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A Convergence Theorem on the Iterative Solution of Nonlinear Two-Point Boundary-Value Systems

NGUYEN CANH

The nonlinear two-point boundary value problem occurs quite naturally in studies in many diverse science branches. For obtaining the approaching solution of the nonlinear problem we often replace the nonlinear problem with a sequence of linear problems in such a manner that the sequence of solutions to the linear problems approach in a limiting sense the solution of the nonlinear problem. The convergence theorem proved here establishes the applying of the modified Newton's method for solving the nonlinear two-point boundary-value problem.

INTRODUCTION

Consider the following nonlinear equation:

$$y = f(x) \tag{1}$$

the equation (1) may be rewritten as

$$F(x) = y - f(x) = 0\,. \tag{2}$$

For given y and an approximate solution $x = x_0$ we wish to find x such that this equation is satisfied.

Starting with x_0, we replace $F(x)$ by

$$F(x_0) + F'(x_0)(x - x_0)\,, \tag{3}$$

setting this relation to zero we solve the resulting linear equation for x_1 and so forth. Generally we have

$$F(x_n) + F'(x_n)(x_{n+1} - x_n) = 0\,, \quad n = 0, 1\ \ldots\,, \tag{4}$$

or

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}\,. \tag{5}$$

Each x_n is an approximate solution of Eq. (1) and under appropriate condition the sequence $\{x_n\}$ converges to a solution of Eq. (1).

The method setting the sequence $\{x_n\}$ as above is called the original Newton's method.

If the sequence $\{x_n\}$ converges to the solution x^* and x_0 is selected sufficiently near x^*, then, since the continuous of $F'(x_n)$ then $F'(x_0)$ and $F'(x_n)$ are different only a little . therefore we may replace $F'(x_n)$ with $F'(x_0)$.

The sequence (4) then becomes

$$F(x_n) + F'(x_0)(x_{n+1} - x_n) = 0 \tag{6}$$

or

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_0)}. \tag{7}$$

The method setting this sequence $\{x_n\}$ is called the modified Newton's method.

Note. If x is an n-dimensional vector ($x = (x^1, \ldots, x^n)$) then f and F are n-dimensional vectors and

$$F'(x) = [\partial F^i / \partial x^j] .$$

We now turn our attention to the study of nonlinear second order differential equation with nonhomogeneous boundary conditions:

$$F(v'', v', v, x) = 0 , \quad v(a) = v_a , \quad v(b) = v_b . \tag{8}$$

Let $v_0(x)$ is an approximate solution for the nonlinear equation. By analogy with the previous case we obtain:

$$\begin{aligned} F(v_n'' \; v_n', v_n, x) &+ F_v(v_n'', v_n', v_n, x)\,[v_{n+1}(x) - v_n(x)] + \\ &+ F_{v'}(v_n'', v_n', v_n, x)\,[v_{n+1}'(x) - v_n'(x)] + \\ &+ F_{v''}(v_n'', v_n', v_n, x)\,[v_{n+1}''(x) - v_n''(x)] = 0 , \\ &n = 0, 1, \ldots \end{aligned} \tag{9}$$

Suppose the original equation may be written as

$$F(v'', v', v, x) = v'' - f(v', v, x) = 0 . \tag{10}$$

Then we have $F_v = -f_v$, $F_{v'} = -f_{v'}$, and $F_{v''} = 1$, which yields

$$\begin{aligned} v_{n+1}''(x) &= f(v_n', v_n, x) + f_v(v_n', v_n, x)\,[v_{n+1}(x) - v_n(x)] + \\ &+ f_{v'}(v_n', v_n, x)\,[v_{n+1}'(x) - v_n'(x)] \\ v_n(a) &= v_a , \quad v_n(b) = v_b , \quad n = 0, 1, \ldots \end{aligned} \tag{11}$$

A convergence theorem on this iterative solution of above nonlinear two-point boundary-value systems was suggested by R. McGill and P. Kenneth [2].

