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A Generalization of the Propositional Calculus for Purposes of the Theory of Logical Nets with Probabilistic Elements

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The definition of logical nets with probabilistic elements is given in the following paper. To describe these nets, the logical—probabilistic expressions, as a probabilistic extension of propositional calculus, are introduced. Some fundamental properties of these logical-probabilistic expressions are investigated.

This paper is an attemp to fill a certain gap in the structural theory of finite automata. The theory of logical nets connected with the theory of deterministic automata is well known. This paper is concerned with the theory of logical nets generalized in such a way that the new theory will deal with logical nets in which probabilistic elements can occur. Propositional calculus is widely used for the description of deterministic logical nets. For our purposes it was necessary to extend propositional calculus in a probabilistic way and to develop the description of nets with probabilistic elements. In this extended propositional calculus, certain theorems which can also be considered as theorems dealing with nets, are formulated. The whole theory can be understood in a more general way as a probabilistic extension of propopositional calculus, without relating it to logical nets, and it can also be applied in a different way.

In the first part of this paper, certain important notions — in particular, the notion of the logical-probabilistic expression — will be defined, and certain assertions will be made about these notions.

We will also add an interpretation to the theory of logical nets (it will be necessary to define the logical-probabilistic nets). The second part will contain proofs of these theorems and certain other theorems necessary for various calculations and for the proofs of the theorems from the first part.

I. DEFINITIONS AND THEOREMS

For reasons of intuitive intelligibility the account will proceed from the definitions concerning the nets to the definition of the corresponding logical or logical-probabilistic expressions. We shall see that the definitions and properties of the logical-probabilistic expressions do not depend upon the concept of a net and so we could limit ourselves in our account to them alone. However, for the sake of clearness of the presentation, it is convenient to bear in mind the applications of the defined notions to nets.

Definition 1. Let us consider the following two kinds of elements. The first one (see Fig. 1a) will be called a *primitive element of the first kind*, the second one (see Fig. 1b) will be called a *primitive element of the second kind*. We call a_1 or a_2, a_3 the inputs of an element, A_1 or A_2 the nucleus of an element, and b_1 and b_2 the outputs of an element.

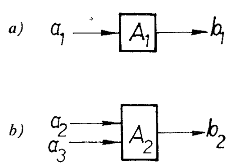

Fig. 1.

We define a *net* as follows:

1) A primitive element of the first or second kind is a net with input a_1 or a_2, a_3 and output b_1 or b_2.

2) Let N_1 be a net with output b_3 and inputs $a_1, \ldots, a_k$,

a) given a primitive element of the first kind with the input a_1 and output b_1 (not contained in the net N_1) then by connecting b_3 with a_1, we obtain a net with inputs $a_1, \ldots, a_k$ and output b_1;

b) if we connect two different inputs a_i, a_j of the net N_1, we obtain a net with the same output and inputs $a_1, \ldots, a_i, \ldots, a_{j-1}, a_{j+1}, a_{j+2}, \ldots, a_k$.

3) Let N_1 and N_2 be two distinct nets with outputs b_3, b_4 and inputs $a_1^1, \ldots, a_k^1$ $a_1^2, \ldots, a_l^2$ respectively. Given a primitive element of the second kind with the inputs a_1, a_2 and output b_2 (which is not contained in the nets N_1, N_2), then connectig b_3 with a_1 and b_4 with a_2 (or a_1 with b_3 only) we obtain a net with inputs $a_1^1, \ldots, a_k^1$, $a_1^2, \ldots, a_l^2$ (or $a_1^1, \ldots, a_k^1, a_2$) and output b_2.

A net defined in this way consists, therefore, of two kinds of elements. Through the net there can propagate $0-1$ pulses which are treated by elements in a different way. The elements of the second kind can have a different function and must, therefore, be differently denoted. So we shall have two things: a net with differently described elements and the function of this net determined by the function of individual elements.

Definition 2. A net N (with inputs $a_1, \ldots, a_k$ and output b) will be called a *labeled net* iff:

1) every primitive element of the first kind is denoted by the symbol $\bigwedge$ or $\sim$,

2) every primitive element of the second kind is denoted by one and only one of the symbols $\alpha_0, \ldots, \alpha_{15}$,

3) the input a_i is denoted by x_i (for $i = 1, \ldots, k$) (moreover we can denote the output b by y).

