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Closed-Loop Stability of Discrete Linear Single-Variable Systems

VLADIMÍR KUČERA

The paper provides rigorous foundations for the analysis and synthesis of discrete linear single-variable feedback systems in the frequency domain. Given transfer functions of the system to be compensated and of the compensator, the characteristic polynomial of the feedback system is computed. Further conditions to be satisfied by the transfer functions of the feedback system in order to guarantee stability are established. Then applications are discussed to the problems of assigning a desired characteristic polynomial by dynamical compensation, the existence of a stabilizing compensation, and the synthesis of optimal feedback control and filtering systems.

INTRODUCTION

During the sixties the mathematical system theory was established using the axiomatic concept of state. This approach, called the state-space method, can also be taken when analyzing or synthesizing control and filtering systems provided the state of the system to be compensated is available. This method then yields nice and useful results and the attained performance is the best one possible since the state contains all information about the system. Unfortunately, in most process control applications the state of the system is not available for measurement and must be recovered by means of an observer, Kalman filter, etc. The overall system may then be quite complex and certainly not optimal in the original sense for the state is reconstructed not immediately but in a finite time at best.

For the above reasons the engineers tend to use the frequency-domain approach to control and filtering problems, at least for linear constant systems. Such a system is usually characterized by its transfer function or impulse response. True, this description is inadequate from the standpoint of mathematical system theory inasmuch as it reflects just the input-output properties of the system. In other words, it fully describes only those systems that are both completely reachable and completely observable, i.e. minimal realizations of the transfer function. This description, however, is adequate for control and filtering purposes when we intend to feed back just the available output information. Naturally, one cannot, in general, expect as good performance when utilizing just the system output as can be obtained on the basis of state. But when the whole state is unknown, it is usually better to act on the available output by a dynamical compensator than construct an observer or another state-estimating system.

All this is responsible for the come-back of the frequency-domain methods in synthesizing feedback optimal control systems. To lend mathematical respectability to those methods, however, we have to start with the axiomatic state-space description of the system and then for the purposes of synthesizing a feedback system we prefer to work with abstract polynomials instead of transfer functions. This approach has been called the algebraic one and used to great advantage in [7], [8], [9], [10]. It makes it possible to treat general (not necessarily stable) discrete linear constant systems over an arbitrary field (i.e. including linear finite automata). Furthermore, it yields effective and simple computational algorithms well adapted for machine processing.

The purpose of this paper is to provide rigorous foundations for the synthesis of discrete linear feedback systems in the frequency domain. The crucial step in all synthesis procedures is to make the closed-loop system stable. When using a dynamical compensator the optimum system synthesis usually calls for the "zero-pole" cancellations. As a result, the overall system is not a minimal realization even if the original components are and as such it cannot be adequately characterized by its transfer function. Specifically, the transfer function contains insufficient information about the most important properties of the synthesized system – about the characteristic polynomial and its stability. Therefore, some elaborations to be described further are necessary in order that the frequency-domain approach can be reliably and effectively used in feedback system synthesis.

PRELIMINARIES

Referring for details to [7] we first recall several algebraic concepts.

Consider a set $\mathfrak{G}$ in which two laws of composition are given, the first written additively and the second multiplicatively. If the composition laws are associative and commutative, if multiplication distributes over addition, and if zero and identity elements as well as additive inverse exist in $\mathfrak{G}$, we call $\mathfrak{G}$ a (commutative) *ring*.

If an element $e \in \mathfrak{G}$ has a multiplicative inverse, we call e a *unit* of $\mathfrak{G}$. If every nonzero element of $\mathfrak{G}$ has a multiplicative inverse, then $\mathfrak{G}$ is called a *field*.

For example, the set $\mathfrak{Z}$ of integers constitutes a ring, while the rationals $\mathfrak{Q}$, reals $\mathfrak{R}$, and complex numbers $\mathfrak{C}$ all form fields. The set $\mathfrak{Z}_p$ of residue classes of integers modulo a prime p is an example of a finite field.

If $a, b \in \mathfrak{G}$, $b \neq 0$, we say that b *divides* a, and write $b \mid a$, if there exists a $c \in \mathfrak{G}$ such that $a = bc$. For $a, b \in \mathfrak{G}$, a *greatest common divisor* of a and b is an element $d \in \mathfrak{G}$, denoted by (a, b), which is defined as follows.

$$(1) \qquad d \mid a\,, \quad d \mid b\,,$$

$$(2) \qquad c \in \mathfrak{G}\,, \quad c \mid a, \quad c \mid b \quad \text{implies} \quad c \mid d\,.$$

The greatest common divisor is uniquely determined to within a unit of $\mathfrak{G}$. If $(a, b) = 1$ (modulo a unit of $\mathfrak{G}$) the elements a, b are said to be *relatively prime.*

Given a field $\mathfrak{F}$, let $\mathfrak{F}[z]$ denote the *ring of polynomials over* $\mathfrak{F}$ *in the indeterminate* z. If

$$a = \alpha_0 + \alpha_1 z + \ldots \alpha_n z^n \in \mathfrak{F}[z]$$

and $\alpha_n \neq 0$, then n is the *degree* of a denoted as ∂a. We define $\partial 0 = -\infty$. If $\alpha_n = 1$, the a is a *monic* polynomial. The units of $\mathfrak{F}[z]$ are polynomials of zero degree, which are viewed as isomorphic with $\mathfrak{F}$.

Let $\mathfrak{F}(z^{-1})$ denote the quotient field of $\mathfrak{F}[z]$, called the field of *rational functions over* $\mathfrak{F}$, whose elements can be written as

$$A = \alpha_n z^{-n} + \alpha_{n+1} z^{-(n+1)} + \ldots, \quad \alpha_k \in \mathfrak{F}\,,$$

for all integers n. If $\alpha_n \neq 0$ then n is the *order* of A.

The rational functions with nonnegative order form the ring of *realizable rational functions over* $\mathfrak{F}$ denoted by $\mathfrak{F}\{z^{-1}\}$. They have the form

$$A = \alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots, \quad \alpha_k \in \mathfrak{F}\,.$$

The units of $\mathfrak{F}\{z^{-1}\}$ are elements of order 0.

The elements $A \in \mathfrak{F}\{z^{-1}\}$ for which the sequence $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ converges to zero form the ring of *stable realizable rational functions over* $\mathfrak{F}$, denoted by $\mathfrak{F}^+\{z^{-1}\}$.

This motivates the following definition. A polynomial $a \in \mathfrak{F}[z]$ is *stable* if $1/a \in \mathfrak{F}^+\{z^{-1}\}$. Evidently, any unit of $\mathfrak{F}[z]$ is stable.

It is important to emphasize that we regard a polynomial or a rational function as an algebraic object, not as a function of a complex variable z. They are simply an alternate way of viewing finite or infinite sequences in $\mathfrak{F}$, the indeterminate z being a position-marker.