By analogy with the modified Newton's method we obtain

$$F(v''_n, v'_n, v_n, x) + F_v(v''_0, v'_0, v_0, x)\,[v_{n+1}(x) - v_n(x)] + + F_{v'}(v''_0, v'_0, v_0, x)\,[v'_{n+1}(x) - v'_n(x)] + + F_{v''}(v''_0, v'_0, v_0, x)\,[v''_{n+1}(x) - v''_n(x)] = 0\,, \quad n = 0, 1, \ldots \tag{12}$$

For the equation

$$F(v'', v', v, x) = v'' - f(v', v, x) = 0\,, \tag{13}$$

we have

$$v''_{n+1}(x) = f(v'_n, v_n, x) + f_v(v'_0, v_0, x)\,[v_{n+1}(x) - v_n(x)] + + f_{v'}(v'_0, v_0, x)\,[v'_{n+1}(x) - v'_n(x)]\,, \quad v_n(a) = v_a\,, \quad v_n(b) = v_b\,, \quad n = 0, 1, \ldots \tag{14}$$

For simplicity and clarity of presentation, we shall first consider a single equation of the form

$$v''(x) = f(v, x)\,, \quad v(a) = v_a\,, \quad v(b) = v_b\,. \tag{15}$$

Now we may state the following theorem.

Theorem. *Given the nonlinear two-point boundary-value problem*

$$\frac{d^2v}{dx^2} = f(v, x)\,, \quad v(a) = v_a\,, \quad v(b) = v_b\,, \tag{16}$$

with 1) $f(v, x)$ *is continuous,* 2) $f_v(v, x) = [\partial f(v, x)]/\partial v$ *exists and is continuous.*

Let

$$f_v(v_0, x) = \left.\frac{\partial f(v, x)}{\partial v}\right|_{v=v_0}\,,$$

$$v_{ab}(x) = \frac{1}{b - a}\,[(v_b - v_a)\,x + bv_a - av_b]\,.$$

Define the following sequence of linear differential equations

$$\frac{d^2v_{n+1}}{dx^2} = f_v(v_0, x)\,[v_{n+1} - v_n] + f(v_n, x)$$

$$v_n(a) = v_a\,, \quad v_n(b) = v_b\,, \quad n = 0, 1, \ldots\,,$$

and $v_0(x)$ *is an arbitrary continuous function on* $[a, b]$ *such that*

$$\max_{x\in[a,b]} |v_0(x) - v_{ab}(x)| \leqq L .$$

Then for a sufficiently small interval $[a, b]$ *the nonlinear equation* (16) *has a unique solution and*

– *the sequence* $\{v_n(x)\}$ *converges to it;*

– *the convergence speed of the sequence* $\{v_n(x)\}$ *to the solution of equation* (16) *is given by the inequality*

$$\varrho(v_n, v^*) \leqq \frac{\alpha^n}{1-\alpha} \varrho(v_1, v_0) ;$$

– *a bound on the error is given by*

$$\max_{x\in[a,b]} |v_{n+1} - v^*| \leqq \frac{\alpha}{1-\alpha} \max_{x\in[a,b]} |v_{n+1} - v_n|$$

where α *is a positive number given below and* $v^*(x)$ *is the solution of equation* (16).

Proof. It follows from the hypotheses in the theorem that there exist M_1 and $M_2 > 0$ such that $|f(v, x)| \leqq M_1$, $|f_v(v, x)| \leqq M_2$. Let $m = \max\{M_1, M_2\}$.

Define the following complete metric space S:

$$S = \{v(x) \mid v(x) \text{ continuous on } [a, b],\ v(a) = v_a,\ v(b) = v_b,\ \varrho(v, v_{ab}) \leqq L\}$$

where

$$\varrho(v_1, v_2) = \max_{x\in[a,b]} |v_1(x) - v_2(x)| .$$

Define the operator P on S:

$$P(v(x)) = v_{ab}(x) - \int_a^b K(x, s)\{f_v(v_0, s)[P(v(s)) - v(s)] + f(v, s)\}\, ds$$

where $K(x, s)$ is the Green's function,

$$K(x, s) = \begin{cases} \dfrac{b-s}{b-a}(x-a) & \text{for } x \leqq s, \\[2ex] \dfrac{a-s}{b-a}(x-b) & \text{for } x \geqq s . \end{cases}$$