Definition 3. 1) For every primitive element of the first kind denoted by $\bigwedge$ we define the associated function $\bigwedge^*$ ($\bigwedge^* : \{0, 1\} \to \{0, 1\}$) by the table

y	$\bigwedge^*(y)$
0	0
1	1

2) For every primitive element denoted by $\sim$ we define the associated function $\sim^*$ ($\sim^* : \{0, 1\} \to \{0, 1\}$) by the table

y	$\sim^*(y)$
0	1
1	0

3) For every primitive element denoted by α_j we define the associated function α_j^* ($\alpha_j^* : \{0, 1\}^2 \to \{0, 1\}$) by the table ($j = 0, \ldots, 15$):

$\alpha_j^*(y_1, y_2)$

y_1	y_2	$j = 0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

4) For every labeled net N we define, following the inductive definition of a net, a mapping $func_N$ ($func_N : \{0, 1\}^k \to \{0, 1\}$, where k is the number of inputs of the net N):

a) If a net N has the structure of Fig. 2a, then we define $func_N(x_1, \ldots, x_k) = \bigwedge^*(func_{N_1}(x_1, \ldots, x_k))$;

b) if a net N has the structure of Fig. 2b, then we define $func_N(x_1, \ldots, x_k) = \sim^*(func_{N_1}(x_1, \ldots, x_k))$;

c) if a net N has the structure of Fig. 2c, then we define $func_N(x_1, \ldots, x_k) = \alpha_j^*(func_{N_1}(x_1, \ldots, x_{k_1}), func_{N_2}(x_1, \ldots, x_{k_2}))$; ($j = 0, \ldots, 15$);

d) if a net N has the structure of Fig. 2d, then we define

$$func_N(x_1, \ldots, x_i, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k) = func_{N_1}(x_1, \ldots, x_k) .$$

We then call a logical net (L-net) the pair

$$[N, func_N] .$$

Note. This definition does not admit nets of the type from Fig. 3. In our later account we shall see the necessity of this limitation.

We can consider a net as an oriented tree. For this purposes it is useful to treat every pair of inputs, connected as in point 2. b) of Def. 1, as two different inputs. Then the inputs of the net are terminal nodes of the tree, the primitive elements are nodes, and the output is a root of this tree. If the net is labeled then the tree is called a labeled tree.

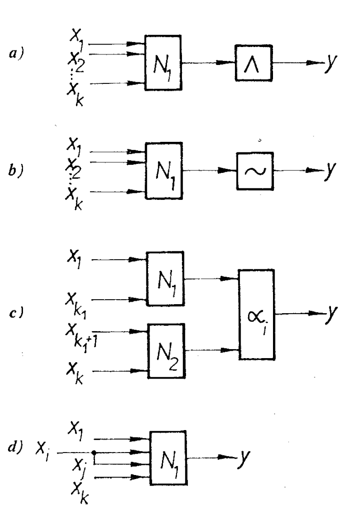

Fig. 2.

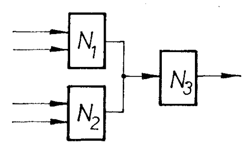

Fig. 3.

Let N be some net, treated as a tree, then we can call a *subnet* of N every branch N' of N having one or more nodes (shortly $N' \prec N$). Inputs of this subnet are terminal nodes of N and the output is an edge connecting the first node of the branch with the tree N (with the rest of tree N).

A logical expression can be associated with a denoted net in the following way.

Definition 4. Let us consider symbols $x_1, x_2, \ldots$ (individual variables), $\sim$ (negation), $\alpha_0, \ldots, \alpha_{15}$ (binary logical connectives), (,),,,.

Let $F, F_1, F_2, \ldots$ be names of finite sequences of these symbols. We define the *logical form* (L-form) as follows:

1) x_i for $i = 1, 2, \ldots$ is an L-form;

2) if F_1 is an L-form, then $\sim F_1$ is an L-form;

3) if F_1, F_2 are L-forms, then $\alpha_i(F_1, F_2)$, for $i = 0, \ldots, 15$, is an L-form.

For an L-form we can define the mapping $func_F$ as the usual evaluation function of L-forms, i.e. following the steps of Def. 3.

We shall call the pair $[F, func_F]$ a *logical expression* (L-expression).