Now consider the equation

$$(3) \qquad ax + by = c$$

over $\mathfrak{F}[z]$ for unknowns x and y. This equation has been called a *linear Diophantine equation* in [7] and it plays a fundamental role in the analysis and synthesis of discrete linear constant systems.

It is well-known [7] that equation (3) has a solution if and only if $(a, b) \mid c$. If x_0 and y_0 is a particular solution of (3) then all solutions are given as

$$x = x_0 + \frac{b}{(a, b)} t , \qquad (4)$$

$$y = y_0 - \frac{a}{(a, b)} t ,$$

where t is an arbitrary polynomial of $\mathfrak{F}[z]$.

A very efficient method of finding x_0 and y_0 is presented in [7].

It is to be noted that when equation (3) is viewed over $\mathfrak{F}^+\{z^{-1}\}$ the formulas (4) read

$$x = x_0 + \frac{b}{(a, b)} T , \qquad (5)$$

$$y = y_0 - \frac{a}{(a, b)} T ,$$

where T is an arbitrary element of $\mathfrak{F}^+\{z^{-1}\}$.

SYSTEM DESCRIPTION

Following [6], [7] let

$\mathscr{T}$ = time set = $\mathfrak{Z}$ = (ordered) set of integers,
$\mathscr{U}$ = input values = $\mathfrak{F}$ = arbitrary field viewed as a vector space over itself,
$\mathscr{Y}$ = output values = $\mathfrak{F}$,
$\mathscr{X}$ = state space = $\mathfrak{F}^n$ = vector space of n-tuples over a field $\mathfrak{F}$.

Then a *finite dimensional, discrete, constant, linear, single-input, single-output system* $\mathscr{S}$ *over a field* $\mathfrak{F}$ is a quadruple $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ of linear maps

$$\mathbf{A} : \mathscr{X} \to \mathscr{X} ,$$

$$\mathbf{B} : \mathscr{U} \to \mathscr{X} ,$$

$$\mathbf{C} : \mathscr{X} \to \mathscr{Y} ,$$

$$\mathbf{D} : \mathscr{U} \to \mathscr{Y} ,$$

defining the equations

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k , \qquad (6)$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k ,$$

where $k \in \mathfrak{Z}$, $\mathbf{x} \in \mathscr{X}$, $\mathbf{u} \in \mathscr{U}$, $\mathbf{y} \in \mathscr{Y}$. The n is the dimension of the system.

We shall usually not make a distinction between **A**, **B**, **C** and **D** as linear maps or as matrices representing these maps with respect to a given basis.

The matrix

$$S = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\,,$$

characterizing the input-output behavior of $\mathcal{S}$, is called the *transfer function* of $\mathcal{S}$.

The system $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ is said to be completely reachable and completely observable, or equivalently, to be a *minimal realization* of S, if

$$\operatorname{rank}\,[\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}] = n\,,$$

$$\operatorname{rank}\begin{bmatrix}\mathbf{C}\\ \mathbf{CA}\\ \mathbf{CA}^{n-1}\end{bmatrix} = n\,.$$

The minimal realization has least dimension among all possible realizations of S see [6]. It is well-known [6] that there is a one-to-one correspondence between S, and its realization if and only if the realization is minimal. Otherwise speaking, nonminimal realizations contain certain parts which have no relaition to S.

The monic polynomial $\det(z\mathbf{I} - \mathbf{A}) \in \mathfrak{F}[z]$ is called the *characteristic polynomial* of $\mathcal{S}$ and its degree is the system dimension. Further, the $\mathcal{S}$ is defined to be *stable* if $\det(z\mathbf{I} - \mathbf{A})$ is stable.

The transfer function S of the system is a realizable rational function and, hence, it can be written as a ratio of two polynomials. However, there are two ways of doing so. First let

$$S = \frac{\hat{b}}{\hat{a}}\,, \tag{7}$$

where $\hat{a}$, $\hat{b}$ are polynomials of $\mathfrak{F}[z]$ that satisfy

$$(\hat{a}, \hat{b}) = 1\,, \qquad \partial\hat{b} \leqq \partial\hat{a} \tag{8}$$

and where $\hat{a}$ is the characteristic polynomial of the minimal realization of S.

Second we write

$$S = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D} = \mathbf{C}z^{-1}(\mathbf{I} - z^{-1}\mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

and introduce two polynomials a, $b \in \mathfrak{F}[z^{-1}]$ such that

$$S = \frac{b}{a}\,. \tag{9}$$

It follows that

$$a = z^{-n}\hat{a}\,, \qquad b = z^{-n}\hat{b}\,, \tag{10}$$

and (8) implies

$$(a, b) = 1 ,$$

$$(a, z^{-1}) = 1 .$$

Representation (7), in particular the $\hat{a}$, is useful when analyzing the system. For the purpose of system synthesis, however, we prefer to use representation (9). The reader will have noticed it in [7], [8], [9], where the symbol ζ stands for z^{-1}. The main advantage of this representation stems from the fact that any polynomial of $\mathfrak{F}[z^{-1}]$ can be realized as a system (6). This is not true of polynomials of $\mathfrak{F}[z]$. It means that the system can be synthesized in terms of a and b and the physical realizability of the resulting system is inherently guaranteed, while synthesis procedures based on $\hat{a}$ and $\hat{b}$ should manipulate only the ratio $\hat{b}/\hat{a}$ as a whole. Since polynomial manipulations are much simpler than manipulations with rational functions, it is the *polynomial algebra* that seems to be a natural mathematical tool for the discrete system analysis and synthesis.

The polynomial $\det(\mathbf{I} - z^{-1}\mathbf{A}) = z^{-n} \det(z\mathbf{I} - \mathbf{A})$ will be called the *pseudocharacteristic polynomial* of S. Hence a in (9) is the pseudocharacteristic polynomial of the minimal realization of S.

The pseudocharacteristic polynomial may be quite different from the characteristic polynomial. Example:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \hat{a} = \det(z\mathbf{I} - \mathbf{A}) = z^2 , \quad a = \det(\mathbf{I} - z^{-1}\mathbf{A}) = 1 .$$

Nevertheless, the pseudocharacteristic polynomial is as good for determining the system stability as the characteristic polynomial. In view of the definition of stability in $\mathfrak{F}[z]$ it is clear that $\det(z\mathbf{I} - \mathbf{A})$ is stable if and only if $\det(\mathbf{I} - z^{-1}\mathbf{A})$ is stable, see [8].

It should be emphasized that the expressions $\hat{b}/\hat{a}$ and b/a, even if called the transfer functions, will not be regarded as functions of complex variables but simply as rational functions over $\mathfrak{F}$ in the indeterminates z and z^{-1}, respectively.

THE CHARACTERISTIC POLYNOMIAL

As the name implies the characteristic polynomial is one of the most important characteristics of a system. It conveys information about the order of the system, its stability, and to some extent about the zero-input dynamical behavior.

It is to be noted that the characteristic polynomial cannot be inferred from the transfer function of the system unless the system is its minimal realization. Nonetheless we shall show that under certain natural assumptions a formula for the characteristic polynomial of the feedback system can be developed starting from the transfer function description of its components.