Firstly we shall show that, the Green's function

$$|K(x, s)| \leqq \tfrac{1}{4}(b-a) .$$

It means that:

a) For $x \leqq s$ implies

$$\left|\frac{b - s}{b - a}(x - a)\right| \leqq \tfrac{1}{4}(b - a).$$

In fact, we get

$$(x - a) = \delta(b - a), \quad 0 < \delta < 1,$$
$$b - s = \eta(b - a), \quad 0 < \eta \leqq 1 - \delta.$$

From that we have

$$\left|\frac{(b - s)(x - a)}{(b - a)^2}\right| = \delta\eta,$$
$$\delta\eta \leqq \delta(1 - \delta) = \delta - \delta^2;$$

when $\delta = \frac{1}{2}$ the product $\delta\eta$ achieves the maximum value and $\delta\eta \leqq \frac{1}{4}$, which is obvious. Finally we have

(17) $$\left|\frac{b - s}{b - a}(x - a)\right| \leqq \tfrac{1}{4}(b - a).$$

b) For $x \geqq s$, by the similar proof, implies that

(18) $$\left|\frac{a - s}{b - a}(x - b)\right| \leqq \tfrac{1}{4}(b - a).$$

Combining the both relations (17), (18) implies that

$$|K(x, s)| \leqq \tfrac{1}{4}(b - a).$$

The operator equation $Pv = v$ has a unique solution in S. P maps S into S, for arbitrary $v \in S$ we have

$$\varrho(Pv, v_{ab}) = \max|P\,v(x) - v_{ab}(x)| \leqq \frac{m}{4}(b - a)^2\left[\varrho(Pv, v) + 1\right] \leqq$$
$$\leqq \frac{m}{4}(b - a)^2\left[\varrho(Pv, v_{ab}) + \varrho(v, v_{ab}) + 1\right]$$

or

$$\varrho(Pv, v_{ab}) \leqq \frac{(m/4)(b - a)^2(L + 1)}{1 - (m/4)(b - a)^2} \leqq L,$$

for $(b - a)$ sufficiently small. This implies $P\,v(x) \in S$. For two arbitrary elements v_1,

 $v_2 \in S$ we have

$$Pv_1 - Pv_2 = \int_a^b K(x, s)\{f_v(v_0, s)[P\,v_2(s) - v_2(s)] -$$

$$- f_v(v_0, s)[P\,v_1(s) - v_1(s)] - [f(v_1, s) - f(v_2, s)]\}\,ds\,,$$

$$Pv_1 - Pv_2 = \int_a^b K(x, s)\{f_v(v_0, s)[P\,v_2(s) - P\,v_1(s) + v_1(s) -$$

$$- v_2(s)] - f(v_1, s) + f(v_2, s)\}\,ds\,.$$

$f(v_1, s) - f(v_2, s)$ is replaced by $f_v(\bar{v}, s)(v_1 - v_2)$ where $\bar{v}(s)$ is such that

$$\varrho(\bar{v}, v_2) \leqq \varrho(v_1, v_2)\,.$$

It follows that

$$\varrho(Pv_1, Pv_2) \leqq \frac{m}{4}(b - a)^2\,[\varrho(Pv_2, Pv_1) + 2\varrho(v_1, v_2)]$$

or

$$\varrho(Pv_1, Pv_2) \leqq \frac{(m/2)(b - a)^2}{1 - (m/4)(b - a)^2}\,\varrho(v_1, v_2)\,.$$

From which we see that when the condition

$$\frac{m}{2}(b - a)^2 \Big/ \left[1 - \frac{m}{4}(b - a)^2\right] = \alpha < 1$$

is satisfied, it means that $(b - a)$ is sufficiently small, then P is a contraction mapping.

From the theorem 1 Chapter 14 [3] that the operator equation $Pv = v$ has a unique solution v^* in S, v^* may be obtained as limit of the sequence $\{v_n\}$

$$v^*(x) = \lim_{n \to \infty} v_n(x)\,,$$

where $v_{n+1}(x) = P\,v_n(x)$ and v_0 is an arbitrary element in S. Part 1 of the theorem is proved.