Note. It is possible to introduce the symbol $\bigwedge$ — an empty symbol. Then we can use following rule:

4) if F_1 is an L-form, then $\bigwedge F_1$ is an L-form. F_1, $\bigwedge F_1$, and $\bigwedge(F_1)$ are the same L-forms.

There exists a one to one correspondence between the L-nets and L-expressions.

If the L-net $[N, func_N]$ has n distinct inputs denoted by $x_1, \ldots, x_n$, then we take $x_1, \ldots, x_n$ as variables of the corresponding expression, and we construct the corresponding L-expression $[F, func_F]$ by induction:

Let N_1 be a subnet of N, let $\sim N_1$ be a subnet of N, and let F_1 correspond to N_1; then $\sim F_1$ corresponds to $\sim N_1$. Either F_1 or $\bigwedge F_1$ correspond to $\bigwedge N_1$ (see Def. 4, point 4); we do not distinquish between L-forms $\bigwedge F_1$ and F_1). We proced with the connectives $\alpha_0, \ldots, \alpha_{15}$ analogically. The relation between L-nets and L-expression is described in detail in other writings; see [5].

It is evident that for an L-net $[N, func_N]$ and the corresponding L-expression $[F, func_F]$ we have $func_F = func_N$.

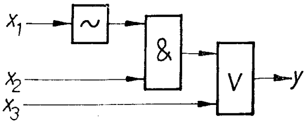

Fig. 4.

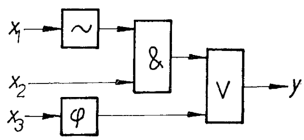

Fig. 5.

Example 1. Let N be the labelled net from Fig. 4, then the corresponding L-form is $(\sim x_1 \,\&\, x_2) \vee x_3$ and we obtain the following values of functions:

x_1	x_2	x_3	$func_F = func_N$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

We can make the following generalization of the concept of L-net:

We add a primitive element of the first kind, denoted by φ, to the net as in Fig. 5. This element can then have the associated function $\varphi^*(y, \omega)$, $\varphi^* : \{0, 1\} \times \Omega \rightarrow$

$\rightarrow \{0, 1\}$, where Ω is some non-empty set of random events. Let $\varphi^*(0, \omega) = \varphi^*(1, \omega)$ for every $\omega \in \Omega$. So we can write $\varphi^*(\omega)$ for $\varphi^*(y, \omega)$. Let P_1, P_2 be two distribution of probabilities on $[\Omega, A]$ (A is some σ-field of subsets of Ω). Let $P_i(\omega)$ be the probability of random event ω if the value i ($i = 0, 1$) is on the input of element φ. For our purposes, of course, only the events $\omega_0 = (\varphi^*)^{-1}(0)$, $\omega_1 = (\varphi^*)^{-1}(1)$ and the probabilities $p_0 = P_0(\omega_0)$ and $p_1 = P_1(\omega_0)$ are important ($\omega_0 \cup \omega_1 = \Omega$).

So we have now the probabilistic element φ in the net. The values on the output of this element do not depend on the input values; the probabilities of the output values are dependent on the input values only (the word "dependence" is here to be understood as functional dependence).

For instance for $p_0 = 1$, $p_1 = 0{\cdot}05$ this element can simulate the unreliability of the conductor. By means of these elements and the elements having the function of logical connectives we can describe even unreliable elements which realize logical operations as we shall see later.

Now we shall give an exact definition of the logical-probabilistic net (LP-net).

Definition 5. Let us consider a logical net $[N', func_{N'}]$. Let us denote some of the elements of the first kind (denoted so far by $\bigwedge$) by one (and only one) of the symbols $\varphi_1, \varphi_2, \ldots$ (the new net will be called N) obeying the following two conditions:

1) no symbol from $\{\varphi_1, \varphi_2, \ldots\}$ can occur in the net N more than once,

2) if we assume that an element of N denoted by φ_i is connected to the output of a subnet N_1 of N and that N_1 contains an element denoted by φ_j, then $j < i$.

Let the L-net $[N', func_{N'}]$ have n distinct inputs and let there be m probabilistic elements in the new net N; let us denote $\Sigma = \{0, 1\}^n$ and $\Omega_N = \{0, 1\}^m$, $\boldsymbol{\sigma}$ elements from Σ and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ elements from Ω_N.