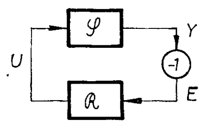

Fig. 1. The feedback system configuration.

For this purpose consider the feedback configuration shown in Fig. 1, where $\mathscr{S}$ denotes the system to be compensated and $\mathscr{R}$ is the compensator.

Let the $\mathscr{S}$ be defined by the equations

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k\,, \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k\,, \end{aligned} \tag{11}$$

where $\mathbf{x} \in \mathfrak{F}^n$ and let the $\mathscr{R}$ be defined by the equations

$$\begin{aligned} \mathbf{z}_{k+1} &= \mathbf{F}\mathbf{z}_k + \mathbf{G}\mathbf{e}_k\,, \\ \mathbf{u}_k &= \mathbf{H}\mathbf{z}_k + \mathbf{J}\mathbf{e}_k\,, \end{aligned} \tag{12}$$

where $\mathbf{z} \in \mathfrak{F}^m$.

Since the discrete closed loop must contain a delay of at least one time unit to be physically realizable, we shall agree on including this delay into the system to be compensated by assuming that $\mathbf{D} = \mathbf{0}$ in (11), or that $\partial\hat{b} < \partial\hat{a}$ in (7), or that $z^{-1} \mid b$ in (9).

A detailed representation of the feedback system is given in Fig. 2. The state equation of the system shown therein becomes

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix},$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} - \mathbf{BJC} & \mathbf{BH} \\ -\mathbf{GC} & \mathbf{F} \end{bmatrix}. \tag{13}$$

The characteristic polynomial of the system is defined as

$$\hat{c} = \det(z\mathbf{I} - \mathbf{K}) \in \mathfrak{F}[z]$$

and has the degree

$$\partial\hat{c} = m + n\,. \tag{14}$$

Assuming that both $\mathcal{S}$ and $\mathcal{R}$ are minimal realizations of the transfer functions

$$S = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} = \frac{\hat{b}}{\hat{a}},$$

(15)
$$R = \mathbf{H}(z\mathbf{I} - \mathbf{F})^{-1}\,\mathbf{G} + \mathbf{J} = \frac{\hat{s}}{\hat{r}},$$

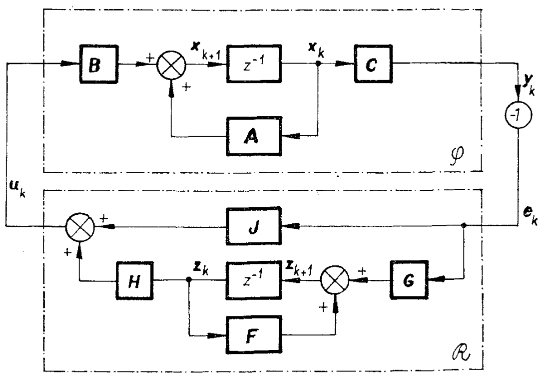

Fig. 2. A detail of the feedback system.

we have

$$\hat{a} = \det(z\mathbf{I} - \mathbf{A}),$$

$$\hat{r} = \det(z\mathbf{I} - \mathbf{F}).$$

Then we claim

Theorem 1. *Given the feedback system shown in Fig.* 1, *where both* $\mathcal{S}$ *and* $\mathcal{R}$ *are minimal realizations of* $S = \hat{b}/\hat{a} \in \mathfrak{F}\{z^{-1}\}$ *and* $R = \hat{s}/\hat{r} \in \mathfrak{F}\{z^{-1}\}$, *respectively. Then the characteristic polynomial* $\hat{c}$ *of the feedback system is given as*

$$\hat{c} = \hat{a}\hat{r} + \hat{b}\hat{s}\,.$$

Proof. We apply the well-known formula [3]

$$\det\begin{bmatrix}\mathbf{U} & \mathbf{V}\\ \mathbf{W} & \mathbf{Z}\end{bmatrix} = \det\mathbf{Z}\,.\,\det(\mathbf{U} - \mathbf{V}\mathbf{Z}^{-1}\mathbf{W})$$

 to compute

$$\begin{aligned}\hat{c} &= \det(z\mathbf{I} - \mathbf{K}) \\ &= \det(z\mathbf{I} - \mathbf{F}) \,.\, \det[z\mathbf{I} - \mathbf{A} + \mathbf{BJC} + \mathbf{BH}(z\mathbf{I} - \mathbf{F})^{-1}\,\mathbf{GC}] \\ &= \det(z\mathbf{I} - \mathbf{F}) \,.\, \det(z\mathbf{I} - \mathbf{A} + \mathbf{B}R\mathbf{C})\end{aligned}$$

on using (13) and (15).

Making use of the formula [5]

$$\det(z\mathbf{I} - \mathbf{A} + \mathbf{BC}) = \det(z\mathbf{I} - \mathbf{A}) \,.\, \det[\mathbf{I} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}]$$

we obtain

$$\hat{c} = \det(z\mathbf{I} - \mathbf{F}) \,.\, \det(z\mathbf{I} - \mathbf{A}) \,.\, \det(1 + SR)$$

and hence

$$\hat{c} = \hat{r}\hat{a} \det \frac{\hat{a}\hat{r} + \hat{b}\hat{s}}{\hat{a}\hat{r}} = \hat{a}\hat{r} + \hat{b}\hat{s}$$

by the assumption of minimal realizations. □

The pseudocharacteristic polynomial of the feedback system is defined as $c = \det(\mathbf{I} - z^{-1}\mathbf{K}) \in \mathfrak{F}[z^{-1}]$ and by reasoning analogous to that in Theorem 1 we obtain

$$c = ar + bs\,, \tag{16}$$

where $S = b/a$, $R = s/r$ and $a = \det(\mathbf{I} - z^{-1}\mathbf{A})$, $r = \det(\mathbf{I} - z^{-1}\mathbf{F})$.

In view of (10) it follows that

$$c = (z^{-n}\hat{a})(z^{-m}\hat{r}) + (z^{-n}\hat{b})(z^{-m}\hat{s}) = z^{-(m+n)}\,\hat{c}\,.$$

Expression (16) has been obtained in [9] using a different argument.