Since

$$v_{n+1}(x) = P\,v_n(x)\,, \quad v_n(x) = P\,v_{n-1}(x)\,,$$

and

$$\varrho(Pv_n, Pv_{n-1}) \leqq \alpha\,\varrho(v_n, v_{n-1})$$

or

$$\varrho(v_{n+1}, v_n) \leqq \alpha\,\varrho(v_n, v_{n-1})\,.$$

By using continuously the similar inequalities, we have

$$\varrho(v_{n+p}, v_n) \leqq \varrho(v_{n+p}, v_{n+p-1}) + \ldots + \varrho(v_{n+1}, v_n) \leqq$$

$$\leqq (\alpha^{n+p-1} + \ldots + \alpha^n)\,\varrho(v_1, v_0)\,.$$

Finally we have

$$v^* = \lim_{p \to \infty} v_{p+n}$$

and

$$\varrho(v_n, v^*) \leqq \frac{\alpha^n}{1 - \alpha} \varrho(v_1, v_0) .$$

Part 2 of the theorem is proved.

We now consider the expression

$$|v_{n+1}(x) - v^*(x)| = \left| \int_a^b K(x, s) \{f_v(v_0, s) [v_{n+1}(s) - v_n(s)] + \right.$$
$$\left. + [f(v_n, s) - f(v^*, s)]\} \, \mathrm{d}s \right| .$$

By the mean value theorem, it follows that

$$|v_{n+1}(x) - v^*(x)| = \left| \int_a^b K(x, s) \{f_v(v_0, s) [v_{n+1}(s) - v_n(s) + \right.$$
$$\left. + f_v(\bar{v}, s) [v_n(s) - v^*(s)] \, \mathrm{d}s \right| ,$$

where $\bar{v}(s)$ is such that

$$\varrho(\bar{v}, v^*) \leqq \varrho(v_n, v^*) ,$$

therefore we have

$$|v_{n+1}(x) - v^*(x)| = \left| \int_a^b K(x, s) \{f_v(v_0, s) [v_{n+1}(s) - v^*(s) + v^*(s) - v_n(s)] + \right.$$
$$\left. + f_v(\bar{v}, s) [v_n(s) - v^*(s)]\} \, \mathrm{d}s \right|$$

or

$$\varrho(v_{n+1}, v^*) \leqq \frac{m}{4} (b - a)^2 [\varrho(v_{n+1}, v^*) + 2\varrho(v_n, v^*)]$$

and

$$\varrho(v_{n+1}, v^*) \leqq \alpha \, \varrho(v_n, v^*) . \tag{19}$$

We now observe that

$$\varrho(v_{n+1}, v_n) \leqq \alpha \, \varrho(v_n, v_{n-1})$$

and

$$\varrho(v_{n+p}, v_n) \leqq \varrho(v_n, v_{n+1}) + \varrho(v_{n+1}, v_{n+2}) + \ldots + \varrho(v_{n+p-1}, v_{n+p}) ,$$

or

$$\varrho(v_{n+p}, v_n) \leqq \varrho(v_n, v_{n+1}) (1 + \alpha + \alpha^2 + \ldots + \alpha^p) ,$$

 when p intends to ∞ we have

$$\lim_{p\to\infty} v_{n+p} = v^*$$

and

$$\varrho(v_n, v^*) \leqq \frac{1}{1-\alpha}\,\varrho(v_{n+1}, v_n)\,.$$

This inequality together with the inequality (19) imply

$$\varrho(v_{n+1}, v^*) \leqq \frac{\alpha}{1-\alpha}\,\varrho(v_{n+1}, v_n)\,.$$

The theorem is completely proved.

We now extend the above results to the system of equations. Consider the system of equations

$$\frac{d^2\mathbf{V}}{dx^2} = \mathbf{F}(\mathbf{V}, x)\,,$$

$$\mathbf{V}(a) = \mathbf{V}_a\,, \quad \mathbf{V}(b) = \mathbf{V}_b\,,$$

where

$$\mathbf{V}(x) = \begin{pmatrix} v^1(x) \\ \cdots \\ \cdots \\ v^N(x) \end{pmatrix}, \quad \mathbf{F}(\mathbf{V}, x) = \begin{pmatrix} f^1(v^1, \ldots, v^N, x) \\ \cdots\cdots \\ \cdots\cdots \\ f^N(v^1, \ldots, v^N, x) \end{pmatrix};$$

the f^i are defined on the $N + 1$ dimensional closed domain D, which is given by

$$|v^i - v^i_{ab}| \leqq L, \quad x \in [a, b]\,, \quad i = 1, \ldots, N\,,$$

and

$$v^i_{ab}(x) = \frac{1}{b-a}\left[(v^i_b - v^i_a)\,x + bv^i_a - av^i_b\right].$$