Now we define a mapping $func_N N'$ from $\Sigma \times \Omega_N$ to $\{0, 1\}$ for each subnet N' of N by induction in the following way: ($x_1, \ldots, x_n$ will be denoted by $\boldsymbol{x}$)

a) the value σ_i is on the input x_i for the value $(\boldsymbol{\sigma}, \boldsymbol{\omega})$;

b) for a subnet N' having the structure of Fig. 2a we define $func_N N'(\boldsymbol{\sigma}, \boldsymbol{\omega}) = func_N N_1(\boldsymbol{\sigma}, \boldsymbol{\omega})$;

c) for a subnet N' having the structure of Fig. 2b we define $func_N N'(\boldsymbol{\sigma}, \boldsymbol{\omega}) = 1 - func_N N_1(\boldsymbol{\sigma}, \boldsymbol{\omega})$;

d) for a subnet N' having the structure of Fig. 2c we define

$$func_N N'(\boldsymbol{\sigma}, \boldsymbol{\omega}) \doteq \alpha_j^*(func_N N_1(\boldsymbol{\sigma}, \boldsymbol{\omega}), func_N N_2(\boldsymbol{\sigma}, \boldsymbol{\omega}),$$

where α_j^* is the associated function from point 3) of Def. 3;

e) for a subnet N' having the structure of Fig. 6 we define $func_N N'(\boldsymbol{\sigma}, \boldsymbol{\omega}) = \omega_i$, where i is the rank of the φ_j in the vector $(\varphi_{j_1}, \ldots, \varphi_{j_m})$ ranked with respect to the increasing indices;

f) for the net N' with connected inputs x_i, x_j $(i \neq j)$ we define

$$func_N\, N'(\sigma, \omega) = func_N\, N_1(\sigma_1, \ldots, \sigma_i, \ldots, \sigma_{j-1}, \sigma_i, \sigma_{j+1}, \ldots, \sigma_n, \omega)\,.$$

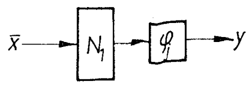

Fig. 6.

For any $\gamma = (\gamma_1, \ldots, \gamma_m) \in \{0, 1\}^m$ let $P_N(\gamma_1, \ldots, \gamma_m;\ .)$ be a mapping from Ω_N to $\langle 0, 1\rangle$ for which:

3) $\sum_{\omega \in \Omega_N} P_N(\gamma; \omega) = 1$;

4) for every $\omega \in \Omega_N$, $k \leqq m$, $\gamma \in \{0, 1\}^m$, $\gamma' \in \{0, 1\}^m$ for every $\{j_1', \ldots, j_k'\} \subset \subset \{j_1, \ldots, j_m\}$ and for every $\omega^* \in (0, 1)^k$ if $\gamma_{j_1'} = \gamma'_{j_1'}, \ldots, \gamma_{j_k'} = \gamma'_{j_k'}$, then

$$\sum_{\{\omega;(\omega_{j_1'}, \ldots, \omega_{j_k'}) = \omega^*\}} P_N(\gamma; \omega) = \sum_{\{\omega;(\omega_{j_1'}, \ldots, \omega_{j_k'}) = \omega^*\}} P_N(\gamma'; \omega)\,.$$

We define a system of functions

$$\mathscr{P}_N = \{P_N(\gamma;\ .)\}_{\gamma \in \{0,1\}^m}\,.$$

For a given value of inputs let $\gamma_j = func_N\, N_1'(\sigma, \omega)$, where N_1' is the subnet, the output of which is connected to the input of the element denoted by φ_i; j is the rank of φ_i $(i = 1, \ldots, m)$.

Then we shall call any such triplet $[N, \Omega_N, \mathscr{P}_N]$ a *logical-probabilistic net* (LP-net).

The numbers p_0^j, p_1^j, where

$$p_0^j = \sum_{\{\omega;\omega_j = 1\}} P_N(\gamma_1, \ldots, \gamma_{j-1}, 0, \gamma_{j+1}, \ldots, \gamma_m; \omega)\,,$$
$$p_1^j = \sum_{\{\omega;\omega_j = 1\}} P_N(\gamma_1, \ldots, \gamma_{j-1}, 0, \gamma_{j+1}, \ldots, \gamma_m; \omega)\,,$$

will be called the *probabilistic parameters* of the element φ_{i_j} in LP-net $[N, \Omega_N, \mathscr{P}_N]$.

