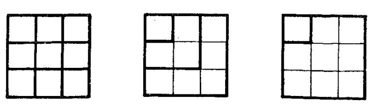
(i) $u \equiv_E w$ and $u \geqq_{CC} v \geqq_{CC} w$ implies $u \equiv_E v \equiv_E w$;
(ii) $u \equiv_E v$ implies $u \equiv_E u \,\&\, v \equiv_E w$;
(iii) $uUv$ iff $u \geqq_{CC} v$ and $u \equiv_E v$.

Proof. (1) Consider $U$ as a graph on $\{0, \times, 1\}^2$ and let $u \equiv_E v$ mean that $u$ and $v$ are in the same component of $U$. Suppose $U$ to be a CC-closure set. We prove (iii). If $uUv$ then $u \geqq_{CC} v$ by (II) and evidently $u \equiv_E v$; to prove the converse suppose we have a chain $u_0, \ldots, u_n$ such that, for each $i$, either $u_i U u_{i+1}$ or $u_{i+1} U u_i$ (i.e. $u_0$ and $u_n$ are in the same component). We prove $u_0 U(u_0 \,\&\, u_i)$ for $i = 1, \ldots, n$. This is obvious for $i = 1$ by (I). Let the statement hold for some $i$ and consider $i + 1$. If $u_i \geqq_{CC} u_{i+1}$ then $u_i U u_{i+1}$ hence $u_i U(u_0 \,\&\, u_i \,\&\, u_{i+1})$ by (V), hence $u_i U(u_0 \,\&\, u_{i+1})$ and $u_0 \,\&\, u_i U u_0 \,\&\, u_{i+1}$ by (IV). Then $u_0 U(u_0 \,\&\, u_{i+1})$ by (III) and $u_{i+1} U(u_0 \,\&\, u_{i+1})$ by (IV) (from $u_i U(u_0 \& u_{i+1})$). On the other hand, if $u_{i+1} \geqq_{CC} u_i$ then $u_{i+1} U u_i U u_0 \& \mu_i$; hence $u_{i+1} U u_0 \,\&\, u_i$; furthermore, $u_{i+1} \geqq_{CC} u_0 \,\&\, u_{i+1} \geqq_{CC} u_0 \,\&\, u_i$, which implies $u_{i+1} U(u_0 \,\&\, u_{i+1})$ by (IV). Similarly, $u_0 U(u_0 \,\&\, u_i)$ and $u_0 \geqq_{CC} u_0 \,\&\, u_{i+1} \geqq_{CC} u_0 \,\&\, u_i$ implies $u_0 U(u_0 \,\&\, u_{i+1})$ by (IV). This proves (iii), since if $u = u_0$, $v = u_n$ and $u \leqq_{CC} v$ then $uUv$ ($v = u \,\&\, v$).

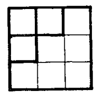
We prove (i). Let $u \equiv_E w$ and $u \geqq_{CC} v \geqq_{CC} w$. By (iii), $uUw$; hence $uUvUw$ by (*IV*) and hence $u \equiv_E v \equiv_E w$. We prove (ii), Let $u \equiv_E v$, $u = u_0$, $v = u_n$, $u_0, \ldots, u_n$ as above. Then it follows $u_0 U(u_0 \,\&\, u_i)$ as above; we obtain $uU(u \,\&\, v)$ and $vU(v \,\&\, u)$. This completes the proof if the implication $\rightarrow$ in the theorem.

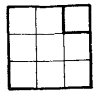
(2) Suppose now that (i)–(iii) are satisfied; we prove that $U$ is a closure set. We verify (I)–(V). (I) and (II) are evident from (iii). (III) follows from (iii) using transitivity of $E$ and of $\geqq_{CC}$. (*IV*) follows by (i) and (V) follows by (I). This completes the proof.

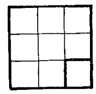
**14.14. Examples.** The previous theorem enables us to represent a closure set as an appropriate decomposition of $\{0, \times, 1\}^2$. The set $\{0, \times, 1\}^2$ is represented as a square matrix where the first row (column) corresponds to the value 1, the second to the value $\times$ and the third to the value 0. Thick lines define subsets of $\{0, \times, 1\}^2$ that form the decomposition. The conditions that must be satisfied are (i) and (ii) of 14.13: (iii) defines $U$.



(a) (b) (c)


(d)


(e)


(f)

Comments: (a) $u \equiv_E v$ iff $u = v$;

(b) $u \equiv_E v$ iff $u_1 \& u_2 = v_1 \& v_2$ $(u = \langle u_1, u_2 \rangle,\ v = \langle v_1, v_2 \rangle)$;

(c) $u \equiv_E v$ iff $(u = \langle 1, 1 \rangle$ is equivalent to $v = \langle 1, 1 \rangle)$;

(d) $u \equiv_E v$ iff(whenever $u$ or $v$ are in $\{\langle 1, 1 \rangle, \langle \times, 1 \rangle, \langle 1, \times \rangle\}$ then $u = v$).

(e) This equivalence does not determine a CC-closure set since (i) is violated: $\langle 1, \times \rangle \equiv_E \langle \times, 0, \rangle$, $\langle 1, \times \rangle \geqq_{CC} \langle 1, 0 \rangle \geqq_{CC} \langle \times, 0 \rangle$ but not $\langle 1, \times \rangle \equiv_E \langle 1, 0 \rangle$.

(f) Here (ii) is violated: $\langle \times, 0 \rangle \equiv_E \langle 0, \times \rangle$, $\langle \times, 0 \rangle \& \langle 0, \times \rangle = \langle 0, 0 \rangle$ but not $\langle \times, 0 \rangle \equiv_E \langle 0, 0 \rangle$.

**14.15. Remark.** (1) We shall meet some of the above examples in Sections 15 and 16.

(2) We are interested in EC's; we are dealing with psEC's since they are closed under supremum (w.r.t. $\lessdot$). Having a CC-closure operator, two questions arise:

(i) If $\langle \kappa, \lambda \rangle$ is a pair of EC's, is $Reg \lll_M(\kappa, \lambda) = \langle \bar{\kappa}, \bar{\lambda} \rangle$ a pair of EC's?

(ii) Is $\bar{\kappa}, \bar{\lambda}$ a pair of incompressible conjunctions? Is $\bar{\kappa}$ & $\bar{\lambda}$ incompressible? The following theorems give some information.

**14.16. Theorem.** Let $U$ be a CC-closure set defining a CC-closure operator $\lll$. The following are equivalent:

(i) $\langle 1, 1, u, v \rangle \in U$ implies $\langle u, v \rangle = \langle 1, 1 \rangle$;

(ii) For each $M$ such that $(\exists a \in M)(\|\kappa \& \lambda\|_M (a) = 1$ ($\kappa, \lambda$ are EC), $Reg \lll_M(\kappa, \lambda)$ is a pair of EC's.

Proof. Suppose (i). Let $\|(\kappa, \lambda) \lll (\kappa \& (X) F, \lambda\|_M = 1$ and $\|\kappa \& \lambda\|_M (a) = 1$. We want to show that $X = \emptyset$ is impossible. Suppose $X = \emptyset$. Then $\|\kappa \& (\emptyset) F\|_M (a) \in \{0, \times\}$, hence $\langle \|\kappa\|_M (a), \|\lambda\|_M (a), \|\kappa \& (\emptyset) F\|_M (a), \|\lambda\|_M (a) \rangle \notin U$, a contradiction.

Suppose non (i). Then either $\langle 1, 1, 1, \times \rangle \in U$ or $\langle 1, 1, \times, 1 \rangle \in U$; suppose $\langle 1, 1, 1, \times \rangle \in U$. Let $F$ be a functor not in $\kappa \& \lambda$ and let $M$ be a model in which, for each object $a$, $\|\kappa\|_M (a) = \|\lambda\|_M (a) = 1$ and $\|F\|_M (a) = \times$. Then $\|(\kappa, \lambda) \lll (\kappa \& (\emptyset) F, \lambda)\|_M = 1$, hence $Reg \lll_M(\kappa, \lambda)$ is not a pair of EC's.

**14.17. Theorem.** (1) If $\lll$ is a CC-closure operator and if $\langle \kappa, \lambda \rangle$ is a pair of psEC's then for each $M$, $Reg \lll_M(\kappa, \lambda)$ is a pair of $M$-incompressible psEC's.

(2) Let $\lll$ be universally defined by a CC-closure set $U$. The following are equivalent:

(i) $\langle 1, 0, 0, 0\rangle$, $\langle 0, 1, 0, 0\rangle \in U$ (and hence all pairs containing at least one 0 are equivalent in the equivalence $E$ of 14.13). 

(ii) For each $M$ and each pair $\langle \kappa, \lambda\rangle$ of psC's, if we put $Reg \lessdot_{M}(\kappa, \lambda) = \bar{\kappa}, \bar{\lambda}\rangle$ then $\kappa \,\&\, \lambda$ is $M$-incompressible.

Proof. (1) follows from 14.7.

(2) Suppose (i). It suffices to verify the following: If $\|(\kappa, \lambda) \lessdot (\kappa \,\&\, (X)\, F, \lambda)\|_M = 1$ and $\|\kappa \,\&\, \lambda \,\&\, (X)\, F \cong \kappa \,\&\, \lambda \,\&\, (X_0)\, F\|_M = 1$ for and $X_0 \subseteq X$ then $\|(\kappa, \lambda) \lessdot (\kappa \,\&\, (X_0)\, F, \lambda)\|_M = 1$ (and similarly for $\kappa$, $\lambda \,\&\, (X)\, F$). Indeed, if for an object $a$ the value of $\kappa \,\&\, \lambda \,\&\, (X)\, F$ is 1 then the value of $\kappa \,\&\, \lambda \,\&\, (X_0)\, F$ is also 1 and hence $\|(X_0)\, F\|_M\,(a) = \|(X)\, F\|_M\,(a) = 1$; hence the quadruple $\langle u, v, \bar{u}, \bar{v}\rangle$ of values of $\kappa$, $\lambda$, $\kappa \,\&\, (X)\, F$, $\lambda$ equals to the quadruple $\langle u, v, \hat{u}, \hat{v}\rangle$ of values of $\kappa$, $\lambda$, $\kappa \,\&\, (X_0)\, F$, $\lambda$ and so the latter one is in $U$. If the value of $\kappa \,\&\, \lambda \,\&\, (X)\, F$ is $\times$ then either $\|F\|_M\,(a) = \times$ and hence $\|(X)\, F\|_M\,(a) = \|(X_0)\, F\|_M\,(a) = \times$ or $\|(X)\, F\|_M\,(a) = 1$ and then $\|(X_0)\, F\|_M\,(a) = 1$ which follows from $\kappa \,\&\, \lambda \,\&\, (X)\, F \cong_M \kappa \,\&\, \lambda \,\&\, (X_0)\, F$. If $\|\kappa \,\&\, \lambda \,\&\, (X)\, F\|_M\,(a) = 0$ then $\|(X_0)\, F\|_M\,(a)$ can be different from $\|(X)\, F\|_M(a)$ and we still have $\kappa \,\&\, \lambda \,\&\, (X)\, F \cong M\, \kappa \,\&\, \lambda \,\&\, (X_0)\, F$. But in the present case we have ($\bar{u} = 0$ or $\bar{v} = 0$ and ($\hat{u} = 0$) or $\hat{v} = 0$), hence $\langle \bar{u}, \bar{v}\rangle \equiv_E \langle \hat{u}, \hat{v}\rangle$; furthermore, $\langle \bar{u}, \bar{v}\rangle \geqq_{CC} \langle \hat{u}, \hat{v}\rangle$, hence $\langle \bar{u}, \bar{v}\rangle\, U \langle \hat{u}, \hat{v}\rangle$, which together with $\langle u, v\rangle\, U \langle \bar{u}, \bar{v}\rangle$ yields $\langle u, v, \hat{u}, \hat{v}\rangle \in U$.