Example 1. Given the system $\mathscr{S}$ to be compensated as a minimal realization of

$$S = \frac{1}{z - 1} = \frac{z^{-1}}{1 - z^{-1}}$$

and the compensator $\mathscr{R}$ as a minimal realization of

$$R = \frac{0{\cdot}5z + 1}{z^2 + 1{\cdot}5z + 1} = \frac{0{\cdot}5z^{-1} + z^{-2}}{1 + 1{\cdot}5z^{-1} + z^{-2}}\,,$$

both over the field $\mathfrak{R}$. Then the characteristic polynomial of the closed-loop system is

$$\hat{c} = (z - 1)(z^2 + 1{\cdot}5z + 1) + (0{\cdot}5z + 1) = z^2(z + 0{\cdot}5)$$

and the pseudocharacteristic polynomial becomes

$$c = (1 - z^{-1})(1 + 1{\cdot}5z^{-1} + z^{-2}) + z^{-1}(0{\cdot}5z^{-1} + z^{-2}) = 1 + 0{\cdot}5z^{-1}\,.$$

Example 2. Consider the system $\mathscr{S} = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ over $\mathfrak{R}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\mathbf{C} = [1 \; 0]\,, \quad \mathbf{D} = [0]\,,$$

$$S = \frac{1}{z - 1}$$

and the compensator $\mathscr{R} = \{\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}\}$ over $\mathfrak{R}$, where

$$\mathbf{F} = [-1]\,, \quad \mathbf{G} = [1]\,,$$

$$\mathbf{H} = [1]\,, \quad \mathbf{J} = [0]\,,$$

$$R = \frac{1}{z + 1}\,.$$

It is to be noted that $\mathscr{S}$ is not a minimal realization of S.

Then by definition

$$\hat{c} = \det(z\mathbf{I} - \mathbf{K}) = \det \begin{bmatrix} z - 1 & 0 & -1 \\ 0 & z - 1 & -1 \\ 1 & 0 & z + 1 \end{bmatrix} = z^2(z - 1)$$

while

$$\hat{a}\hat{r} + \hat{b}\hat{s} = (z - 1)(z + 1) + 1 = z^2\,.$$

The two polynomials do not coincide due to the nonminimal realization of S and there is no way of computing the actual characteristic polynomial via the transfer function representations.

ASSIGNING A CHARACTERISTIC POLYNOMIAL

Having established an expression for the characteristic polynomial of the feedback system we are interested in solving the problem of assigning a desired characteristic polynomial. This problem is also referred to as that of pole assignment since, in fact, we are assigning desired eigenvalues to the $\mathbf{K}$ matrix.

The pole assignment by state-variable feedback has been solved in [2]. We recall that given a system (11) there exists a state feedback $\mathbf{u}_k = -\mathbf{L}\mathbf{x}_k$ such that $\det(z\mathbf{I} - \mathbf{A} + \mathbf{BL})$ is a preassigned monic polynomial of degree n belonging to $\mathfrak{F}[z]$ if and only if system (11) is completely reachable.

Using a constant output feedback $\mathbf{u}_k = -\mathbf{J}\mathbf{y}_k$ we cannot make $\det(z\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{J}\mathbf{C})$ equal to an arbitrary monic polynomial of degree n belonging to $\mathfrak{F}[z]$, even under the stronger assumption that system (11) be a minimal realization [1], [4].

Thus we are naturally led to use a dynamical output feedback realized by a compensator (12). This is the problem of *pole assignment by dynamical compensation* which is formally defined as follows:

(17) Given the configuration of Fig. 1, where $\mathscr{S}$ is a minimal realization of $S = \hat{b}/\hat{a} \in \mathfrak{F}\{z^{-1}\}$. Find a compensator $\mathscr{R}$ which is a minimal realization of some $R \in \mathfrak{F}\{z^{-1}\}$ such that the characteristic polynomial of the feedback system be equal to a desired monic polynomial $\hat{c} \in \mathfrak{F}[z]$.

It will be shown that this is impossible in general. A condition for a minimally realized compensator to exist will be derived together with an effective algorithm to solve for the compensator. An interesting observation is that the compensator, if it exists, need not be unique. The restriction that $\mathscr{R}$ be a minimal realization is essential for otherwise the actual characteristic polynomial would not be given by Theorem 1.

Theorem 2. *Problem* (17) *has a solution if and only if the linear Diophantine equation*

$$\hat{b}x + \hat{a}y = \hat{c} \tag{18}$$

has a solution x^0, y^0 *such that*

$$(x^0, y^0) = 1\,,$$
$$\partial x^0 \leqq \partial y^0 = \partial \hat{c} - \partial \hat{a}\,.$$

The compensator is then obtained as a minimal realization of

$$R = \frac{x^0}{y^0}$$

and it is not unique in general.

Proof. The proof is trivial in view of Theorem 1. It just remains to check that $\mathscr{R}$ is a system according to our definition. Indeed, the condition $\partial x^0 \leqq \partial y^0$ makes R physically realizable while $(x^0, y^0) = 1$ guarantees that $\mathscr{R}$ is a minimal realization of R. The condition $\partial y^0 = \partial \hat{c} - \partial \hat{a}$ is then equivalent to (14). □

It is to be noted that given a system $\mathscr{S}$ of dimension n, we cannot assign a characteristic polynomial $\hat{c}$ of a degree less than n due to (14). Such a $\hat{c}$ would leave no room for the compensator.

Example 3. Consider the system over $\mathfrak{R}$ which is a minimal realization of

$$S = \frac{1}{z - 1}$$

and find a compensator $\mathfrak{R}$ of minimal realization so that $\hat{c} = z$.

We are to solve the equation

$$x + (z - 1)\, y = z\,,$$

which gives us

$$x = 1 + (z - 1)\, t\,,$$

$$y = 1 - t\,,$$

for an arbitrary $t \in \mathfrak{R}[z]$.

Since $\partial\hat{a} = 1$, $\partial\hat{c} = 1$, we have to take a t such that $\partial y = 0$, i.e. $t = \tau_0 \neq 1$. Then

$$x = \tau_0 z + (1 - \tau_0)\,,$$

$$y = 1 - \tau_0\,,$$

Further $\partial x \leqq \partial y$ necessitates the choice $\tau_0 = 0$. Hence

$$x^0 = 1\,,$$

$$y^0 = 1\,,$$

and the minimal realization of $R = 1$ is the unique solution of our problem.

Example 4. Now consider a minimal realization of

$$S = \frac{1}{z}$$

over $\mathfrak{R}$ and find a compensator $\mathscr{R}$ of minimal realization which makes $\hat{c} = z^2 - 1$.

Equation (18) reads

$$x + zy = z^2 - 1$$

and the solution obtains as

$$x = -1 - zt\,,$$

$$y = z + t\,,$$

for any $t \in \mathfrak{R}[z]$.

Since $\partial\hat{a} = 1$, $\partial\hat{c} = 2$, we have to take a t such that $\partial y = 1$, i.e. $t = \tau_1 z + \tau_0$, $\tau_1 \neq 1$. Then

$$x = -\tau_1 z^2 - \tau_0 z - 1\,,$$

$$y = (1 + \tau_1)\, z + \tau_0\,,$$

and we have to confine ourselves to $\tau_1 = 0$ to get $\partial x \leqq \partial y$. Hence all compensators result as a minimal realization of

$$R = \frac{-\tau_0 z - 1}{z + \tau_0}$$

for any real $\tau_0 \neq \pm 1$. For $\tau_0 = \pm 1$ the $x^0 = \pm z - 1$, and $y^0 = z \pm 1$ would not be relatively prime and hence the minimal realization of R would yield a different characteristic polynomial $\hat{c} = z \pm 1$.