Note. An important property of LP-net is that the value of γ_j is dependent only on ω_i for $i < j$.

Given a LP-net $\mathscr{N} = [N, \Omega_N, \mathscr{P}_N]$ we say: 1) that a LP-net corresponding to the labeled subnet N_1, output of which is connected with the input of element φ_i, is the ***interior*** of probabilistic element φ_i in the LP-net $\mathscr{N}$; 2) that an element φ_i is stochastically independent of his interior in the LP-net $\mathscr{N}$ iff

$$\sum_{\{\omega;\omega_{j_1'}, \ldots, \omega_{j_k'} = \omega^*, \omega_j = 1\}} P_N(\gamma_1, \ldots, \gamma_m; \omega) =$$
$$= \sum_{\{\omega;\omega_{j_1'}, \ldots, \omega_{j_k'} = \omega^*\}} P_N(\gamma_1, \ldots, \gamma_m; \omega)\, p^i_{\gamma_j}\,,$$

where j is the rank of φ_i, and $j_1', \ldots, j_k'$ are ranks of probabilistic elements from N_1.

In this case we can interpret the probabilistic parametres of the elements φ_i as

$$p_0^i = P(func_N\, \varphi_i(N_1)\,(\sigma, \omega) = 1/func_N\, N_1(\sigma, \omega) = 1)\,,$$
$$p_1^i = P(func_N\, \varphi_i(N_1)\,(\sigma, \omega) = 1/func_N\, N_1(\sigma, \omega) = 0)\,.$$

It is necessary for further considerations to define a LP-subnet.

Definition 6. Let $\mathcal{N} = [N, \Omega_N, \mathscr{P}_N]$ be an LP-net, let N_1 be a subnet of the net N, let $\Omega_N = \Omega_1 \times \ldots \times \Omega_m = \mathsf{X}_{i=1}^m \Omega_i$. Let N_1 contain probabilistic elements denoted by $\varphi_{k_1}, \ldots, \varphi_{k_l}$. Let i_j be the rank of φ_{k_j} in $(\varphi_{i_1}, \ldots, \varphi_{i_m})$ and j' the rank of φ_{k_j} in $(\varphi_{k_1}, \ldots, \varphi_{k_l})$. We denote $\Omega_{N_1} = \mathsf{X}_{j'=1}^l \Omega'_{j'} = \mathsf{X}_{j=1}^l \Omega_{i_j}$ $(\Omega'_{j'} = \Omega_{i_j})$, $\omega' \in \Omega_{N_1}$ and for every $\gamma' \in \{0, 1\}^l$, $\gamma'_{j'} = \gamma_j$ we define

$$P_{N_1}(\gamma'; \omega') = \sum_{\{\omega; (\omega_{i_1}, \ldots, \omega_{il}) = \omega'\}} P_N(\gamma; \omega)\,.$$

Then we can denote

$$\mathscr{P}_{N_1} = \{P_{N_1}(\gamma'; .)\}_{\gamma' \in \{0,1\}^l}$$

and we shall call the triplet $\mathcal{N}' = [N_1, \Omega_{N_1}, \mathscr{P}_{N_1}]$ a *LP-subnet* of the LP-net $\mathcal{N}$.

We shall write $\mathcal{N}' \prec \mathcal{N}$.

The consistency of the previous definition is guaranteed by the condition 4) from Def. 5.

Now we shall give a definition which is very important for the aplications in papers [11], [12].

Definition 7. We say that a LP-net is *stochastically independent* if for every $\gamma \in \{0, 1\}^m$

$$P_N(\gamma; \omega) = \prod_{i=1}^m P_i(\gamma_i; \omega_i)\,.$$

For further considerations the following convention will be useful.

Convention 1. We shall assume in all cases, supposing it could not cause confusion, that the rank and the index of every probabilistic element in the given LP-net are equal.

The usual logical nets will be described by logical expressions. These logical expressions are, however, not sufficient for the decription of LP-nets. This is the reason for extending also the notion of logical expression in a corresponding way, and defining the logical-probabilistic expressions (LP-expressions). It is necessary to point out that these expressions can be defined, quite independently of the notion of the net, as a certain probabilistic extension of the propositional calculus. In our later account we shall deal with these expressions without stressing their net interpretation. We shall investigate the question of equivalence of these expressions to certain normal forms and consider to what extent the laws of propositional calculus are conservated (see also [11]). As a notion corresponding to the probabilistic elements we shall introduce a new kind of an unary logical connective (more exactly a whole class of unary connectives).