Suppose not (i) and let e.g. $\langle 1, 0\rangle \not\equiv_E \langle 0, 0\rangle$. Let $M$, $\kappa$, $\lambda$, $X_0 \subset X$ be such that for each $a$, $\|\kappa\|_M\,(a) = \|(X)\, F\|_M\,(a) = 1$, $\|\lambda\|_M\,(a) = \|(X_0)\, F\|_M\,(a) = 0$. Then $\|(\kappa, \lambda) \lessdot (\kappa \,\&\, (X)\, F, \lambda)\|_M = 1$, $\|(\kappa, \lambda) \lessdot (\kappa \,\&\, (X_0)\, F, \lambda)\|_M = 0$, $\kappa \,\&\, \lambda(X)\, F \cong_M \kappa \,\&\, \lambda \,\&\, (X_0)\, F$. Hence if $\langle \bar{\kappa}, \bar{\lambda}\rangle = Reg \lessdot_{M}(\kappa, \lambda)$ then $\bar{\kappa} \,\&\, \bar{\lambda}$ is $M$-compressible.

**14.18. Theorem.** If both 14.16 (i) and (14.17 (i) holds and if $\kappa \,\&\, \lambda$ is an EC then for $Reg \lessdot_{M}(\kappa, \lambda) = \langle \bar{\kappa}, \bar{\lambda}\rangle$ we have: $\bar{\kappa} \,\&\, \bar{\lambda}$ is an EC (in the following sense: whenever $(X)\, F$ occurs in $\bar{\kappa}$ and $(Y)\, F$ occurs in $\bar{\lambda}$ then $X = Y$; all coefficients are non-empty).

Proof. The fact that all coefficients are non-empty follows by 14.16 If $\|(\kappa, \lambda) \lessdot (\kappa \,\&\, (X)\, F, \lambda \,\&\, (Y)\, F)\|_M = 1$ observe that $(\kappa \,\&\, \lambda \,\&\, (X)F) \,\&\, (Y)\, F \cong_M (\kappa \,\&\, \lambda \,\&\, (X)\, F) \,\&\, (X \cap Y)\, F$ so that, by the proof of 14.17, we have $\|(\kappa, \lambda) \lessdot (\kappa \,\&\, (X)\, F, \lambda \,\&\, (X \cap Y)\, F\|_M = 1$. Similarly we obtain $\|(\kappa. \lambda) \lessdot (\kappa \,\&\, (X \cap Y)\, F, \lambda \,\&\, (X \cap Y)\, F)\|_M = 1$.

**14.19. Definition.** Let $U$ be an CC-closure set defining a closure operator $\lessdot$. Define

$$\|Ant(\varphi_1, \varphi_2, \psi\|_M = \|(\varphi_1, \varphi_2) \lessdot (\varphi_1 \,\&\, \psi, \varphi_2)\|_M\,,$$

$$\|Suc(\varphi_1, \varphi_2, \psi\|_M = \|(\varphi_1, \varphi_2) \lessdot (\varphi_1 \,\&\, \psi)\|_M \quad \text{for each } M\,.$$

(Hence *Ant* and *Suc* are operators of the type $1^3$ called the *antecedent operator* and the *succedent operator* corresponding to $\lessdot$).

**14.20. Theorem.** If $\leqslant$ is a CC-closure operator and if $\kappa$, $\lambda$ are psEC's then $Reg\leqslant_M(\kappa, \lambda) = \langle \bar{\kappa}, \bar{\lambda} \rangle$ where $\bar{\kappa}$ is the conjunction of all pseudoliterals $(X)\,F$ such that $(X)$ is the smallest coefficient of all pseudoliterals $(Z)$ such that $\|Ant(\kappa, \lambda, (Z)\,F\|_M = 1$ and $\bar{\lambda}$ is the conjunction of all pseudoliterals $(X)\,F$ such that $(X)$ is the smallest coefficient $(Z)$ such that $\|Suc(\kappa, \lambda, (Z)\,F\|_M = 1$.

Proof. Put $Reg\leqslant_M(\kappa, \lambda) = \langle \hat{\kappa}, \hat{\lambda} \rangle$ and let $\bar{\kappa}$, $\bar{\lambda}$ be defined as in the theorem. Evidently, $\|(\kappa, \lambda) \leqslant (\bar{\kappa}, \bar{\lambda})\|_M = 1$, hence $\bar{\kappa} \lessdot \hat{\kappa}$ and $\bar{\lambda} \lessdot \hat{\lambda}$. On the other hand, let $(X)\,F$ be a literal from $\hat{\kappa}$; then $\|Ant(\kappa, \lambda,(X)\,F\|_M = 1$ by the definition of ***Ant***. Suppose that there is an $X_0 \subset X$ such that $\|Ant(\kappa, \lambda, (X_0)\,F\|_M = 1$. Then obviously $\|(\kappa, \lambda) \leqslant (\hat{\kappa}((X_0)\,F/(X)\,F), \hat{\lambda})\|_M = 1$ ($\hat{\kappa}(\ldots)$ results from $\hat{\kappa}$ by replacing $(X)\,F$ by $(X_0)\,F$) and $\hat{\kappa}((X_0)\,F/(X)\,F) \gtrsim_M \hat{\kappa}$, which contradicts to $\langle \hat{\kappa}, \hat{\lambda} \rangle = Reg\leqslant_M(\kappa, \lambda)$. Hence $(X)$ is the smallest coefficient $(Z)$ such that $\|Ant(\kappa, \lambda, (Z)\,F\|_M = 1$ and $(X)\,F$ is in $\bar{\kappa}$. We have proved $\bar{\kappa} = \hat{\kappa}$ (the proof of $\bar{\lambda} = \hat{\lambda}$ is similar).

**14.21. Remark.** The last theorem shows that one can find $Reg\leqslant_M(\kappa, \lambda)$ quickly; one has to consider separate pseudoliterals and not all pairs $\langle \bar{\kappa}, \bar{\lambda} \rangle$ such that $\langle \kappa, \lambda \rangle \lessdot \langle \bar{\kappa}, \bar{\lambda} \rangle$.

**14.22. Theorem.** Let both 14.16 (i) and 14.17 (i) hold. The following are equivalent:

(i) $\langle u, v, \bar{u}, \bar{v} \rangle \in U$ implies $u \,\&\, v = \bar{u} \,\&\, \bar{v}$

$$\left(\text{hence } \frac{(\kappa, \lambda) \leqslant (\bar{\kappa}, \bar{\lambda})}{\kappa \,\&\, \lambda \cong \bar{\kappa} \,\&\, \bar{\lambda}} \text{ is sound}\right).$$

(ii) If $\kappa \,\&\, \lambda$ is ***M***-incompressible and $\langle \bar{\kappa}, \bar{\lambda} \rangle = Reg\leqslant_M(\kappa, \lambda)$ then $\kappa \,\&\, \lambda \subseteq \bar{\kappa} \,\&\, \bar{\lambda}$ (subconjunction).

Proof. Suppose (i) We have $\kappa \,\& \cong_M \bar{\kappa} \,\&\, \bar{\lambda}$ and $\kappa \,\&\, \lambda \subseteq \bar{\kappa} \,\&\, \bar{\lambda}$. Let $\kappa_0 \,\&\, \lambda_0$ be the subconjunction of $\bar{\kappa} \,\&\, \bar{\lambda}$ having the same functors as $\kappa \,\&\, \lambda$. Then obviously $\kappa \,\&\, \lambda \cong_M \kappa_0 \,\&\, \lambda_0$ and $\kappa_0 \,\&\, \lambda_0$ is poorer than $\kappa \,\&\, \lambda$, so $\kappa_0 \,\&\, \lambda_0$ equals to $\kappa \,\&\, \lambda$, since $\kappa \,\&\, \lambda$ is incompressible.

Suppose not (i). Consider e.g. the case $\langle \times, 1, 0, 1 \rangle \in U$ (other cases similarly). Consider a $\{0, 1, 2, \times\}$ – model of the type $1^3$ with two objects given by the following table. Put $\kappa = (1, 2)\,F \,\&\, (1)\,G$, $\bar{\kappa} = 1(1)\,F \,\&\, (1)\,G$, $\lambda = (1)\,H$.

| $F$ | $G$ | $H$ | $(X)\,F$ | $(X_0)\,F$ | $\kappa$ | $\bar{\kappa}$ | $\lambda$ |
|---|---|---|---|---|---|---|---|
| 2 | $\times$ | 1 | 1 | 0 | $\times$ | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Clearly, $\kappa$ is ***M***-incompressible, $\bar{\kappa}$ is strictly poorer than $\kappa$, and $\|(\kappa, \lambda) \leqslant (\bar{\kappa}, \bar{\lambda})\|_M = 1$.

In the present section we are going to study secured (a fortiori, qualitative) operators of the type $1^2$ that for $\{0, 1\}$-models mean: coincidence of the two properties predominates over difference. It should be clear that this meaning can be made precise in many ways, both statistical and non-statistical.

**15.1. Definition.** (1) If $M$ is a $\{0, 1\}$-model then $a_M, b_M, c_M, d_M$ denote the cardinality of the set of objects having the card $\langle 1, 1\rangle$, $\langle 1, 0\rangle$, $\langle 0, 1\rangle$, $\langle 0, 0\rangle$ respectively. (As usual, the *card* of an object $a \in M$ in $\boldsymbol{M} = \langle M, f_1, f_2\rangle$ is $\langle f_1(a), f_2(a)\rangle$. We put $q_M = \langle a_M, b_M, c_M, d_M\rangle$.

(2) In the sequel saying "*quadruple*" we mean a quadruple of natural numbers whose sum is positive (so that for such a quadruple $\langle a, b, c, d\rangle$ there is an $\boldsymbol{M}$ such that $a = a_M, \ldots, d = d_M$.

**15.2. Definition.** (1) A quadruple $q_1 = \langle a_1, b_1, c_1, d_1\rangle$ is a-*better* than $q_2 = \langle a_2, b_2, c_2, d_2\rangle$ if $a_1 + b_1 + c_1 + d_1 = a_2 + b_2 + c_2 + d_2$ and $a_1 \geqq a_2$, $b_1 \leqq b_2$, $c_1 \leqq c_2$, $d_1 \geqq d_2$.

(2) A secured operator $\sim$ of the type $1^2$ is *associational* if the following holds for any $\{0, 1\}$-models $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ of the type $1^2$: If $Asf_\sim(\boldsymbol{M}_1) = 1$ and if $q_{M_2}$ is a-better than $q_{M_1}$ then $Asf_\sim(\boldsymbol{M}_2) = 1$.

**15.3. Examples** of associational operators: (1) the "relatively more" operator of [3] 8.9: $Asf_\sim(\boldsymbol{M}) = 1$ iff $a_M d_M > b_M c_M$.

(2) The Fisher operator (see [5] p. 430–431): $Asf_{\sim\sigma}(\boldsymbol{M}) = 1$ iff $a_M d_M > b_M c_M$ and $\Delta(a_M, b_M, c_M\, d_M) \leqq \sigma$ where $\Delta$ is the Fisher statistic.