Example 5. Let us have the finite automaton over $\mathfrak{Z}_2$ described as a minimal realization of

$$S = \frac{1}{z^2}$$

and try to assign the polynomial $\hat{c} = z^3$.

We are to solve the equation

$$x + z^2 y = z^3 .$$

Remembering that all computations are to be carried out in the modulo 2 arithmetics, we obtain

$$x = z^3 + z^2 t ,$$

$$y = t$$

for an arbitrary $t \in \mathfrak{Z}_2[z]$.

Since $\partial\hat{a} = 2$, $\partial\hat{c} = 3$, we have to take a t such that $\partial y = 1$, i.e. $t = \tau_1 z + \tau_0$, $\tau_1 \neq 0$, Then

$$x = (1 + \tau_1) z^3 + \tau_0 z^2 ,$$

$$y = \tau_1 z + \tau_0$$

and it seen that the only choice to get $\partial x \leqq \partial y$ is $\tau_1 = 1$, $\tau_0 = 0$. Then, however, $x^0 = 0$, $y^0 = z$ and we have destroyed the primeness of x^0 and y^0 because $(0, z) = z$. We conclude that the problem has no solution in the class of minimum-realized compensators.

Indeed, $R = 0/z = 0$ would have the minimal realization $\mathscr{R} = \{\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\}$ and the associated characteristic polynomial would become $\hat{c} = z^2$.

On the other hand, there are nonminimal realizations of $R = 0$, e.g. $\mathscr{R} = \{\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}\}$ with

$$\mathbf{F} = [0] , \quad \mathbf{G} = [0] ,$$

$$\mathbf{H} = [1] , \quad \mathbf{J} = [0] ,$$

that do yield the desired $\hat{c} = z^3$. This solution cannot be found on the basis of transfer function description, however. The resulting feedback system is degenerated and it is shown in Fig. 3.

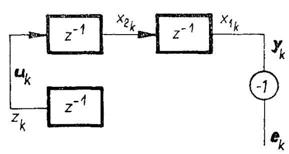

Fig. 3. The feedback system in Example 5.

The problem of assigning a given pseudocharacteristic polynomial has been solved in [9]. It has always a solution since the compensator is not restricted by relation (14).

CLOSED-LOOP STABILITY

Consider the closed-loop system shown in Fig. 1 and apply an external input signal W to the system, see Fig. 4. Then all possible transfer functions are listed below in (20), the most important of them being denoted as

(19) $$K = \frac{SR}{1 + SR}:$$

(20) $$Y = KW,$$
$$E = (1 - K)\,W,$$
$$U = R(1 - K)\,W.$$

As mentioned in the Introduction the feedback system need not be a minimal realization of the transfer function K, even if both $\mathcal{S}$ and $\mathcal{R}$ are minimal realizations of S and R respectively. Then the K does not fully describe the closed-loop system any more in that the system may contain certain parts which have no relation to K. Therefore, the K may conceal the closed-loop system instability.

To illustrate the difficulties arising in the feedback system stability analysis, consider the following example.

Example 6. Given the system $\mathcal{S}$ to be compensated by

$$S = \frac{0{\cdot}5}{z - 1}$$

and the compensator $\mathcal{R}$ by

$$R = \frac{z - 1}{z},$$

both over the field $\mathfrak{R}$.

Then

$$K = \frac{0{\cdot}5}{z + 0{\cdot}5},$$

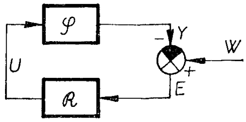

Fig. 4. The feedback system with an input W.

that is,

$$Y = \frac{0\cdot5}{z + 0\cdot5} W\,, \quad E = \frac{z}{z + 0\cdot5} W\,, \quad U = \frac{z - 1}{z + 0\cdot5} W$$

and one might get the impression that the overall system is stable. This is false, however. The characteristic polynomial of the system is given by Theorem 1 as

$$\hat{c} = (z - 1)\, z + 0\cdot5(z - 1) = (z - 1)\,(z + 0\cdot5)$$

and it is *not* stable.

What has happened? A minimal realization of S is

$$\mathbf{A} = [1]\,, \quad \mathbf{B} = [1]\,,$$
$$\mathbf{C} = [0\cdot5]\,, \quad \mathbf{D} = [0]\,,$$

and that of R becomes

$$\mathbf{F} = [0]\,, \quad \mathbf{G} = [1]\,,$$
$$\mathbf{H} = [-1]\,, \quad \mathbf{J} = [1]\,.$$

The state-space equation of the overall system, which has dimension 2, reads

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} + \mathbf{L}\mathbf{w}_k\,,$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} - \mathbf{BJC} & \mathbf{BH} \\ -\mathbf{GC} & \mathbf{F} \end{bmatrix} = \begin{bmatrix} 0\cdot5 & -1 \\ -0\cdot5 & 0 \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} \mathbf{BJ} \\ \mathbf{G} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since

$$\operatorname{rank}\,[\mathbf{L}, \mathbf{KL}] = \operatorname{rank} \begin{bmatrix} 1 & -0\cdot5 \\ 1 & -0\cdot5 \end{bmatrix} = 1 < 2\,,$$

the feedback system is not a minimal realization of K, $1 - K$, or $R(1 - K)$. Hence the transfer functions are insufficient to describe the system.

We have shown that stability of the transfer function K does not, in general, imply stability of the closed-loop system. Our next task is, therefore, to find additional conditions for K that would guarantee the closed-loop stability.

One result frequently used in the literature but not always rigorously proven states that the feedback system shown in Fig. 4 is stable if and only if the transfer function K can be written in the form $K = SM$, where M is an arbitrary element of $\mathfrak{F}^+\{z^{-1}\}$. This is true only for *stable* systems as the following counterexample indicates.

Example 7. Consider the system over $\mathfrak{R}$ described by

$$S = \frac{0{\cdot}5}{z - 1}$$

and choose

$$M = \frac{z - 1}{z},$$

which *is* stable. Then

$$K = \frac{0{\cdot}5}{z}, \quad R = \frac{z - 1}{z - 0{\cdot}5}$$

by virtue of (19) and the characteristic polynomial

$$\hat{c} = (z - 1)(z - 0{\cdot}5) + 0{\cdot}5(z - 1) = z(z - 1)$$

is *not* stable.

Otherwise speaking, the class of all transfer functions K that yield a stable closed-loop system is, in general, *less* than $K = SM$ and it is given in the following theorem.

Theorem 3. *Given the feedback system shown in Fig. 4, where $\mathscr{S}$ and $\mathscr{R}$ are minimal realizations of $S = b/a \in \mathfrak{F}\{z^{-1}\}$ and $R = s/r \in \mathfrak{F}\{z^{-1}\}$, respectively. Then the feedback system is stable if and only if the transfer function K has the form*

$$K = bM\,, \quad 1 - K = aN\,,$$

where M and N are elements of $\mathfrak{F}^+\{z^{-1}\}$ that satisfy the linear Diophantine equation

$$bM + aN = 1\,. \tag{21}$$

Proof. Stability of the feedback system is equivalent to stability of its pseudo-characteristic polynomial c.