Note. 1) A particular case of the definition of the LP-expression (stochastical independence) was published by the author in [9], 2) The notion of a subform is defined in accordance to the notion of a subnet (labeled); i.e., in the usual way. 3) The notion of the interior of a probabilistic connective corresponds to the notion of interior of a probabilistic element. We shall write $F_1 = int(\varphi_i, F)$, if F_1 is the interior of probabilistic connective φ_i.

Definition 8. Let us consider a logical expression $[F', func_{F'}]$. Let us substitute into F' some of the symbols $\varphi_1, \varphi_2, \ldots$ in place of some symbols $\bigwedge$.

The new form will be denoted by F.

Let the following conditions be satisfied:

1) any symbol from $\{\varphi_1, \varphi_2, \ldots\}$ cannot occur in F more than once, 2) the symbols $\varphi_1, \varphi_2, \ldots$ are used in such a way, that for every φ_i no φ_j, $i \leqq j$, can occur in the interior of φ_i in F.

The symbols $\varphi_1, \varphi_2, \ldots$ will be called probabilistic connectives.

Let the form F contain n different variables, let in F occur m probabilistic connectives, and let Ω_N, Σ, ω, σ be the same as in Def. 5 and $\Omega_F = \Omega_N$.

Then we define a mapping $func_F\, F_1$ from $\Sigma \times \Omega_F$ to $\langle 0, 1 \rangle$ for each subform F_1 by induction in the following way:

a) $func_F\, x_i(\sigma, \omega) = \sigma_i$;
b) $func_F \bigwedge F_1(\sigma, \omega) = func_F\, F_1(\sigma, \omega)$;
c) $func_F \sim F_1(\sigma, \omega) = 1 - func_F\, F_1(\sigma, \omega)$;
d) $func_F\, \alpha_j(F_1, F_2)(\sigma, \omega) = \alpha_j^*(func_F\, F_1(\sigma, \omega), func_F\, F_2(\sigma, \omega))$,
where α_j^* is the associated function from the poin d) of Def. 3;
e) $func_F\, \varphi_i(F_1)(\sigma, \omega) = \omega_j$ (j is the rank of φ_i);
f) if $x_i = x_j$ then

$$func_F\, F_1(\sigma_1, \ldots, \sigma_i, \ldots, \sigma_{j-1}, \sigma_{j+1}, \ldots, \sigma_n, \omega) =$$
$$= func_F\, F_1(\sigma_1, \ldots, \sigma_i, \ldots, \sigma_{j-1}, \sigma_i, \sigma_{j+1}, \ldots, \sigma_n, \omega) \, ;$$

g) $func_F(\sigma, \omega) = func_F\, F(\sigma, \omega)$.

Let us now define P_F, $\mathscr{P}_F$ in the same way as P_N and $\mathscr{P}_N$ in Definition 5.

We shall call any such triplet $\Phi = [F, \Omega_F, \mathscr{P}_F]$ a *logical-probabilistic expression* (*LP-expression*).

The notion of probabilistic parametres of probabilistic connectives and the notion of stochastical independence are the same as in the case of LP-net. It is possible to give a direct definition of LP-expression (without defining first LP-nets and L-expressions) but for intuitive reasons it seems better to give the definition in the above form. The direct definition was used in [9] and [10]. Now it is possible to consider the correspondence between LP-nets and LP-expressions.

Theorem 1. For every LP-net $\mathcal{N} = [N, \Omega_N, \mathcal{P}_N]$ there exists a unique LP-expression $\Phi = [F, \Omega_F, \mathcal{P}_F]$ so that

$$\Omega_N = \Omega_F\,, \quad func_N = func_F \quad \text{and} \quad \mathcal{P}_F = \mathcal{P}_N\,.$$

Note. The converse of this theorem is also true.

Proof. The proof is based on the fact that for every L-net we can construct (in a unique way) a corresponding LP-expression. Let us consider an LP-net $\mathcal{N}$. When constructing a corresponding LP-expression we proceed in the following way:

1) We substitute $\bigwedge$ for all symbols $\varphi_1, \varphi_2, \ldots$ describing the probabilistic elements of the denoted net N. However, we distiquish symbols $\bigwedge$ by the same indices as φ_i (i.e., $\bigwedge_i \to \varphi_i$); all $\bigwedge_i$ have the same function as $\bigwedge$.