The complete metric space S is defined as

$$S = \{\mathbf{V}(x) \mid v^i(x) \text{ continuous on } [a, b],\ v^i(a) = v^i_a,$$
$$v^i(b) = v^i_b,\ \max |v^i(x) - v^i_{ab}(x)| \leqq L,\ i = 1, \ldots, N\}$$

with the distance function $\varrho(\mathbf{V}_1, \mathbf{V}_2)$ given by

$$\varrho(\mathbf{V}_1, \mathbf{V}_2) = \sum_{i=1}^{N} \max_x |v^i_1(x) - v^i_2(x)|$$

and

$$J(\mathbf{V}_0, x) = \begin{bmatrix} f^1_{v^1}(v^1_0, \ldots, v^N_0, x), \ldots, f^1_{v^N}(v^1_0, \ldots, v^N_0, x) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f^N_{v^1}(v^1_0, \ldots, v^N_0, x), \ldots, f^N_{v^N}(v^1_0, \ldots, v^N_0, x) \end{bmatrix}.$$

We may now state and proof the following theorem.

Theorem. *Given the system of nonlinear differential equations with two-point boundary conditions*

$$\text{(20)} \qquad \frac{d^2\mathbf{V}}{dx^2} = \mathbf{F}(\mathbf{V}, x), \quad \mathbf{V}(a) = \mathbf{V}_a, \quad \mathbf{V}(b) = \mathbf{V}_b,$$

where the $f^i(v^1, \ldots, v^N, x)$, $i = 1, \ldots, N$, *have the following properties on* D:

1) $f^i(v^1, \ldots, v^N, x)$ *are continuous;*
2) $f^i_{v^j}(v^1, \ldots, v^N, x) = [\partial f^i(v^1, \ldots, v^N, x)]/\partial v^j$ *exist and are continuous.*

Define the following sequence of system of linear differential equations

$$\frac{d^2\mathbf{V}_{n+1}}{dx^2} = J(\mathbf{V}_0, x)\,[\mathbf{V}_{n+1}(x) - \mathbf{V}_n(x)] + \mathbf{F}(\mathbf{V}_n, x),$$

$$\mathbf{V}_n(a) = \mathbf{V}_a, \quad \mathbf{V}_n(b) = \mathbf{V}_b, \quad n = 0, 1, \ldots,$$

and $\mathbf{V}_0(x)$ *is such that* $v^i_0(x)$, $i = 1, \ldots, N$, *are continuous on* $[a, b]$ *and*

$$\max |v^i_0(x) - v^i_{ab}(x)| \leqq L, \quad i = 1, \ldots, N.$$

Then for a sufficiently small interval $[a, b]$ *the unique solution to system* (20) *exists and*

– *the sequence* $\{\mathbf{V}_n(x)\}$ *converges to it;*

– *the convergence speed of the sequence* $\{\mathbf{V}_n(x)\}$ *to the solution of* (20) *is given by the inequality*

$$\varrho(\mathbf{V}_n, \mathbf{V}^*) \leqq \frac{\beta^n}{1 - \beta}\,\varrho(\mathbf{V}_1, \mathbf{V}_0);$$

– *a bound on the error is given by*

$$\varrho(\mathbf{V}_{n+1}, \mathbf{V}^*) \leqq \frac{\beta}{1 - \beta}\,\varrho(\mathbf{V}_{n+1}, \mathbf{V}^*)$$

where $\mathbf{V}^*(x)$ *is the solution of system* (20) *and the number* β *is defined below.*

Proof. It follows from the hypotheses of the theorem above that there exist the numbers Q_i, R_{ij}, U_i such that

$$|f^i(v^1, \ldots, v^N, x)| \leqq Q_i,$$
$$|f^i_v j(v^1, \ldots, v^N, x)| \leqq R_{ij},$$

and

$$|f^i(v_1^1, \ldots, v_1^N, x) - f^i(v_2^1, \ldots, v_2^N, x)| \leqq U_i \sum_{i=1}^{N} |v_1^i - v_2^i| .$$