(3) The $\chi^2$-operator: $Asf\chi^2_\alpha(\boldsymbol{M}) = 1$ iff $a_M d_M > b_M c_M$ and $[(a_M d_M - b_M c_M)^2] : : [(a_M + b_M)(a_M + c_M)(b_M + d_M)(c_M + d_M)] \geqq \chi^2_\alpha$ (see e.g. [7]).

**15.4. Definition.** (1) A $\{0, 1\}$-model $\boldsymbol{M}_2$ is a-*better* than a $\{0, 1\}$-model $\boldsymbol{M}_1$ if $q_{M_2}$ is a-better than $q_{M_1}$.

(2) A $\{0, \times, 1\}$-model $\boldsymbol{M}_2$ is a-*better* than a $\{0, \times, 1\}$-model $\boldsymbol{M}_1$ if for each completion $\boldsymbol{N}_2$ of $\boldsymbol{M}_2$ there is a completion $\boldsymbol{N}_2$ of $\boldsymbol{M}_2$ there is a completion $\boldsymbol{N}_1$ of $\boldsymbol{M}_1$ such that $\boldsymbol{N}_2$ is a-better than $\boldsymbol{N}_1$.

(3) $\boldsymbol{M}_1$ is a-*equivalent* to $\boldsymbol{M}_2$ if $\boldsymbol{M}_1$ is a-better than $\boldsymbol{M}_2$ and vice versa.

**15.5. Remark.** Evidently, both definitions of "a-better" coincide for $\{0, 1\}$-models. The "a-better" relation is reflexive and transitive, hence a quasiordering; a-equivalence is the canonical equivalence given by the "a-better" quasiordering. (cf. [2] 3.6).

**15.6. Lemma** (1) Let $M_1, M_2$ be qualitative models (i.e. $\{0, \times, 1\}$-models) of the type $1^2$ and let $\sim$ be an associational operator. If $M_2$ is a-better than $M_1$ and if $Asf_\sim(M_1) = 1$ then $Asf_\sim(M_2) = 1$.

(2) If $M_2$ is not a-better than $M_1$ then one can define an associational operator $\sim$ such that $Asf_\sim(M_1) = 1$ but $Asf_\sim(M_2) \neq 1$.

Proof. (1) Note that the assertion is obvious if $M_1, M_2$ have complete information (= are $\{0, 1\}$-models). If $N_2$ is a completion of $M_2$ then there is a completion $N_1$ of $M_1$ such that $N_2$ is a-better than $N_1$. Since $Asf_\sim(M_1) = 1$ and $\sim$ is secured we have $Asf_\sim(N_1) = 1$; since $\sim$ is associational and $N_1$ is a $\{0, 1\}$-model we have $Asf(N_2) = 1$. Since $N_2$ was an arbitrary completion of $M_2$ we have $Asf_\sim(M_2) = 1$.

(2) First suppose that $M_1, M_2$ have complete information. Then put for each $\{0, 1\}$-model $M$ $Asf_\sim(M) = 1$ iff $q_M$ is a-better than $q_{M_1}$. By the transitivity of "a-better", this is an associational operator; $Asf_\sim(M_1) = 1$ and $Asf_\sim(M_2) = 0$. In the general case there exists a completion $N_2$ of $M_2$ such that for no completion $N_1$ of $M_1$, $N_2$ is a-better than $N_1$. Put for each $\{0, 1\}$-model $M$, $Asf_\sim(M) = 1$ iff there is a completion $N_1$ of $M_1$ such that $M$ is a-better than $N_1$. Then $Asf_\sim(M_1) = 1$ by securedness but $Asf_\sim(M_2) \neq 1$ since $Asf_\sim(N_2) = 0$.

**15.7. Definition.** If $M = \langle M, F_1, f_2 \rangle$ is a qualitative model, if $A \subseteq M$ and if $u = \langle u_1, u_2 \rangle \in \{0, \times, 1\}^2$ then $M(A : u)$ is the model $\langle M, g_1, g_2 \rangle$ where $g_i(a) = f_i(a)$ for $a \notin A$ and $g_i(a) = u_i$ for $a \in A$ (cards of elements of $A$ are changed to be $u$). In particular, if $a \in M$ then $M(a : u)$ means $M(\{a\} : u)$ and if $v \in \{0, \times, 1\}^2$ then $M(v : u)$ means $M(A : u)$ for $A = \{a \in M$; the card of $a$ is $v\}$.

**15.8. Remark.** (1) If $a_1 \neq a_2$ then $M(a_1 : u)(a_2 : v) = M(a_2 : v)(a_1 : u)$.

(2) If $A = \{a_1, \ldots, a_n\}$ then $M(A : u) = M(a_1 : u) \ldots (a_n : u)$.

**15.9. Definition.** Let $u, v \in \{0, \times, 1\}^2$. $v$ is *a-better* than $u$ ($v \geqq_{ab} u$) if for each qualitative model $M$ of the type $1^2$ and each $a \in M$ we have the following: If the card of a is $u$ then $M(a : v)$ is a-better than $M$.

**15.10. Remark** (1) Evidently $\geqq_{ab}$ is a quasiordering.

(2) An alternative definition reads as follows: $v \geqq_{ab} u$ iff for each qualitative $M$ of the type $1^2$ and each $a \in M$ we have: if the card of $a$ is $v$ the $M$ is a-better than $M(a : u)$.

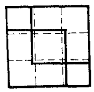
**15.11. Theorem.** The quasiordering $\geqq_{ab}$ is completely described by the following conditions:

(a) $\langle 1, 0 \rangle \equiv_{ab} \langle 1, \times \rangle \equiv_{ab} \langle \times, 0 \rangle$ ; $\langle 0, 1 \rangle \equiv_{ab} \langle \times, 1 \rangle \equiv_{ab} \langle 0, \times \rangle$ .

(b) $\langle \times, \times \rangle <_{ab} \langle 0, 1 \rangle$ , $\langle \times\, \times \rangle <_{ab} \langle 1, 0 \rangle$ , $\langle 1, 0 \rangle <_{ab} \langle 1, 1 \rangle$ ,
$\langle 1, 0 \rangle <_{ab} \langle 0, 0 \rangle$ , $\langle 0, 1 \rangle <_{ab} \langle 1, 1 \rangle$ , $\langle 0, 1 \rangle <_{ab} \langle 0, 0 \rangle$ .

(c) $\{\langle 1, 1 \rangle , \langle 0, 0 \rangle\}$ and $\{\langle 1, 0 \rangle , \langle 0, 1 \rangle\}$ are incomparable pairs.

We visualize $\geqq_{ab}$ in the following diagram where a dashed line means equivalence and a dashed line together with a thick line means that the transition from the side of the dashed line to the side of the thick line makes the pair (strictly) a-better (drives understand).



Proof. $K(a)$ denotes the card of $a$ in an arbitrary fixed model $M$ of the type $1^2$.

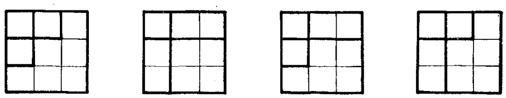
(a) Let $K(a) = \langle 1, \times \rangle$ and consider $M' = M(a : \langle 1, 0 \rangle)$. Each completion of $M'$ is a completion of $M$, hence $M'$ is a-better than $M$. Conversely, if $N$ is a completion of $M$ then the card of $a$ in $N$ is either $\langle 1, 0 \rangle$ or $\langle 0, 1 \rangle$. In the former case $N$ is a completion of $M'$; in the latter case $N' = N(a : \langle 1, 0 \rangle)$ is a completion of $M'$ and $N$ is a-better than $N'$. Remaining cases are treated similarly.

(c) Let $K(a) = \langle 1, 1 \rangle$ in a $\{0, 1\}$-model $M$, put $N = M\,(a : \langle 0, 0 \rangle)$. Then $q_N = \langle a_M - 1, b_M, c_M, d_M + 1 \rangle$ so that neither $q_M$ is a-better than $q_N$ nor $q_N$ is a-better than $q_M$. This shows that $\langle 1, 1 \rangle$, $\langle 0, 0 \rangle$ are incomparable.

(b) Evidently, $\langle 1, 1 \rangle \geqq_{ab} \langle 1, 0 \rangle$. To prove that $\langle 1, 0 \rangle \geqq_{ab} \langle 1, 1 \rangle$ does not hold, it suffices to take a $\{0, 1\}$-model $M$ of the type $1^2$ with at least one card $\langle 1, 0 \rangle = K(a)$ and observe that for $N = M(a : \langle 1, 1 \rangle$ $q_M$ is not a-better than $q_N$.

If $K(a) = \langle \times, \times \rangle$ then for any $u$ each completion of $M(a : u)$ is a completion of $M$, hence $M(a : u)$ is a-better that $M$. To prove that e.g. $\langle \times, \times \rangle \geqq_{ab} \langle 1, 1 \rangle$ does not hold suppose that $K(a) = \langle \times, \times \rangle$ in a model $M_0$ and suppose that all other cards of $M_0$ are in $\{0, 1\}^2$. Put $M = M_0(a : \langle 1, 1 \rangle)$ and $N = M(a : \langle 0, 0 \rangle)$. We showed in (c) that neither $M$ is a-better than $N$ not $N$ is a-better than $M$. This shows that $\langle \times, \times \rangle$ is not a-better than $\langle 1,1 \rangle$. All remaining cases are treated similarly.

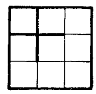
**15.12. Theorem.** There are four $\subseteq$-maximal CC-closure sets such that the corresponding closure operator $\lll$ has the following property: For each associational operator $\sim$, $\lll$ is an improvement operator for $\sim$, $CC$ and $\lessdot$. They are defined by the following tables (i)–(iv):



(i) (ii) (iii) (iv)

All of them satisfy 14.16 (i); but only the first one satisfies 14.17 (i). (Hence we prefer the first operator, cf. 14.18.)

Proof. Consider the table in 15.9 and recall that we are interested only in quadruples $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ where $\boldsymbol{u} \geqq_{CC} \boldsymbol{v}$. Since $U$ has to define an improvement operator, $\langle \boldsymbol{u}, \boldsymbol{v} \rangle \in U$ must imply $\boldsymbol{u} \geqq_{ab} \boldsymbol{v}$ (cf. 15.6 (2)). Consequently, the following quadruples must not be in $U$ : $\langle 1, 1, 1, \times \rangle$, $\langle 1, 1, \times, 1 \rangle$, $\langle 1, \times, \times, \times \rangle$, $\langle \times, 1, \times, \times \rangle$ (see table (v)).



(v)

The tables (i)–(iv) show four possible maximal equivalence relations in which none of the above quadruples is a pair of equivalent pairs. Cf. 14.13.