Necessity: Let c be stable. We have

$$K = \frac{SR}{1 + SR} = \frac{bs}{ar + bs} = b\,\frac{s}{c}\,,$$

$$1 - K = \frac{1}{1 + SR} = \frac{ar}{ar + bs} = a\,\frac{r}{c}\,.$$

Denoting

$$M = \frac{s}{c}, \quad N = \frac{r}{c}$$

we obtain $K = bM, 1 - K = aN$. Since the c is stable both M and N are stable, and since $K + (1 - K) = 1$ equation (21) follows.

Sufficiency: Let $K = bM$, $1 - K = aN$, where

$$M = \frac{s}{c}, \quad N = \frac{r}{c}$$

are stable and suppose to the contrary of what is to be proved that c has an instable factor e, $c = c_0 e$. Then

$$M = \frac{s}{c_0 e}, \quad N = \frac{r}{c_0 e}$$

and due to the stability assumption the e must be cancelled in both M and N.

It follows that $e \mid r$, $e \mid s$ and since $(r, s) = 1$ by the assumption of minimal realizations, the e is a unit of $\mathfrak{F}[z^{-1}]$. Hence e is stable contradicting our hypothesis and, in turn, the c is stable. □

The above theorem specifies just all possible transfer functions K that yield a stable feedback system. Referring to Example 7 with

$$S = \frac{0{\cdot}5}{z - 1} = \frac{0{\cdot}5z^{-1}}{1 - z^{-1}},$$

all admissible K are obtained by solving the equation

$$0{\cdot}5z^{-1}M + (1 - z^{-1})N = 1 .$$

By virtue of (5) the general solution is

$$M = 2 + (1 - z^{-1})\,T,$$
$$N = 1 - 0{\cdot}5z^{-1}T$$

for an arbitrary $T \in \mathfrak{F}^+\{z^{-1}\}$ and, therefore, only the K of the form

$$K = z^{-1}[1 + 0{\cdot}5(1 - z^{-1})\,T]$$

yield a stable system. The $K = 0{\cdot}5z^{-1}$ in Example 7 does not evidently fall within the class.

An interesting interpretation of Theorem 3 is as follows.

Corollary 1. *With the notation used in Theorem* 3, *let*

$$M = \frac{m_2}{m_1}, \quad (m_1, m_2) = 1 ,$$

$$N = \frac{n_2}{n_1}, \quad (n_1, n_2) = 1 ,$$

$$K = \frac{k_2}{k_1}, \quad (k_1, k_2) = 1\,,$$

$$1 - K = \frac{l_2}{l_1}, \quad (l_1, l_2) = 1\,.$$

Then we have

$$c = m_1(a, s)\,, \tag{22}$$

$$= n_1(b, r)\,, \tag{23}$$

$$= k_1(a, s)\,(b, r)\,, \tag{24}$$

$$= l_1(a, s)\,(b, r)\,. \tag{25}$$

Proof. By Theorem 3, $M = s/c$ and $N = r/c$. Then $c = m_1(c, s) = n_1(c, r)$. Since $c = ar + bs$ and $(r, s) = 1$ we conclude that $(c, s) = (a, s)$ and $(c, r) = (b, r)$. Hence (22) and (23) follow.

Further $K = bs/c$ by (19). Then $c = k_1(c, bs)$ and we are to prove that $(c, bs) = (a, s)\,(b, r)$. To do so denote

$$a = a_1(a, s)\,, \quad b = b_1(b, r)\,,$$

$$r = r_1(b, r)\,, \quad s = s_1(a, s)\,.$$

Then

$$c = (a_1 r_1 + b_1 s_1)\,(a, s)\,(b, r)\,,$$

$$bs = b_1 s_1 (a, s)\,(b, r)$$

and because $(a_1, s_1) = 1, (b_1, r_1) = 1$ by definition and also $(a_1, b_1) = 1, (r_1, s_1) = 1$ we conclude that the polynomials $b_1 s_1$ and $a_1 r_1 + b_1 s_1$ are relatively prime. Hence $(c, bs) = (a, s)\,(b, r)$ and (24) follows.

Further, $l_2/l_1 = 1 - k_2/k_1 = (k_1 - k_2)/k_1$ and hence $l_1 = k_1$ up to a unit of $\mathfrak{F}[z^{-1}]$. Then (25) follows. □

Interpreting the (a, s) and (b, r) as the "zero-pole" cancellations in the cascade $\mathscr{S}\mathscr{R}$ and remembering that different polynomials c and k_1 indicate a nonminimal realization, we can say that the closed-loop system shown in Fig. 3 is a minimal realization of K if and only if there are no "zero-pole" cancellations in the cascade $\mathscr{S}\mathscr{R}$. In view of this interpretation Theorem 3 guarantees the closed-loop stability by prohibiting unstable "zero-pole" cancellations. See Example 6.

Before concluding this section we shall give further consequences of Theorem 3. Given a polynomial $m \in \mathfrak{F}[z^{-1}]$ we consider the factorization

$$m = m^+ m^-\,,$$

where m^+ is the stable factor of m having highest degree and belonging to $\mathfrak{F}[z^{-1}]$. This factorization is unique up to units of $\mathfrak{F}[z^{-1}]$, see [7].

Corollary 2. *Given the feedback system shown in Fig. 4 where $\mathscr{S}$ and $\mathscr{R}$ are minimal realizations of $S = b/a \in \mathfrak{F}\{z^{-1}\}$ and $R = s/r \in \mathfrak{F}\{z^{-1}\}$ respectively. Then the characteristic polynomial of the feedback system is stable if and only if the K has the form*

$$K = b^- M_1 \,, \quad 1 - K = a^- N_1 \,,$$

where M_1 and N_1 are elements of $\mathfrak{F}^+\{z^{-1}\}$ that satisfy the equation

$$b^- M_1 + a^- N_1 = 1 \,. \tag{26}$$

Proof. Set

$$M = \frac{M_1}{b^+} \,, \quad N = \frac{N_1}{a^+}$$

in (21). Since $1/a^+$ and $1/b^+$ are units of $\mathfrak{F}^+\{z^{-1}\}$, the M and N are equal to the M_1 and N_1 modulo units of $\mathfrak{F}^+\{z^{-1}\}$. Therefore, it makes no difference to solve equation (26) instead of (21), they are essentially the same. □

This corollary gives visually the least necessary predetermination of K and $1 - K$. It is interesting that the stability condition in the form (26) has been first obtained in [11] using completely different arguments.

In case $\mathscr{S}$ is a stable system the statement of Theorem 3 greatly simplifies.