Let

$$m = \max_{\substack{i=1,\ldots,N \\ j=1,\ldots,N}} \{R_{ij}, Q_i, U_i\} .$$

Define the operator P on S,

$$P\mathbf{V} = \mathbf{V}_{ab}(x) - \int_a^b K(x, s) \{J(\mathbf{V}_0, s) [P\, \mathbf{V}(s) - \mathbf{V}(s)] + \mathbf{F}(\mathbf{V}, s)\} \,\mathrm{d}s$$

where

$$K(x, s) = \begin{cases} \dfrac{b - s}{b - a}(x - a) & \text{for} \quad x \leqq s , \\[2ex] \dfrac{a - s}{b - a}(x - b) & \text{for} \quad x \geqq s , \end{cases}$$

therefore

$$|K(x, s)| \leqq \tfrac{1}{4}(b - a) .$$

Firstly we shall show that the operator equation $P\mathbf{V} = \mathbf{V}$ has a unique solution on S. P maps S into S, for arbitrary $\mathbf{V} \in S$ we have:

$$\begin{aligned} \varrho(P\mathbf{V}, \mathbf{V}_{ab}) = \sum_{i=1}^{N} \max_x |Pv^i - v_{ab}^i| &\leqq N \frac{m}{4} (b - a)^2 [\varrho(P\mathbf{V}, \mathbf{V}) + 1] \leqq \\ &\leqq N \frac{m}{4} (b - a)^2 [\varrho(P\mathbf{V}, \mathbf{V}_{ab}) + \varrho(\mathbf{V}, \mathbf{V}_{ab}) + 1] \leqq \\ &\leqq N \frac{m}{4} (b - a)^2 [\varrho(P\mathbf{V}, \mathbf{V}_{ab}) + NL + 1] \end{aligned}$$

or

$$\varrho(P\mathbf{V}, \mathbf{V}_{ab}) \leqq \frac{N(m/4)(b - a)^2 (NL + 1)}{1 - N(m/4)(b - a)^2} \leqq L$$

for $(b - a)$ sufficiently small. This implies $P\mathbf{V} \in S$. Furthermore, for two arbitrary elements $\mathbf{V}_1$ and $\mathbf{V}_2$ in S, we have

$$\begin{aligned} P\mathbf{V}_1 - P\mathbf{V}_2 = &\int_a^b K(x, s) \{J(\mathbf{V}_0, s) [P\mathbf{V}_2 - \mathbf{V}_2] - J(\mathbf{V}_0, s) [P\mathbf{V}_1 - \mathbf{V}_1] - \\ &- \mathbf{F}(\mathbf{V}_1, s) + \mathbf{F}(\mathbf{V}_2, s)\} \,\mathrm{d}s = \\ = &\int_a^b K(x, s) \{J(\mathbf{V}_0, s) [P\mathbf{V}_2 - P\mathbf{V}_1 + \mathbf{V}_1 - \mathbf{V}_2] - \\ &- \mathbf{F}(\mathbf{V}_1, s) + \mathbf{F}(\mathbf{V}_2, s)\} \,\mathrm{d}s . \end{aligned}$$

We replace $\mathbf{F}(\mathbf{V}_2, s) - \mathbf{F}(\mathbf{V}_1, s)$ by $J(\mathbf{V}, s)(\mathbf{V}_2 - \mathbf{V}_1)$, $\mathbf{V} \in (\mathbf{V}_1, \mathbf{V}_2)$, i.e., $\varrho(\mathbf{V}, \mathbf{V}_2) < < \varrho(\mathbf{V}_1, \mathbf{V}_2)$.