**15.13. Definition.** $\lll^a$ means the closure operator defined by the table (i) of 15.2.

**15.14. Theorem.** Let $\|Ant^a(\kappa, \lambda, \kappa')\|_M = \|(\kappa, \lambda) \lll^a (\kappa \,\&\, \kappa', \lambda)\|_M$ and analogously for $Suc^a$ (cf. 14.19). Then (1) $Ant^a$ is universally defined by the set

$$U_A^a = \{\langle u, v, \bar{u} \rangle \in \{0, \times, 1\}^3; [(\langle u, v \rangle = \langle 1\ 1 \rangle \vee \langle u, v \rangle = \langle 1, \times \rangle) \rightarrow \bar{u} = 1] \,\&\, \& [\langle u, v \rangle = \langle \times, 1 \rangle \rightarrow \bar{u} \in \{1, \times\}]\}.$$

(2) $Suc^a$ is universally defined by the set

$$U_S^a = \{\langle u, v, \bar{v} \rangle \in \{0, \times, 1\}^3; [(\langle u, v \rangle = \langle 1, 1 \rangle \vee \langle u, v \rangle = \langle \times, 1 \rangle \rightarrow \bar{v} = 1] \,\&\, \& [\langle u, v \rangle = \langle 1, \times \rangle \rightarrow \bar{v} \in \{1, \times\}]\}.$$

Proof. In the following table we list all possible tuples of values of $\kappa$, $\lambda$, $\kappa \,\&\, \kappa'$ such that the quadruple of values of $\kappa$, $\lambda$, $\kappa \,\&\, \kappa'$, $\lambda$ is in $U$; the fourth line contains the corresponding possible values of $\kappa'$. "arb" means arbitrary.

| | | | | | | |
|---|---|---|---|---|---|---|
| $\kappa$ | 1 | 1 | $\times$ | $\times$ | 0 | arb |
| $\lambda$ | 1 | $\times$ | 1 | $\times$ | arb | 0 |
| $\kappa \,\&\, \kappa'$ | 1 | 1 | $\times$ | $\leqq \times$ | 0 | $\leqq \|\kappa\|$ |
| $\kappa'$ | 1 | 1 | 1, $\times$ | arb | arb | arb |

**15.15. Theorem.** Let $\kappa$, $\lambda$ be psEC's and let $\boldsymbol{M}$ be a model. (1) (a) There is an $X$ such that $\|Ant^a(\kappa, \lambda, (X) F)\|_M = 1$ iff there is no object $a \in M$ such that $[(\langle \|\kappa\|_M(a), \|\lambda\|_M(a) \rangle$ is $\langle 1, 1 \rangle$ or $\langle 1, \times \rangle)$ and $\|F\|_M(a) = \times]$.
(b) Suppose the last condition holds; call a $u \in \{0, \times, 1\}^2$ critical if $u \in \{\langle 1, 1 \rangle, \langle 1, \times \rangle, \langle \times, 1 \rangle\}$. The least $X$ such that $\|Ant^e(\kappa, \lambda, (X) F\|_M = 1$ is $X = \{\|F\|_M(a); \langle \|\kappa\|_M(a), \|\lambda\|_M(a) \rangle$ critical and $\|F\|_M(a) \in V_{reg}\}$.

(2) (a) There is an $X$ such that $\|Suc^a(\kappa, \lambda, (X) F\|_M = 1$ iff there is no object $a \in M$ such that $[(\langle\|\kappa\|_M (a), \|\lambda\|_M (a)\rangle$ is $\langle 1, 1\rangle$ or $\langle \times, 1\rangle$ and $\|F\|_M (a) = \times]$. 
(b) If the last condition holds then the least $X$ such that $\|Suc^a(\kappa, \lambda), (X) F\|_M = 1$ is the $X$ defined in (1) (b) above.

Proof. Use the table for $Ant^a$ in 15.14. For each $a \in M$ put $u_a = \|\kappa\|_M (a)$, $v_a = \|\lambda\|_M (a)$, $f_a = \|F\|_M (a)$. If there is an object with $\langle u_a, v_a\rangle = \langle 1, 1\rangle$ or $\langle 1, \times\rangle$ and $f_a = \times$ then, for any $X$, $\langle u_a, v_a, \|(X) F\|_M (a)\rangle \notin U^a_A$ and hence $\|Ant^a(\kappa, \lambda, (X) F\|_M = 0$. If there is no such $a$ take the coefficient defined in (1) (b); we show $\|Ant^a(\kappa, \lambda, (X) F\|_M = 1$. Let $a \in M$. If $\langle u_a, v_a\rangle$ is not critical nothing is to prove. If $\langle u_a, v_a\rangle$ is $\langle 1, 1\rangle$ or $\langle 1, \times\rangle$ then $f_a \in V_{reg}$, hence $f_a \in X$ and $\|(X) F\|_M (a) = 1$ hence $\langle u_a, v_a, \|(X) F\|_M (a)\rangle \in U^a_A$. If $\langle u_a, v_a\rangle$ is $\langle \times, 1\rangle$ then either $f_a \in V_{reg}$ and $\|(X) F\|_M (a) = 1$ (as above) or $f_a = \times$, hence $\|(X) F\|_M (a) = \times$ and $\langle u_a, v_a, \|(X) F\|_M (a)\rangle \in U^a_A$. Hence $\|Ant^a(\kappa, \lambda, (X) F\|_M = 1$. It remains to show that $X$ is minimal. Let $j \in X - Y$ and let $a$ be such that $\langle u_a, v_a\rangle$ is critical and $f_a = j$. Then $\|(Y) F\|_M (a) = 0$ and $\langle u_a, v_a, 0\rangle \notin U^a_A$; hence $\|Ant^a(\kappa, \lambda, (Y) F\|_M = 0$.

**15.16. Remark.** The last theorem yields an effective algorithm for finding $Reg \ll^a_M(\kappa, \lambda)$ (cf. 14.20). *Caution*: it may happen that the only $X$ such that $\|Ant^a(\kappa, \lambda, (X) F)\|_M = 1$ is $X = V_{reg}$; then $(X) F$ is not a pseudoliteral and is not put into $\bar{\kappa}$ (where $Reg \ll^a_M(\kappa, \lambda) = \langle\bar{\kappa}, \bar{\lambda}\rangle$).

**15.17. Discussion.** We discuss the relation of the above results to GUHA-problems and their solutions.

(i) Recall the general theory of improvement operators (13.9–13.13). Here we have an arbitrary associational operator $\sim$, the set $CC$ of pairs of psEC's and the ordering $\lessdot$. Call $\sim$ a *strict* associational operator if, for each $M, \kappa, \lambda$ $\|\kappa \sim \lambda\|_M = 1$ implies $(\exists a \in M)(\|\kappa \,\&\, \lambda\|_M (a) = 1)$. (E.g. the Fisher's operator etc.) If $\sim$ is strict then whenever $\langle\kappa, \lambda\rangle \in CC$ and $\|\kappa \sim \lambda\|_M = 1$ then $Reg \ll^a_M(\kappa, \lambda)$ is a pair of EC's (not only psEC's); denoting it by $\langle\hat{\kappa}, \hat{\lambda}\rangle$ we have $\|\hat{\kappa} \sim \hat{\lambda}\|_M = 1$. Hence suppose $\sim$ to be strict. We have the problem $\boldsymbol{P} = \langle CC, \{1\}, IC\rangle$ where $IC$ is the following rule:

$$\frac{(\kappa \sim \lambda) \,\&\, (\kappa, \lambda) \ll (\hat{\kappa}, \hat{\lambda})}{\bar{\kappa} \sim \bar{\lambda}} (\langle\kappa, \lambda\rangle \lessdot \langle\bar{\kappa}, \bar{\lambda}\rangle \lessdot \langle\hat{\kappa}, \hat{\lambda}\rangle) .$$

Recall that $\kappa \sim \lambda$ is $M$-prime if $\langle\kappa, \lambda\rangle$ is the $\lessdot$-smallest pair $\langle\kappa', \lambda'\rangle$ such that $\|(\kappa' \sim \lambda') \,\&\, (K', \lambda') \ll (\kappa, \lambda)\|_M = 1$.

We know that $X_M = \{(\kappa \sim \lambda) \,\&\, (\kappa, \lambda) \ll (\hat{\kappa}, \hat{\lambda}); (\kappa \sim \lambda)$ is $M$-prime and $\langle\hat{\kappa}, \hat{\lambda}\rangle = Reg \ll^a_M(\kappa, \lambda)\}$ is a solution of $\boldsymbol{P}$ in $M$. $Reg \ll^a_M(\kappa, \lambda)$ is found using 15.15 and 14.20.

(ii) Note the following fact concerning the present solution: If we want $X_M$ to determine a solution of a hierarchical problem $\langle\boldsymbol{P}, \boldsymbol{H}\rangle$ then, by 13.13, we must request $R \subseteq R_H$ where $R$ is the ordering described in 13.13. This means that the formula $\kappa \sim \lambda$ should be tested earlier than any $\bar{\kappa} \sim \bar{\lambda}$ with $\langle \kappa, \lambda \rangle \lessdot \langle \bar{\kappa}, \bar{\lambda} \rangle$. In particular, if $\kappa$, $\lambda$ and $\bar{\kappa}$, $\bar{\lambda}$ contain the same functors but the coefficients in $\kappa$, $\lambda$ are supersets of the corresponding coefficients in $\kappa$, $\bar{\lambda}$ then $\kappa \sim \lambda$ is to be tested earlier. This is not in accordance with the natural syntactic "simpler-than" relation $\lhd$ between EC's (cf. 12.8 (4)). We can ignore this fact when $V_{reg}$ is small (e.g. has four elements); in particular, if $V_{reg} = \{0, 1\}$ then everything is all right and $\lessdot$ reduces to $\subseteq$. On the other hand, if $V_{reg}$ is big (e.g. has twenty elements) then one is forced to make some restrictions on the cardinality of coefficients. We are led to the following

**15.18. Definition and Theorem.** Let $1 \leqq k \leqq card(V_{reg})$ and put $\boldsymbol{F}_k = \{\kappa \sim \lambda;\ \langle \kappa, \lambda \rangle \in CC$ and all coefficients in $\kappa$ and $\lambda$ have cardinality at most $k\}$, $\boldsymbol{P}_k = \boldsymbol{F}_k, \{1\}, IC\rangle$. Define $\kappa \sim \lambda$ to be $\boldsymbol{M} - \boldsymbol{F}_k$*-prime* if $(\kappa \sim \lambda) \in \boldsymbol{F}_k$, $\|\kappa \sim \lambda\|_{\boldsymbol{M}} = 1$ and there is no $\langle \kappa_0 \lambda_0 \rangle \lessdot \langle \kappa, \lambda \rangle$, $\langle \kappa_0 \lambda_0 \rangle \neq \langle \kappa, \lambda \rangle$ such that $(\kappa_0 \sim \lambda_0) \in \boldsymbol{F}_k$, $\|\kappa_0 \sim \lambda_0\|_{\boldsymbol{M}} = 1$ and $\|(\kappa, \lambda_0) \lll (\kappa, \lambda)\|_{\boldsymbol{M}} = 1$. Let $Reg \lll^a_{\boldsymbol{M},k}(\kappa, \lambda) = \langle \bar{\kappa}, \bar{\lambda} \rangle$ result from $Reg \lll^a_{\boldsymbol{M}}(\kappa, \lambda) = \langle \hat{\kappa}, \hat{\lambda} \rangle$ by omitting in $\hat{\kappa}$ and $\hat{\lambda}$ all literals with coefficiens of cardinality bigger than $k$. Put $\boldsymbol{X}^k_{\boldsymbol{M}} = \{(\kappa \sim \lambda) \,\&\, (\kappa, \lambda) \lll \langle \bar{\kappa}, \bar{\lambda} \rangle;\ (\kappa \sim \lambda)\ \boldsymbol{M} - \boldsymbol{F}_k$-prime and $\langle \bar{\kappa}, \bar{\lambda} \rangle = Reg \lll^a_{\boldsymbol{M},k}(\kappa, \lambda)\}$. Then $\boldsymbol{X}^k_{\boldsymbol{M}}$ is a $\subseteq$-minimal solution of $\boldsymbol{P}_k$ in $\boldsymbol{M}$.

The proof is left to the reader as an exercise.

To close the present section, we discuss the possibility of applying 14.22, using a weaker improvement operator and restricting one's interest to formulas $\kappa \sim \lambda$ such that $\|\kappa \sim \lambda\|_{\boldsymbol{M}} = 1$ and $\kappa \,\&\, \lambda$ is $\boldsymbol{M}$-incompressible.