Corollary 3. *Given the feedback system shown in Fig. 4, where $\mathscr{S}$ and $\mathscr{R}$ are minimal realizations of $S = b/a \in \mathfrak{F}\{z^{-1}\}$ and $R = s/r \in \mathfrak{F}\{z^{-1}\}$, respectively, and let a be stable. Then the feedback system is stable if and only if the transfer function K has the form*

$$K = bM \,,$$

where M is an arbitrary element of $\mathfrak{F}^+\{z^{-1}\}$.

Proof. The a being stable, $1/a$ is a unit of $\mathfrak{F}^+\{z^{-1}\}$. Therefore, we can set

$$N = \frac{N_2}{a}$$

in (26) to obtain the equivalent equation

$$bM + N_2 = 1 \,.$$

Here M can be taken arbitrarily within $\mathfrak{F}^+\{z^{-1}\}$ since then

$$N_2 = 1 - bM$$

always belong to $\mathfrak{F}^+\{z^{-1}\}$ and is uniquely determined by M. Otherwise speaking, the condition $K = bM$ is sufficient to guarantee the other condition $1 - K = aN$. □

It is further to be noted that if both a and b in the transfer function representation of $\mathcal{S}$ were stable, no predetermination of K would be necessary and the system could be "completely compensated". This is impossible, however, because $z^{-1} \mid b$ due to our agreement on incorporating the necessary delay into $\mathcal{S}$.

The above results are fundamental for the synthesis of feedback systems and their stabilization. The sections to follow are devoted to their applications.

THE EXISTENCE OF A STABILIZING COMPENSATION

We have seen that given a system $\mathcal{S}$ it is not possible to make the closed-loop characteristic polynomial equal to an arbitrary polynomial. The question now is whether or not the characteristic polynomial can be stable. The affirmative answer is plausible but the author is not aware of any direct proof.

Theorem 4. *Given the system $\mathcal{S}$ as a minimal realization of $S = b/a \in \mathfrak{F}\{z^{-1}\}$, there exists a compensator $\mathcal{R}$ which is a minimal realization of some $R \in \mathfrak{F}\{z^{-1}\}$ such that the feedback system shown in Fig.* 4 *is stable.*

Proof. By definition, $(a, b) = 1$. Therefore, elements M, N always exist in $\mathfrak{F}^+\{z^{-1}\}$ that satisfy equation (21). Then the transfer functions $K = bM$ and $1 - K = aN$ satisfy the hypothesis of Theorem 3 and hence the feedback system is stable.

The compensator yielding this K is given as a minimal realization of

$$R = \frac{1}{S}\frac{K}{1-K} = \frac{a}{b}\frac{bM}{aN} = \frac{M}{N}$$

by virtue of (19). □

Example 8. Consider again the system $\mathcal{S}$ over $\mathfrak{R}$ given by

$$S = \frac{0{\cdot}5z^{-1}}{1 - z^{-1}}$$

and find all stabilizing compensators having minimal realization.

The Diophantine equation

$$0{\cdot}5z^{-1}M + (1 - z^{-1})\,N = 1$$

has the general solution

$$M = 2 + (1 - z^{-1})\,T,$$
$$N = 1 - 0{\cdot}5z^{-1}T,$$

where $T \in \mathfrak{F}^+\{z^{-1}\}$ arbitrary. Hence a minimal realization of

$$R = \frac{2 + (1 - z^{-1})\,T}{1 - 0{\cdot}5z^{-1}T}$$

 is a stabilizing compensator for $\mathscr{S}$ regardless of T. Among all solutions, $R_0 = 2$ is the one yielding least dimension of $\mathscr{R}$. The associated characteristic polynomial then becomes $c_0 = z$.

SYNTHESIS OF OPTIMAL FEEDBACK SYSTEMS

The most important applications of Theorem 3 fall within the scope of optimum feedback system synthesis. Generally speaking, the closed-loop system must be stable and minimize an optimality criterion. It is, therefore, natural to first specify all possible transfer functions K that guarantee a stable closed-loop system via Theorem 3. The freedom in choosing M and N can then be exploited for optimization.

It will be shown in what follows that the optimal control strategy requires "zero-pole" cancellations between $\mathscr{S}$ and $\mathscr{R}$ in order to minimize an optimality criterion. Thus the optimal feedback system has intrinsically a nonminimal realization. We have seen earlier, however, that unstable "zero-pole" cancellations would destroy stability of the closed-loop system. Thus the whole synthesis procedure is a compromise between the two, i.e. only stable factors can be cancelled between $\mathscr{S}$ and $\mathscr{R}$.

As a simple and instructive example of the feedback system synthesis we shall consider the *time optimal control problem*:

(27) Given the configuration of Fig. 4, where $\mathscr{S}$, the system to be controlled, is a minimal realization of $S = b/a \in \mathfrak{F}\{z^{-1}\}$ and the external input sequence is described as $W = q/p \in \mathfrak{F}\{z^{-1}\}$. Find a controller $\mathscr{R}$ which is a minimal realization of some $R \in \mathfrak{F}\{z^{-1}\}$ such that the feedback system is stable, the control sequence U is stable and the error sequence E vanishes in a minimum time $k_{\min}$ and thereafter.

For convenience denote $(a, p) = d$ and

(28)
$$a = a_0 d\,,$$
$$p = d p_0\,.$$

Theorem 5. *Problem* (27) *has a solution if and only if* p_0 *is stable. The compensator* $\mathscr{R}$ *is unique and is given as a minimal realization of*

$$R = \frac{a_0^+ x^0}{p_0 b^+ y^0}\,,$$

where x^0, y^0 *is the solution of the linear Diophantine equation*

(29)
$$b^- x + a_0^- p y = q^+$$

such that $\partial y^0 =$ minimum.

Proof. Recalling (20), we have

$$E = (1 - K)\,W.$$

To guarantee a stable closed-loop system we have to set $1 - K = aN$ for some $N \in \mathfrak{F}^+\{z^{-1}\}$. It follows that

$$E = aN\frac{q}{p} = a_0 N \frac{q}{p_0}.$$

Since the error sequence is to vanish in a minimum time and thereafter, E must be a polynomial of minimum degree. Therefore

$$N = \frac{p_0 y}{a_0^+ q^+}, \tag{30}$$

where y is a polynomial of $\mathfrak{F}[z^{-1}]$ to be specified later. This choice yields the error

$$E = a_0^- q^- y. \tag{31}$$

Using the other stability condition of Theorem 3, $K = bM$ with $M \in \mathfrak{F}^+\{z^{-1}\}$, we can write

$$E = W - KW = \frac{q}{p} - bM\frac{q}{p}$$

or

$$pE = q - bMq.$$

The E is a polynomial of minimum degree whenever pE is so. It follows that bMq must be a polynomial of minimum degree. This is effected by the choice

$$M = \frac{x}{b^+ q^+} \tag{32}$$

where x is an unspecified polynomial of $\mathfrak{F}[z^{-1}]$ as yet.