It follows that

$$\varrho(P\mathbf{V}_1, P\mathbf{V}_2) = \sum_{i=1}^{N} \max_x |Pv_1^i - Pv_2^i| \leqq N \frac{m}{4} (b - a)^2 [\varrho(P\mathbf{V}_1, P\mathbf{V}_2) + 2\varrho(\mathbf{V}_1, \mathbf{V}_2)],$$

$$\varrho(P\mathbf{V}_1, P\mathbf{V}_2) \leqq \frac{N(m/2)(b - a)^2}{1 - N(m/4)(b - a)^2} \varrho(\mathbf{V}_1, \mathbf{V}_2),$$

from which we see that when the condition

$$\frac{N(m/2)(b - a)^2}{1 - N(m/4)(b - a)^2} = \beta < 1$$

is satisfied i.e., $(b - a)$ is sufficiently small, then P is a contraction mapping of S into S. Therefore the operator equation $P\mathbf{V} = \mathbf{V}$ has a unique solution V in S, and the sequence $\{\mathbf{V}_n(x)\}$ converges to it, i.e.,

$$\mathbf{V}^*(x) = \lim_{n \to \infty} \mathbf{V}_n(x),$$

where the $\mathbf{V}_n(x)$ are calculated by the equation $\mathbf{V}_{n+1} = P\mathbf{V}_n$, $n = 0, 1, \ldots$, and $\mathbf{V}_0$ has satisfied the condition defined above.

Since P is a contraction mapping of S into S and from the contraction mapping principle we easy to see that

$$\varrho(\mathbf{V}_n, \mathbf{V}^*) \leqq \frac{\beta^n}{1 - \beta} \varrho(\mathbf{V}_1, \mathbf{V}_0).$$

We now consider the expression

$$|\mathbf{V}_{n+1} - \mathbf{V}^*| = \left| \int_a^b K(x, s) \{J(\mathbf{V}_0, s) [\mathbf{V}_{n+1}(s) - \mathbf{V}_n(s)] + [\mathbf{F}(\mathbf{V}_n, s) - \mathbf{F}(\mathbf{V}^*, s)]\} \, \mathrm{d}s \right|.$$

By the mean value theorem for functions of several variables, we may replace every component of $\mathbf{F}(\mathbf{V}_n, s) - \mathbf{F}(\mathbf{V}^*, s)$ by the following form

$$f^j(v_n^1, \ldots, v_n^N, s) - f^j(v^{*1}, \ldots, v^{*N}, s) = f^j_{v^1}({}^jv^1, \ldots, {}^jv^N, s)$$
$$[v_N^1 - v^{*1}] + \ldots + f^j_{v^N}({}^jv^1, \ldots, {}^jv^N, s) [v_n^N - v^{*N}], \quad j = 1, \ldots, N$$

where for each $s \in [a, b]$, ${}^j\mathbf{V}$ is a vector on the line segment joining $\mathbf{V}^*$ to $\mathbf{V}_n$. After some calculation we obtain

$$\varrho(\mathbf{V}_{n+1}, \mathbf{V}^*) \leqq \frac{Nm(b - a)^2}{4} [(\varrho\mathbf{V}_{n+1}, \mathbf{V}_n) + 2\varrho(\mathbf{V}_n, \mathbf{V}^*)]$$

or

$$\varrho(\mathbf{V}_{n+1}, \mathbf{V}^*) \leqq \frac{N(m/2)(b-a)^2}{1 - N(m/4)(b-a)^2}\, \varrho(\mathbf{V}_n, \mathbf{V}^*) = \beta\, \varrho(\mathbf{V}_n, \mathbf{V}^*) . \tag{21}$$

And from the contraction mapping principle we have

$$\varrho(\mathbf{V}_n, \mathbf{V}^*) \leqq \frac{1}{1-\beta}\, \varrho(\mathbf{V}_{n+1}, \mathbf{V}_n) .$$

This inequality together with the inequality (21) implies

$$\varrho(\mathbf{V}_{n+1}, \mathbf{V}^*) \leqq \frac{\beta}{1-\beta}\, \varrho(\mathbf{V}_{n+1}, \mathbf{V}_n) .$$

The theorem is completely proved.

CONCLUSIONS

In this paper we have presented a convergence proof for a proposed method of obtaining numerical solutions to systems of nonlinear differential equations with two-point boundary conditions. By this method some computational effort may be saved, but the convergence will necessary be slower than the method which is base on the original Newton's method.

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Ing. Nguyen Canh; Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 180 76 Praha 8. Czechoslovakia.