**15.19. Theorem and Definition.** There is a uniquely determined CC-closure set satisfying 14.16 (i), 14.17 (i) and 14.22 (i) and such that the corresponding operator is an improvement operator for each associational operator $\sim$, for $CC$ and $\lessdot$. It is given by the following table:

<table><tr><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td></tr></table>

This operator will be denoted by $\lll^b$.

(2) Introduce $Ant^b$ and $Suc^b$ with the usual meaning; then $Ant^b$ is universally defined by

$$U^b_a = \{\langle u, v, \bar{u} \rangle;\ [(u = 1 \,\&\, v \in \{1, \times\}) \to \bar{u} = 1] \,\&\, [(u = \times \,\&\, v \in \{1, \times\}) \to \to \bar{u} \in \{1, \times\}]\}\,.$$

$Suc^b$ is universally defined by

$$U^b_S = \{\langle u, v, \bar{v} \rangle;\ [(v = 1 \,\&\, u \in \{1, \times\}) \to \bar{v} = 1] \,\&\, [(v = \times \,\&\, u = \{1, \times\}) \to \to \bar{v} \in \{1, \times\}]\}\,.$$

(3) Theorem 15.15 holds for the present operator $\lessdot^b$ with the following change: critical pairs are $\langle 1, 1\rangle$, $\langle 1, \times\rangle$, $\langle \times, 1\rangle$, *and* $\langle \times, \times\rangle$.

**15.20. Discussion.** Let $\sim$ be a strict associational operator. Put

$$IC^* = \left\{\frac{(\kappa \sim \lambda) \,\&\, Inc(\kappa \,\&\, \lambda) \,\&\, (\kappa, \lambda) \lessdot^b (\hat{\kappa}, \hat{\lambda})}{(\bar{\kappa} \sim \bar{\lambda}) \,\&\, Inc(\bar{\kappa} \,\&\, \bar{\lambda})};\ \langle \kappa, \lambda\rangle \subseteq \langle \bar{\kappa}, \bar{\lambda}\rangle \subseteq \langle \hat{\kappa}, \hat{\lambda}\rangle\right\},$$

$$F^* = \{(\kappa \sim \lambda) \,\&\, Inc(\kappa \,\&\, \lambda); \langle \kappa, \lambda\rangle \in CC\}$$

($Inc(\kappa)$ is a formula such that $\|Inc(\kappa)\|_M = 1$ iff $\kappa$ is $M$-incompressible, $\boldsymbol{P}^* = \langle \boldsymbol{F}^*, \{1\}, IC^*\rangle$,

$$X_M^* = \{(\kappa \sim \lambda) \,\&\, Inc(\kappa \,\&\, \lambda) \,\&\, (\kappa, \lambda) \lessdot^b (\hat{\kappa}, \hat{\lambda}); (\kappa \sim \lambda)$$
$$M\text{-prime}, \kappa \,\&\, \lambda\ \boldsymbol{M}\text{-incompressible}, \langle \hat{\kappa}, \hat{\lambda}\rangle = Reg \lessdot_{\boldsymbol{M}}^b(\kappa, \lambda)\}.$$

Then, by 15.19 and 14.22, $IC^*$ is a sound rule (see also 12.12 (1) and 14.17) and $X_M^*$ is a solution of $\boldsymbol{P}^*$ in $\boldsymbol{M}$. Put $(\kappa \sim \lambda) \leqq_Q (\bar{\kappa} \sim \bar{\lambda})$ iff $\langle \kappa, \lambda\rangle \subseteq \langle \bar{\kappa}, \bar{\lambda}\rangle$; then $X_M^*$ determines a solution of a hierachical problem $\langle \boldsymbol{P}^*, \boldsymbol{H}\rangle$ whenever $Q \subseteq R_H$. (Hence one can go through $CC$ e.g. in the order $\lhd$ and obtain a hierarchical solution.) Recall that 15.19 (3) gives an effective algorithm for finding $Reg \lessdot_{\boldsymbol{M}}^b(\kappa, \lambda)$. *Caution*: The intuitive adequacy of the restriction to relevance declared by $\boldsymbol{F}^*$ (in particular, to $\boldsymbol{M}$-incompressible conjunctions) remains to be investigated.

## 16. IMPLICATION OPERATORS

We shall study some particular associational operators called implicational operators; they have some properties of the operator of logical implication (inclusion).

**16.1. Definition.** (1) A quadruple $q_2 = \langle a_2, b_2, c_2, d_2\rangle$ is i-*better* than $q_1 = \langle a_1, b_1, c_1, d_1\rangle$ if $a_1 + b_1 + c_1 + d_1 = a_2 + b_2 + c_2 + d_2$, $a_2 \geqq a_1$, $b_2 \leqq b_1$.
(2) $q_2 = \langle a_2, b_2, c_2, d_2\rangle$ is si-better than $q_1 = \langle a_1, b_1, c_1, d_1\rangle$ iff $a_1 + b_1 + c_1 + d_1 = a_2 + b_2 + c_2 + d_2$ and $a_2 b_1 \geqq a_1 b_2$.

**16.2. Definition.** (1) A secured operator $\sim$ of the type $1^2$ is *implicational* if the following holds for any $\{0, 1\}$-models $\boldsymbol{M}_1, \boldsymbol{M}_2$ of the type $1^2$: whenever $Asf_\sim(\boldsymbol{M}_1) = 1$ and $q_{\boldsymbol{M}_2}$ is i-better than $q_{\boldsymbol{M}_1}$ then $Asf_\sim(\boldsymbol{M}_2) = 1$.
(2) $\sim$ is *strictly implicational* if the following holds for any $\{0, 1\}$-models $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ of the type $1^2$: whenever $Asf_\sim(\boldsymbol{M}_1) = 1$ and $q_{\boldsymbol{M}_2}$ is si-better than $q_{\boldsymbol{M}_1}$ then $Asf_\sim(\boldsymbol{M}_2) = 1$ and $a_{\boldsymbol{M}_1} > 0$.

**16.3. Lemma.** (1) If $q_2$ is a-better than $q_1$ then $q_2$ is i-better than $q_1$.
(2) If $q_2$ is i-better than $q_1$ then $q_2$ is si-better than $q_1$.

(3) If $a_1 + b_1 \neq 0 \neq a_2 + b_2$ then $q_2$ is si-better than $q_1$ iff $a_2 : (a_2 + b_2) \geqq a_1 : (a_1 + b_1)$.

(4) Each implicational operator is associational.

(5) Each strictly implicational operator is implicational.

**16.4. Remark and Definition.** (1) If $\sim$ is implicational $M_1$, $M_2$ have the same cardinality and $a_{M_1} = a_{M_2}$, $b_{M_1} = b_{M_2}$ then $Asf_\sim(M_1) = Asf_\sim(M_2)$.

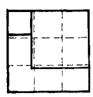
(2) Each of the operators defined in [3] 9.18, Examples (1)–(3) is strictly implicational. (Implication, good almost-implication, probable almost-implication.)

(3) One introduces the notion "$M_2$ is i-*better* (si-*better*) than $M_1$" for qualitative models in the same way as the corresponding a-notion in 15.4. Finally we define in an analogy to 15.9: $v$ is i-better than $u$ ($u, v \in \{0, \times, 1\}^2$; $v \geqq_{ib} u$) if for each qualitative model $M$ of the type $1^2$ we have the following: If the card of $a$ is $u$ then $M(a : v)$ is i-better than $M$ (for each $a \in M$). Similarly for $\geqq_{sib}$.

**16.5. Theorem.** The quasiordering $\geqq_{ib}$ is completely described by the following conditions:

(a) $\langle 1, 0\rangle \equiv_{ib} \langle 1, \times\rangle \equiv_{ib} \langle \times, 0\rangle$; $\langle 0, 0\rangle \equiv_{ib} \langle 0, \times\rangle \equiv_{ib} \langle 0, 1\rangle \equiv_{ib} \langle \times, 1\rangle$;

(b) $$\langle 1, 0\rangle <_{ib} \langle 0, 0\rangle <_{ib} \langle 1, 1\rangle$$



Proof. Note that $v \geqq_{ab} u$ implies $v \geqq_{ib} u$. Hence it suffices to show the following:

(1) $\langle \times, \times\rangle \geqq_{ib} \langle 1, \times\rangle$ (which yields $\langle \times, \times\rangle \geqq_{ib} \langle 1, 0\rangle, \langle 0, \times\rangle$ by transitivity),

(2) $\langle 0, \times\rangle \geqq_{ib} \langle 0, 0\rangle$,

(3) $\langle \times, \times\rangle \ngeqq_{ib} \langle 0, \times\rangle$ (yields $\langle \times, \times\rangle \ngeqq_{ib} \langle 0, 0\rangle, \langle 1, \times\rangle, \langle 1, 0\rangle$),

(4) $\langle 1, \times\rangle \ngeqq_{ib} \langle 1, 1\rangle$ and $\langle \times, 1\rangle \ngeqq_{ib} \langle 1, 1\rangle$.

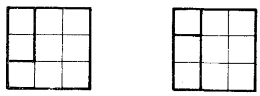
(1) Let $a \in M$, $K_M(a) = \langle 1, \times\rangle$; put $M_1 = M(a : \langle \times, \times\rangle)$ and let $N_1$ be a completion of $M_1$. If $K_{N_1}(a) = \langle 1, v\rangle$ then $N_1$ is a completion of $M$; if $K_{N_1}(a) = \langle 0, v\rangle$ then put $N = N_1(a : \langle 1, 0\rangle)$; $a_N = a_{N_1}$ and $b_N = b_{N_1} + 1$, hence $N_1$ is i-better than $N$ and hence $M_1$ is i-better than $M$.

(3) Let $K_M(a) = \langle 0, \times\rangle$, put $M_1 = M(a : \langle \times, \times\rangle)$ and suppose that for any $b \neq a$, $b \in M$, the card $K_M(b)$ is in $\{0, 1\}^2$. Put $N_1 = M_1(a : \langle 1, 0\rangle)$; we show that $N_1$ is not i-better than any completion $N$ of $M$. We have two possibilities: $N = M(a : \langle 0, 0\rangle)$ or $N = M(a : \langle 0, 1\rangle)$. In both cases we have: $a_{N_1} = a_N$, $b_{N_1} = b_N + 1$, hence $N$ is i-better than $N_1$ and $N_1$ is not i-better than $N$.

(2) Let $K_M(a) = \langle 0, 0\rangle$, put $M_1 = M(a : \langle 0, \times\rangle)$. Let $N_1$ be a completion of $M_1$. If $K_{N_1}(a) = \langle 0, 0\rangle$ then $N_1$ is a completion of $M$; if $K_{N_1}(a) = \langle 0, 1\rangle$ then put $N = N_1(a : \langle 0, 0\rangle)$; it is a completion of $M$ and $M$, $N$ are i-equivalent. 

(4) Let $K_M(a) = \langle 1, 1\rangle$, put $M_1 = M(a : \langle 1, \times\rangle)$, suppose $M$ to be a $\{0, 1\}$-model. Put $N_1 = M_1(a : \langle 1, 0\rangle)$; then $N_1$ is a completion of $M_1$ and $N_1$ is not i-better than $M$ ($M$ is its only own completion).

**16.6. Theorem and Definition.** There are two $\subseteq$-maximal CC-closure sets such that the corresponding closure operator $\lessdot$ has the following property: For each implicational operator $\to^*$ $\lessdot$ is an improvement operator for $\to^*$, $CC$ and $\lessdot$. They are defined by the following tables (i), (ii):



(i) (ii)

Both (i) and (ii) satisfy 14.16 (i); only the first one satisfies 14.17 (i). So we prefer (i). The operator defined by (i) will be denoted by $\lessdot^c$.