To guarantee the closed-loop stability the M and N must obey the linear Diophantine equation (21). Substituting (30) and (32) into (21) and taking (28) into account we end up with equation (29) governing the x and y. Inasmuch as E is to be a polynomial of least degree, equation (29) should be solved for x^0, y^0 such that $\partial y^0 =$ minimum, see (31). This solution, if it exists, is unique and gives us the optimal controller as a minimal realization of

$$R = \frac{M}{N} = \frac{a_0^+ x^0}{p_0 b^+ y^0}$$

 and the optimal performance measure

$$k_{\min} = 1 + \partial a_0^- + \partial q^- + \partial y^0 .$$

In view of (20)

$$U = RE = \frac{a_0 q^- x^0}{p_0 b^+},$$

which is stable if and only if p_0 is stable. Then $(b^-, a_0^- p) = 1$ and, in turn, equation (28) has always a solution. □

It is important that $\mathscr{R}$ be a *minimal* realization of R. Otherwise the characteristic polynomial of the feedback system would not be given by Theorem 1 and the closed-loop stability might be destroyed.

The time optimal control problem has been solved in [9] in a completely different way. Succintly speaking, the explicit formula for the pseudocharacteristic polynomial c has been manipulated there so as to minimize $k_{\min}$ while keeping the c stable. In that way we have avoided the use of Theorem 3.

A similar procedure can be repeated for other problems of optimal control and filtering, namely the least squares control problem, minimum variance filtering problem, etc., see [10].

Example 9. Given the system $\mathscr{S}$ over $\mathfrak{R}$ by

$$S = \frac{z^{-1}(1 + 0{\cdot}5z^{-1})}{(1 - z^{-1})(1 - 0{\cdot}5z^{-1})},$$

find a controller $\mathscr{R}$ that makes the system output follow the reference signal

$$W = \frac{1}{1 - z^{-1}}$$

in a minimum time $k_{\min}$.

By Theorem 5 we are to solve the equation

$$z^{-1}x + (1 - z^{-1})\, y = 1 .$$

The general solution reads

$$x = 1 + (1 - z^{-1})\, t ,$$
$$y = 1 - z^{-1} t ,$$

where $t \in \mathfrak{R}[z^{-1}]$ arbitrary. The solution satisfying $\partial y^0 =$ minimum becomes

$$x^0 = 1 ,$$
$$y^0 = 1 .$$

Therefore

$$R = \frac{1 - 0{\cdot}5z^{-1}}{1 + 0{\cdot}5z^{-1}},$$

while

$$M = \frac{1}{1 + 0{\cdot}5z^{-1}}, \quad N = \frac{1}{1 - 0{\cdot}5z^{-1}}, \quad K = z^{-1}$$

and

$$\hat{c} = z(z + 0{\cdot}5)(z - 0{\cdot}5), \quad k_{\min} = 1.$$

Note the "zero-pole" cancellations in the cascade $\mathscr{S}\mathscr{R}$, which cause a nonminimal realization of the closed-loop system. Since the cancelled factors $1 + 0{\cdot}5z^{-1}$ and $1 - 0{\cdot}5z^{-1}$ are stable, the stability of $\hat{c}$ has not beed destroyed.

Example 10. Given the system $\mathscr{S}$ over $\mathfrak{Q}$ by

$$S = \frac{z^{-1}(1 - 2z^{-1} - z^{-2})}{1 - z^{-1}}$$

and solve the time optimal control problem for the reference signal

$$W = \frac{1}{1 - z^{-1}}.$$

Since $b^- = z^{-1}(1 - 2z^{-1} - z^{-2})$, we solve the equation

$$z^{-1}(1 - 2z^{-1} - z^{-2})\, x + (1 - z^{-1})\, y = 1$$

and obtain

$$x^0 = \tfrac{1}{2},$$
$$y^0 = 1 + \tfrac{3}{2}z^{-1} + \tfrac{1}{2}z^{-2}.$$

Hence

$$R = \frac{-\frac{1}{2}}{1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

and

$$E = 1 + \tfrac{3}{2}z^{-1} + \tfrac{1}{2}z^{-2}, \quad k_{\min} = 3.$$

If the system were defined over the field $\mathfrak{R}$, $b^- = z^{-1}(1 - (1 + \sqrt{2})\, z^{-1})$ and we would obtain

$$R = \frac{-\dfrac{1}{\sqrt{2}}}{(1 - (1 - \sqrt{2})\, z^{-1})\left(1 + \dfrac{1 + \sqrt{2}}{\sqrt{2}}\, z^{-1}\right)},$$

 and

$$E = 1 + \frac{1 + \sqrt{2}}{\sqrt{2}} z^{-1}, \quad k_{\min} = 2 .$$

The difference in optimal strategies over the fields $\mathfrak{Q}$ and $\mathfrak{R}$ is due to the fact that the polynomial $1 - 2z^{-1} - z^{-2}$ is irreducible in $\mathfrak{Q}[z^{-1}]$ while it has a stable factor in $\mathfrak{R}[z^{-1}]$, which may be compensated. This is to emphasize the importance of the underlying field.

An important feature of the paper is that it incorporates unstable systems into the framework of this theory. We shall use the following example to justify that it is no luxury to do so. In some situations we cannot afford to confine ourselves to stable systems.

Example 11. Consider the system $\mathscr{S}$ over the field $\mathfrak{R}$ described by

$$S = \frac{z^{-2}}{1 - 0{\cdot}75z^{-1}}$$

and solve the time optimal control problem for the reference input sequence

$$W = \frac{1 + 0{\cdot}75z^{-1}}{1 - 0{\cdot}75z^{-1}} .$$

The Diophantine equation

$$z^{-2}x + (1 - 0{\cdot}75z^{-1})\, y = 1 + 0{\cdot}75z^{-1}$$

has the solution

$$x^0 = 1{\cdot}125 ,$$

$$y^0 = 1 + 1{\cdot}5z^{-1} ,$$

and, in turn, the controller becomes

$$R = \frac{1{\cdot}125}{1 + 1{\cdot}5z^{-1}} .$$

The associated closed-loop transfer function is

$$K = \frac{1{\cdot}125z^{-2}}{1 + 0{\cdot}75z^{-1}}$$

and the characteristic polynomial

$$\hat{c} = z^2(z + 0{\cdot}75) .$$

It is to be noted that the optimal control strategy requires an *unstable* controller even though both the system $\mathscr{S}$ to be controlled and the reference input W are stable and minimum-phase.

CONCLUSIONS

The paper has given rigorous foundations for the analysis and synthesis of feedback systems in the frequency domain by using the algebraic approach. Potential difficulties in stability analysis of feedback systems have been discussed and a fundamental theorem for the synthesis of feedback systems has been proved. An explicit formula for the closed-loop characteristic polynomial has been developed and its application to the pole assignment problem discussed.

Most of the results established in the paper may have seemed intuitively obvious. This is due to the inherent simplicity of single-input single-output systems. A generalization of the results to multivariable systems, which will be considered in a future paper, is by no means a trivial matter and the results are much less transparent.

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Ing. Vladimír Kučera, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 180 76 Praha 8, Czechoslovakia