The proof is completely analogous to the proof of the corresponding theorem in Section 15 (Theorem 15.12) and so are the proofs of the following theorems 16.7—16.8. (*Erratum:* In Table (i) the line between $\langle 1, 1\rangle$ and $\langle \times, 1\rangle$ should be thick.)

**16.7. Theorem.** Let $\|Ant^c(\kappa, \lambda, \kappa')\|_M = \|(\kappa, \lambda) \lessdot^c(\kappa \,\&\, \kappa', \lambda)\|_M$ and analogously for $Suc^c$. Then (1) $Ant^c$ is universally defined by

$$U_A^c = \{u, v, \bar{u}\rangle \in \{0, \times, 1\}^3; [\langle u, v\rangle = \langle 1, 1\rangle \to \bar{v} = 1] \,\&\, [\langle u, v\rangle = \\ = \langle \times, 1\rangle \to \bar{v} \in \{1, \times\}]\}.$$

(2) $Suc^c$ is universally defined by

$$U_S^c = \{\langle u, v, \bar{v}\rangle \in \{0, \times, 1\}^3; [(\langle u, v\rangle = \langle 1, 1\rangle \vee \langle u, v\rangle = \langle \times, 1\rangle) \to \bar{v} = 1]\}.$$

**16.8. Theorem.** Let $\kappa$, $\lambda$ be psEC's and let $M$ be a model. (1) (a) There is an $X$ such that $\|Ant^c(\kappa, \lambda, (X) F\|_M = 1$ iff there is no object $a \in M$ such that $\langle \|\kappa\|_M(a), \|\lambda\|_M(a)\rangle$ is $\langle 1, 1\rangle$ and $\|F\|_M(a) = \times$.

(b) Suppose the last condition holds; call $\boldsymbol{u}$ critical if $\boldsymbol{u} \in \{\langle 1, 1\rangle, \langle 1, \times\rangle\}$. Then the least $X$ such that $\|Ant^c(\kappa, \lambda, (X) F\|_M = 1$ is $X = \{\|F\|_M(a); \langle \|\kappa\|_M(a), \|\lambda\|_M(a)\rangle$ critical and $\|F\|_M(a) \in V_{reg}\}$.

(2) (a) There is an $X$ such that $\|Suc^c(\kappa, \lambda, (X) F\|_M = 1$ iff there is no object $a \in M$ such that $\langle \|\kappa\|_M(a)\ \|\lambda\|_M(a)\rangle \in \{\langle 1, 1\rangle, \langle \times, 1\rangle\}$ and $\|F\|_M(a) = \times$.

(b) If the last condition holds then the least $X$ such that $\|Suc^c(\kappa, \lambda, (X) F\|_M = 1$ is the $X$ defined in (1) (b).

**16.9. Remark.** (1) Read 15.16 and 15.17 and make the obvious modifications for the present context.

(2) In the rest of the present section we are going to study strictly implicational operators $\to^*$ and formulas $\kappa \to^* \delta$ where $\kappa$ is a (ps)EC and $\delta$ is a (ps)ED. In this case we have a non-trivial deduction rule; this rule is combined with the rule for an improvement operator. In this way we obtain a generalization and improvement of [3] § 9(c).

**16.10. Definition.** (1) We introduce the *empty conjuction* $\wedge\, \emptyset$; this is an open formula and $\|\wedge\, \emptyset\|_M (a) = 1$ for each object of each model. *Generalized* EC's (GEC's) are EC's together with the empty conjuction. The same for generalized psEC's. *GCD* denotes the set of all pairs $\langle \kappa, \delta \rangle$ where $\kappa$ is a generalized psEC and $\delta$ is a psED. $\langle \kappa, \delta \rangle \in GCD$ is an elementary GCD-pair (consisting of a GEC and an ED) if $\kappa$ is a GEC, $\delta$ is an ED and have no functors in common. *EGCD* is the set of all elementary GCD-pairs.

(2) $\langle \kappa_1, \delta_1 \rangle (\lessdot \lhd) \langle \kappa_2, \delta_2 \rangle$ if $\kappa_1 \lessdot \kappa_2$ and $\delta_1 \lhd \delta_2$ (cf. 14.8)

(3) $\langle \kappa_1, \delta_1 \rangle$ results from $\langle \kappa_2, \delta_2 \rangle$ by *specification* if either $\langle \kappa_1, \delta_1 \rangle = \langle \kappa_2, \delta_2 \rangle$ or there is an ED $\delta_0$ such that $\delta_2$ is $\delta_1 \vee \delta_0$ and $\kappa_1$ is $\kappa_2 \,\&\, neg(\delta_0)$. $\langle \kappa_1, \delta_1 \rangle$ results from $\langle \kappa_2, \delta_2 \rangle$ by *reduction* if $\kappa_1 = \kappa_2$ and $\delta_1 \lhd \delta_2$. We put $\langle \kappa_1, \delta_1 \rangle\, SpRd \langle \kappa_2, \delta_2 \rangle$ if there is a $\langle \kappa_3, \delta_3 \rangle$ such that $\langle \kappa_1, \delta_1 \rangle$ results from $\langle \kappa_3, \delta_3 \rangle$ by specification and $\langle \kappa_3, \delta_3 \rangle$ results from $\langle \kappa_2, \delta_2 \rangle$ by reduction. (For $\langle \kappa_i, \delta_i \rangle \in EGCD$.)

**16.11. Theorem.** Let $\langle \kappa_1, \delta_1 \rangle, \langle \kappa_2, \delta_2 \rangle \in EGCD$. If $\langle \kappa_1, \delta_1 \rangle\, SpRd \langle \kappa_2, \delta_2 \rangle$, if $\to^*$ is a strictly implicational operator and if $\|\kappa_1 \to^* \delta_1\|_M = 1$ then $\|\kappa_2 \to^* \delta_2\|_M = 1$.

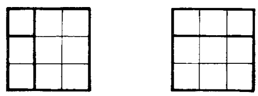
Proof. (1) First suppose $M$ to be a model with complete information. Put $M_1 = \langle M, \|\kappa_1\|_M, \|\delta_1\|_M \rangle$, $M_2 = \langle M, \|\kappa_2\|_M, \|\delta_2\|_M \rangle$. One verifies $a_{M_1} b_{M_2} \leqq a_{M_2} b_{M_1}$. This is evident if $\langle \kappa_1, \delta_1 \rangle$ results from $\langle \kappa_2, \delta_2 \rangle$ by reduction. If $\langle \kappa_1, \delta_1 \rangle$ results from $\langle \kappa_2, \delta_2 \rangle$ by specification denote by $m_{ijk}$ the number of objects $a$ such that $\|\kappa_2\|_M (a) = i$, $\|\delta_1\|_M (a) = j$, $\|\delta_0\|_M (a) = k$. Then $a_{M_1} = m_{110}$, $b_{M_1} = m_{100}$, $a_{M_2} = m_{110} + m_{101} + m_{111}$, $b_{M_2} = m_{100}$. Consequently, $a_{M_1} b_{M_2} \leqq a_{M_2} b_{M_1}$.

(2) Since $\to^*$ is secured, the validity of the theorem for arbitrary models follows from its validity for models with complete information. Indeed, if $N_2$ is a completion of $\langle M, \|\kappa_2\|_M, \|\delta_2\|_M \rangle$ then there is a completion $N$ of $M$ such that $N_2 = \langle M, \|\kappa_2\|_N, \|\delta_2\|_N \rangle$ (here we use the fact that $\langle \kappa, \delta \rangle$ is an elementary pair). Put $N_1 = \langle M, \|\kappa_1\|_N, \|\delta_1\|_N \rangle$. By the assumption ,$Asf{\to^*}(N_1) = 1$ and hence $Asf{\to^*}(N_2) = 1$ by (1).

**16.12. Definition.** A GCD-*closure set* is a closure set $U \subseteq \{0, \times, 1\}$ for GCD and $(\lessdot \lhd)$ satisfying the following condition: $\langle u, v, \bar{u}, \bar{v} \rangle \in U$ implies $u \geqq \bar{u}$ and $v \leqq \bar{v}$

for each $u$, $v$, $\bar{u}$, $\bar{v}$. A GCD-*closure operator* is an operator defined universally by a GCD-closure set. (Cf. 14.8.) 

**16.13. Theorem.** There are two $\subseteq$-maximal GCD-closure sets such that the corresponding closure operator $\preccurlyeq$ is an improvement operator for $\rightarrow^*$, GCD and $(\lessdot \lhd)$. They are defined by the following tables (i), (ii):



(i) (ii)

Proof. Consider the table in 16.5 and recall that we are interested in quadruples $\langle u_1, u_2, v_1, v_2\rangle$ such that $u_1 \geqq v_1$ and $u_2 \leqq v_2$, Since $U$ has to define an improvement operator, $\langle \boldsymbol{u}, \boldsymbol{v}\rangle \in U$ must imply $\boldsymbol{u} \geqq_{ib} \boldsymbol{v}$. Consequently, the quadruple $\langle 1, 1, \times, 1\rangle$ must not be in $U$. Theorem 14.13, after obvious modifications, yields the following condition necessary and sufficient for $U$ to be a GCD-closure set: there is an equivalence $E$ on $\{0, \times, 1\}^2$ such that writing $\langle u_1, u_2\rangle \gtreqless \langle v_1, v_2\rangle$ for ($u_1 \geqq v_1$ and $u_2 \leqq v_2$) and $\langle u_1, u_2\rangle (\& v) \langle v_1, v_2\rangle$ for $\langle u_1, \& v_1, u_2 \vee v_2\rangle$ we have:

(i) $\boldsymbol{u} \equiv_E \boldsymbol{w}$ and $\boldsymbol{u} \gtreqless \boldsymbol{v} \gtreqless \boldsymbol{w}$ implies $\boldsymbol{u} \equiv_E \boldsymbol{v} \equiv_E \boldsymbol{w}$;

(ii) $\boldsymbol{u} \equiv_E \boldsymbol{v}$ implies $\boldsymbol{u} \equiv_E [\boldsymbol{u}(\& v)\, \boldsymbol{v}] \equiv_E \boldsymbol{v}$;

(iii) $\boldsymbol{u}\, U\, \boldsymbol{v}$ iff ($\boldsymbol{u} \gtreqless \boldsymbol{v}$ and $\boldsymbol{u} \equiv_E \boldsymbol{v}$).

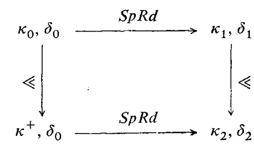
Consequently, the equivalence defined by the tables (i) and (ii) above determine the only two $\subseteq$-maximal GCD-closure sets not containing the quadruple $\langle 1, 1, \times, 1\rangle$.

**16.14. Remark and Definition.** We want now to combine 16.11 with 16.13, so we assume $\rightarrow^*$ to be strictly implicational. Denote for a moment by $\preccurlyeq_1$ the operator defined by 16.13 (i) and similarly for $\preccurlyeq_2$. Having a model $\boldsymbol{M}$ and a formula $\kappa \rightarrow^* \delta$ where $\langle \kappa, \delta\rangle \in EGCD$ (say, $\kappa \rightarrow^* \delta$ is an *elementary *-implication*), let $\|\kappa \rightarrow^* \delta\|_{\boldsymbol{M}} = 1$. Then we can use 16.11 to obtain a formula $\kappa_1 \rightarrow^* \delta_1$ such that $\langle \kappa, \delta\rangle\, SpRd \langle \kappa_1, \delta_1\rangle$, hence $\|\kappa_1 \rightarrow^* \delta_1\|_{\boldsymbol{M}} = 1$, then find a $\langle \kappa_2, \delta_2\rangle$ such that $\|(\kappa_1, \delta_1) \preccurlyeq_i (\kappa_2, \delta_2)\|_{\boldsymbol{M}} = 1$, hence $\|\kappa_2 \rightarrow^* \delta_2\|_{\boldsymbol{M}} = 1$, and perhaps iterate this procedure. The following theorem shows that it suffices to use first the improvement operator with a fixed succedent and then to use 16.11; furthermore, that in the present context $\preccurlyeq_1$ is better. So we are led to a relation of immediate consequence and to the definition of a pleasant GUHA-problem.

**16.15. Lemma.** (1) Let $i = 1, 2$, let $\langle\kappa_j, \delta_j\rangle \in EGCD$ $(j = 0, 1, 2)$ and suppose $\|\kappa_0 \to\to^* \delta_0\|_M = 1$, $\langle\kappa_0, \delta_0\rangle\, SpRd\langle\kappa_1, \delta_1\rangle$, $\langle\kappa_1, \delta_1\rangle\,(\lessdot\, \lhd)\,\langle\kappa_2, \delta_2\rangle$ and $\|(\kappa_1, \delta_1) \lll_i \lll_i (\kappa_2, \delta_2)\|_M = 1$. Then there is an EC $\kappa_3$ such that $\langle\kappa_0, \delta_2\rangle\,(\lessdot\, \lhd)\,\langle\kappa_3, \delta_0\rangle$, $\|(\kappa_0, \delta_0) \lll_i (\kappa_3, \delta_0)\|_M = 1$ and $\langle\kappa_3, \delta_0\rangle\, SpRd\langle\kappa_2, \delta_2\rangle$.

(2) If $\langle\kappa_0, \delta\rangle, \langle\kappa_1, \delta\rangle \in EGCD$ and if $\kappa_0 \lessdot \kappa_1$ then $\|(\kappa_0, \delta) \lll_2 (\kappa_1, \delta)\|_M = 1$ implies $\|(\kappa, \delta) \lll_1 (\kappa_1, \delta)\|_M = 1$.

Proof. (1) Consider the following diagram:




We look for a $\kappa^+$ such that the diagram commutes. We have the following relations: $\kappa_1 \subseteq \kappa_0$, $\delta_0 \lhd \delta_1$, $\kappa_1 \lessdot \kappa_2$, $\delta_1 \lhd \delta_2$. Hence $\delta_0 \lhd \delta_2$. The $\kappa^+$ we have to find must satisfy the following: $\kappa_2 \subseteq \kappa^+$, $\kappa_0 \lessdot \kappa^+$. Put $\kappa^+ = \kappa_0 \,\&\, \kappa_2$ (transformed into an EC: if $F$ occurs both in $\kappa_0$ and in $\kappa_2$, take the intersection of coefficients). $\kappa^+$ is indeed an EC: if $(X)\,F$ is in $\kappa_0$ then either $(X)\,F$ is in $\kappa_1$, $(Z)\,F$ is in $\kappa_2$ for some $Z \subseteq X$ and $Z \neq \emptyset$ since $\to^*$ is strictly implicational and hence $(\exists a \in M)\,(\|\kappa_0 \,\&\, \delta_0\|_M\,(a) = 1$ (since $\|\kappa_0 \to^* \delta_0\|_M = 1$). Note that $\langle 1, 1, \times, 1\rangle \notin U$ and cf. the proof of 14.16. Or $(X)\,F$ is not in $\kappa_1$, then $F$ occurs in $\delta_1$ and hence in $\delta_2$ and consequently $F$ does not occur in $\kappa_2$ (since $\langle\kappa_2, \delta_2\rangle \in EGCD$).

We prove $\|(\kappa_0, \delta_0) \lll_i (\kappa^+\delta_0)\|_M = 1$. Let $i = 1$. We have to verify: If $\|\kappa_0 \,\&\, \&\, \delta_0\|_M\,(a) = 1$ then $\|\kappa^+\|_M\,(a) = 1$. Suppose $\|\kappa_0 \,\&\, \delta_0\|_M\,(a) = 1$; then obviously $\|\kappa_1 \,\&\, \delta_1\|_M\,(a) = 1$ (since $\kappa_1 \subseteq \kappa_0$ and $\delta_0 \lhd \delta_1$). Then $\|\kappa_2 \,\&\, \delta_2\|_M\,(a) = 1$ (using $\lll_1$) and so $\|\kappa^+\|_M\,(a) = \|\kappa_0 \,\&\, \kappa_2\|_M\,(a) = 1$. Let $i = 2$. We have to verify: If $\|\kappa_0\|_M\,(a) = 1$ then $\|\kappa^+\|_M\,(a) = 1$. Let $\|\kappa_0\|_M\,(a) = 1$. Then $\|\kappa_1\|_M\,(a) = 1$ and hence $\|\kappa_2\|_M\,(a) = 1$ (using $\lll_2$). Hence $\|\kappa^+\|_M\,(a) = 1$. This completes the proof of (1).

(2) Suppose $\|(\kappa_0, \delta) \lll_2 (\kappa_1, \delta)\|_M = 1$ and $\kappa_0 \lessdot \kappa_1$. Consequently, for each $a$, $\|\kappa_0\|_M\,(a) = 1$ implies $\|\kappa_1\|_M\,(a) = 1$. Now, $\|(\kappa_0, \delta) \lessdot_1 (\kappa_1, \delta)\|_M = 1$ means: for each $a$, if $\|\kappa_0\|_M\,(a) = 1$ and $\|\delta\|_M\,(a) = 1$ then $\|\kappa_1\|_M\,(a) = 1$. So the assertion follows.

**16.16. Remark and Definition.** In the sequel, $\lll^d$ denotes the operator defined by 16.13 (i) If $\langle\kappa, \delta\rangle \in EGCD$ then $Reg^0 \lll^d_M(\kappa, \delta)$ is the $\lessdot$-supremum of all EC's $\bar{\kappa}$ not containing any functor from $\delta$ and such that $\kappa \lessdot \bar{\kappa}$ and $\|(\kappa, \delta) \lll^d (\bar{\kappa}, \delta)\|_M = 1$. The following is proved in the usual way (cf. 15.14, 15.15 and 16.7, 16.8):

**16.17. Theorem.** (1) Let $\|Ant^d(\kappa, \delta, \kappa')\|_M = \|(\kappa, \delta) \lll^d (\kappa \,\&\, \kappa', \delta)\|_M$. Then $Ant^d$ is universally defined by the set

$$U_A^d = \{\langle u, v, \bar{u}\rangle \in \{0, \times, 1\}^3; [\langle u, v\rangle = \langle 1, 1\rangle \to \bar{v} = 1]\} .$$

(2) Let $\langle \kappa, \delta\rangle \in EGCD$ and let $M$ be a model.

(a) There is an $X$ such that $\|Ant^d(\kappa, \delta, (X)\, F\|_M = 1$ iff there is no $a \in M$ such that $\|\kappa \,\&\, \delta\|_M (a) = 1$ and $\|F\|_M (a) = \times$.

(b) Suppose that the last condition holds; then the least $X$ such that $\|Ant^d(\kappa, \delta, (X)\, F\|_M = 1$ is

$$X = \{\|F\|_M (a) ;\quad \|\kappa \,\&\, \delta\|_M (a) = 1\} .$$

(c) Put $\bar{\kappa} = Reg^0 \lll_M^d(\kappa, \delta)$ then $\bar{\kappa}$ is $M$-incompressible. Moreover, $\bar{\kappa}$ is strongly $M_\delta$-incompressible, where $M_\delta$ is the submodel of $M$ formed by all objects $a \in M$ such that $\|\delta\|_M (a) = 1$.

(d) If $(\exists a \in M)\,(\|\kappa \,\&\, \delta\|_M = 1)$ then $\langle \bar{\kappa}, \delta\rangle \in EGCD$.

Proof. (1) and (2) (a), (b) as usual.

(c) The assertion about $M_\delta$-incompressibility is proved as 14.17. (Show: if $\|(\kappa . \delta) \lll^d \lll^d(\kappa \,\&\, (X)\, F, \delta)\|_M = 1$, $\|\kappa \,\&\, (X)\ \ F \simeq \kappa \,\&\, (X_0)\, F\|_{M_\delta} = 1$ and $X_0 \subseteq X$ then $\|(\kappa, \delta) \lll^d (\kappa \,\&\, (X_0)\, F, \delta)\|_M = 1$.)

(d) Assume $(\exists a \in M)\,(\|\kappa \,\&\, \delta\|_M = 1$. It follows by (b) that $\bar{\kappa}$ is a GEC; by definition $\bar{\kappa}$ does not contain any functor from $\delta$ and so $\langle \kappa, \delta\rangle \in EGCD$.

**16.18. Discussion.** We apply the above results to the description of a GUHA-problem and its solution. Put $F = \{\kappa \to^* \delta; \langle \kappa, \delta\rangle \in EGCD\}$ (relevant questions are elementary *-implications). Call $\kappa \to^* \delta$ *M-prime* if (i) $\|\kappa \to^* \delta\|_M = 1$, (ii) there is no $\langle \kappa_0, \delta_0\rangle\ SpRd\langle \kappa, \delta\rangle$, such that $\langle \kappa_0, \delta_0\rangle \neq \langle \kappa, \delta\rangle$, $\|\kappa_0 \to^* \delta_0\|_M = 1$, (iii) there is no $\kappa_0 \lessdot \kappa$, $\kappa_0 \neq \kappa$ such that $\|\kappa_0 \to^* \delta\|_M = 1$ and $\|Ant^d(\kappa_0, \delta, \kappa)\|_M = 1$. Put

$$IC = \left\{\frac{\kappa \to^* \delta \,\&\, Ant^d(\kappa, \delta, \bar{\kappa})}{\kappa_1 \to^* \delta_1} ;\quad \begin{array}{l}\text{there is a } \kappa_0, \kappa \lessdot \kappa_0 \lessdot \bar{\kappa} \text{ such}\\ \text{that } \langle \kappa_0, \delta\rangle\ SpRd\ \langle \kappa_1, \delta_1\rangle\end{array}\right\} .$$

For each $M$, put $X_M = \{\kappa \to^* \delta \,\&\, Ant^d(\kappa, \delta, \bar{\kappa}); \kappa \to^* \delta$ $M$-prime and $\bar{\kappa} = Reg^0 \lll_M^d(\kappa, \delta)\}$.

**16.19. Theorem.** Under the denotations of 16.18, $X_M$ is a ($\subseteq$-minimal) solution of $P = \langle F, \{1\}, IC\rangle$ in $M$.

Proof. Let $\|\kappa \to^* \delta\|_M = 1$ and let $\kappa \to^* \delta$ be not $M$-prime. Let $\langle \kappa_1, \delta_1\rangle$ be a $SpRd$-minimal EGCD such that $\|\kappa_1 \to^* \delta_1\|_M = 1$ and $\langle \kappa_1, \delta_1\rangle\ SpRd\langle \kappa, \delta\rangle$.

Let $\langle\kappa_2, \delta_1\rangle$ be a $(\prec \lhd)$-minimal EGCD such that $\|\kappa_2, \delta_1) \ll^d (\kappa_1, \delta_1)\|_M = 1$ and $\|\kappa_2 \to^* \delta_1\|_M = 1$. Then obviously

$$\frac{\kappa_2 \to^* \delta_1 \,\&\, Ant^d(\kappa_2, \delta_1, Reg^0 \ll^d_M(\kappa_2, \delta_1))}{\kappa \to^* \delta} \in IC\,.$$

We show that $(\kappa_2 \to^* \delta_1)$ is prime. Evidently, there is no $\kappa_3$ such that $\langle\kappa_3, \delta_1\rangle\ (\prec \lhd)$ $\langle\kappa_2, \delta_1\rangle$, $\|\kappa_3 \to^* \delta_1\|_M = 1$ and $\|Ant^d(\kappa_3, \delta_1, \kappa_2)\|_M = 1$. Suppose that there is a $\langle\kappa_3, \delta_3\rangle\ SpRd\langle\kappa_2, \delta_2\rangle$ such that $\|\kappa_3 \to^* \delta_3\|_M = 1$. Then by 16.15 there is a $\kappa^+$ such that $\|Ant^d(\kappa_3, \delta_3, \kappa^+)\|_M = 1$ and $\langle\kappa^+, \delta_3\rangle\ SpRd\langle\kappa_1, \delta_1\rangle$, hence $\langle\kappa^+, \delta_3\rangle\ SpRd\langle\kappa, \delta\rangle$ and $\|\kappa^+ \to^* \delta_3\|_M = 1$. It follows $\langle\kappa^+, \delta_3\rangle = \langle\kappa_1, \delta_1\rangle$, and hence $\langle\kappa_2, \delta_3\rangle = = \langle\kappa_2, \delta_1\rangle$. Hence $\langle\kappa_2, \delta_1\rangle$ is $\boldsymbol{M}$-prime.

**16.20. Discussion.** We consider the situation of $V_{reg}$ being big. Put $\boldsymbol{F}_k = \{\kappa \to {}^*\delta;$ $\langle\kappa, \delta\rangle \in EGCD$ and all coefficients have at most $k$ elements$\}$. Suppose $2k < card(V_{reg})$; under this assumption we have the following: If $\langle\kappa_1, \delta_1\rangle \neq \langle\kappa_2, \delta_2\rangle$ and $\langle\kappa_1, \delta_1\rangle$ results from $\langle\kappa_2, \delta_2\rangle$ by specification then $\kappa_1 \to^* \delta_1$ and $\kappa_2 \to^* \delta_2$ cannot be simultaneously in $\boldsymbol{F}_k$. Hence we give up any specification and consider a weaker rule

$$IC_1 = \left\{\frac{(\kappa \to^* \delta) \,\&\, Ant^d(\kappa, \delta, \bar{\kappa})}{\kappa_1 \to \delta_1}\,;\ \kappa \prec \bar{\kappa},\ \delta \lhd \delta_1\right\}.$$

Define $\kappa \to^* \delta$ to be $\boldsymbol{M} - \boldsymbol{F}_k$*-prime* if

(i) $(\kappa \to^* \delta) \in \boldsymbol{F}_k$ and $\|\kappa \to^* \delta\|_M = 1$,
(ii) there is no $\kappa_1 \prec \kappa, \kappa_1 \neq \kappa$ such that $(\kappa_1 \to^* \delta) \in \boldsymbol{F}_k$ and $\|\kappa_1 \to^* \delta\|_M = 1$,
(iii) there is no $\delta_0 \lhd \delta$ such that $\|\kappa_1 \to^* \delta_0\|_M = 1$ (irreducibility).

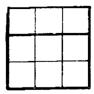
Let $Reg^0 \ll^d_{M,k}(\kappa, \delta) = \bar{\kappa}$ if $\bar{\kappa}$ is the conjunction of all $(X)\ F$ such that card $(X) \leqq k$ and $X$ minimal such that $\|Ant^d(\kappa, \delta, (X)\ F\|_M = 1$. Put

$$\boldsymbol{X}^k_M = \{(\kappa \to^* \delta) \,\&\, Ant^d(\kappa, \delta, \bar{\kappa});\ \kappa \to^* \delta,\ \boldsymbol{M} - \boldsymbol{F}_k\text{-prime and } \bar{\kappa} = Reg^0 \ll^d_{M,k}(\kappa, \delta)\}.$$

**16.21. Theorem.** Under the denotations of 16.20, $\boldsymbol{X}^k_M$ is a ($\subseteq$-minimal solution) of $\boldsymbol{P}_k = \langle\boldsymbol{F}_k, \{1\}, IC_1\rangle$ in $\boldsymbol{M}$.

Proof. Let $(\kappa \to^* \delta) \in \boldsymbol{F}_k$, $\|\kappa \to^* \delta\|_M = 1$. Let $\delta_0$ be a $\lhd$-minimal disjunction such that $\|\kappa \to^* \delta_0\|_M = 1$. Let $\kappa_0$ be a $\prec$-minimal element such that $\kappa_0 \prec \kappa$, $\|\kappa_0 \to^* \delta_0\|_M = 1$, $(\kappa_0 \to^* \delta_0) \in \boldsymbol{F}_k$ and $\|(\kappa_0, \delta_0) \ll^d (\kappa, \delta_0)\|_M = 1$. Then $(\kappa \to^* \delta)$ $IC_1(\kappa_0 \to^* \delta_0)$; we prove that for $\bar{\kappa} = \mathrm{Reg}^0 \ll^d_{M,k}(\kappa_0, \delta_0)$ we have: $\kappa_0 \prec \kappa \prec \bar{\kappa}$ and $(\kappa_0 \to^* \delta_0)$ is $\boldsymbol{M} - \boldsymbol{F}_k$-prime. The assertion concerning $\prec$ is obvious. $(\kappa_0 \to^* \delta_0) \in$ $\in \boldsymbol{F}_k$, $\|\kappa_0 \to^* \delta_0\|_M = 1$ by the definition of $\kappa_0$; evidently, there is no $\kappa_1 \prec \kappa_0$, $\kappa_1 \neq \kappa_0$ such that $\|\kappa_1 \to^* \delta_0\|_M = 1$, $(\kappa_1 \to^* \delta_0) \in \boldsymbol{F}_k$ and $\|(\kappa_1, \delta_0) \ll^d (\kappa_0, \delta_0)\|_M =$ $= 1$. If there is a $\delta_1$ such that $\delta_1 \lhd \delta_0$, $\delta_1 \neq \delta_0$ and $\|\kappa_0 \to^* \delta_1\|_M = 1$ then one shows as in 16.15 (1) such that $\langle\kappa, \delta_1\rangle$ results from $\langle\kappa, \delta_0\rangle$ by reduction and that $\|(\kappa_0, \delta_1) \ll$ $\ll^d (\kappa, \delta_1)\|_M = 1$. Hence $(\kappa_0 \to^* \delta_0)$ is $\boldsymbol{M} - \boldsymbol{F}_k$-prime. This completes the proof.

**16.22. Remark and Definition.** As an analogy to 15.19 (2) we consider the following diagram:



This is the operator (ii) from 16.13; it will be denoted by $<^e$.) Define $Reg^0 \ll^e_M(\kappa, \delta)$ as in 16.16 as the $\prec$-supremum of all $\bar{\kappa}$ such that $\kappa \prec \bar{\kappa}$ and $\|(\kappa, \delta) \ll^e (\bar{\kappa}, \delta)\|_M = 1$. We have the following

**16.23. Theorem.** (1) If $Ant^e$ has the usual meaning then $Ant^e$ is universally defined by

$$U^e_A = \{\langle u, v, \bar{u}\rangle;\ u = 1 \rightarrow \bar{\kappa} = 1\}\,.$$

(2) There is an $X$ such that $\|Ant^e(\kappa, \delta, (X)\,F\|_M = 1$ iff there is no $a \in M$ such that $\|\kappa\|_M\,(a) = 1$ and $\|F\|_M\,(a) = \times$. If the last condition holds the least $X$ such that $\|Ant^e(\kappa, \delta, (X)\,F\|_M = 1$ is $X = \{\|F\|_M\,(a);\ \|\kappa\|_M\,(a) = 1\}$.

(3) If $\langle\kappa, \delta\rangle \in EGCD$ and $\bar{\kappa} = Reg^0 <^e_M(\kappa, \delta)$ then $\bar{\kappa}$ is strongly $M$-incompressible and $\|\kappa \simeq \bar{\kappa}\|_M = 1$; if $(\exists a \in M)\,(\kappa\|_M\,(a) = 1$ then $\bar{\kappa}$ is an EC and $\langle\kappa, \delta\rangle \in EGCD$.

Proof. The proofs of (1) and (2) are routine and completely analogous to previous proofs of analogous theorems. Then it follows by (2) that if $\langle\kappa.\delta\rangle \in EGCD$ then $\bar{\kappa}$ is an GEC and hence $\langle\bar{\kappa}, \delta\rangle \in EGCD$. To prove the strong incompressibility of $\bar{\kappa}$ one observes the following: If $\|(\kappa, \delta) \ll^e (\kappa\,\&\,(X)\,F, \delta)\|_M = 1$ and if $X_0 \subseteq X$, $\|\kappa\,\&\,(X)\,F \simeq \kappa\,\&\,(X_0)\,F\|_M = 1$ then $\|(\kappa, \delta) \ll^e (\kappa\,\&\,(X_0)\,F, \delta)\|_M = 1$. The assertion concerning $\simeq$ is obvious.

**16.24. Corollary.** If $\kappa$ is strongly incompressible and if $\bar{\kappa} = Reg^0 \ll^e_M(\kappa, \delta)$ then $\kappa \subseteq \bar{\kappa}$.

Proof. We have $\kappa \prec \bar{\kappa}$ and $\|\kappa \simeq \bar{\kappa}\|_M = 1$, hence if $\kappa_0$ is the subconhunction of $\kappa$ having the same functors as $\kappa$ then $\kappa_0$ is poorer than $\kappa$ and $\|\kappa \simeq \kappa_0\|_M = 1$, hence $\kappa = \kappa_0$ by strong incompressibility.

**16.25. Definition and Theorem.** Let $S\,Inc(\kappa)$ be a formula such that $\|S\,Inc(\kappa)\|_M = 1$ iff $\kappa$ is strongly $M$-incompressible. Let $F^* = \{(\kappa \rightarrow^* \delta)\,\&\,S\,Inc(\kappa);\ \langle\kappa, \delta\rangle \in EGCD\}$ and

$$IC^* = \left\{\frac{(\kappa \rightarrow^* \delta)\,\&\,S\,Inc(\kappa)\,\&\,(\kappa, \delta) \ll^e(\hat{\kappa}, \delta)}{\kappa_1 \rightarrow^* \delta_1\,\&\,S\,Inc(\kappa_1)}\,;\quad \kappa \subseteq \kappa_1 \subseteq \kappa,\ \delta \lhd \delta_1\right\}.$$

For each $M$ let $X^*_M$ be the set of all formulas $(\kappa \rightarrow^* \delta)\,\&\,S\,Inc(\kappa)\,\&\,(\kappa, \delta) \ll^e (\hat{\kappa}, \delta)$ such that $\kappa \rightarrow^* \delta$ is $M$-prime (in the sense of 16.18 but with the present meaning of $\leqslant$ and *Ant*), $\kappa$ strongly $\boldsymbol{M}$-incompressible and $\bar{\kappa} = Reg^0 \leqslant^e_M(\kappa, \delta)$. Then $X^*_M$ is a ($\subseteq$-minimal) solution of $\boldsymbol{P}^* = \langle F^*, \{1\}, IC^* \rangle$ in $\boldsymbol{M}$.

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