2) For the obtained L-net we find the corresponding L-expression (which describes the structure of L-net);

3) in this L-expression we carry out the substitution $\varphi_i \to \bigwedge_i$; the obtained form can be denoted by F.

4) We put $\Omega_F = \Omega_N$ and for F we construct $func_F$ according to Def. 8, points a) to g). With regards to the construction of $func_N$ is $func_F = func_N$.

5) We put $\mathcal{P}_F = \mathcal{P}_N$. □

In further considerations we shall restrict ourselves only to LP-expressions: Occasionally, however, the corresponding LP-net will also be given for purposes of illustration. Given an LP-net $\mathcal{N}$ we call the expression produced by the algorithm roughly described in the proof of the previous theorem the *corresponding LP-expression* (to $\mathcal{N}$). This expression (as well as the net) is distinctly determined.

Example 2. Let N be the labeled net from Fig. 7, $\Omega_N = \{0, 1\}^7$, $\Sigma = \{0, 1\}^3$, let the probabilistic parametres be $p_0^j = 0{\cdot}1$, $p_1^j = 0{\cdot}9$ for $j = 1, \ldots, 7$, and let this LP-net be stochastically independent, i.e.

$$P_N(\gamma; \omega) = \prod_{i=1}^{7} P_i(\gamma_i; \omega_i)\,, \tag{1}$$

where

$$P_i(0; 0) = 0{\cdot}9\,, \quad P_i(0; 1) = 0{\cdot}1\,, \tag{2}$$
$$P_i(1; 0) = 0{\cdot}1\,, \quad P_i(1; 1) = 0{\cdot}9\,,$$

for $i = 1, \ldots, 7$.

Then the system of probabilities $\mathscr{P}_N$ is determined by the probabilistic parametres (with the help of formulas (1) and (2)). However, we do not know anything about the output probabilities of this net. The corresponding LP-form is

$$\varphi_7((\varphi_5(\sim x_1 \vee x_3 \vee \varphi_1(\sim x_3)) \& x_2) \vee$$
$$\vee \varphi_6(x_1 \& \varphi_2(\sim x_2) \& \varphi_3(x_1)) \& \varphi(x_3)) \vee \varphi_4(x_2 \& x_3)\,.$$

Let us now summarize the foregoing considerationes:

Every LP-net $[N, \Omega_N \mathscr{P}_N]$ corresponds to the LP-expression $[F(N), \Omega_N, \mathscr{P}_N]$, where $F(N)$ is the LP-form decribing the labeled net N. Then we can interpret every theorem concerning LP-expressions as a theorem concerning LP-nets.

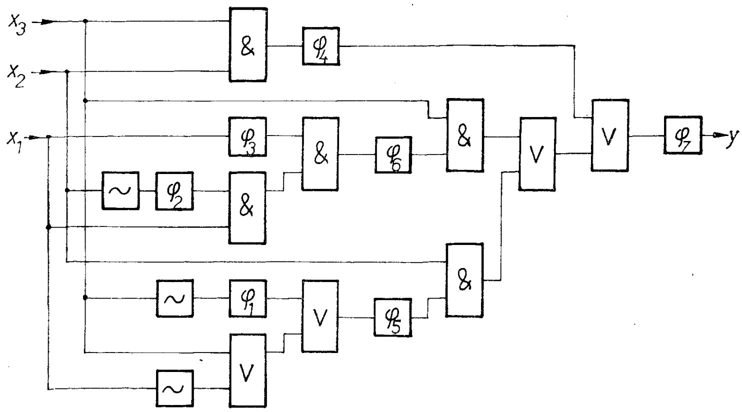

Fig. 7.

Note. If we have an LP-net we can call the LP-expression $[F(N), \Omega_N, \mathscr{P}_N]$ the *canonical expression* of the LP-net (see [5] by N. E. Kobrinskij and B. A. Trachtenbrot). In some works (e.g., in [6] by V. I. Levin) logical nets with elements corresponding to Λ, &, $\sim$ are considered, where these elements work with errors. Later we shall show that we can replace these nets by our LP-nets.

Now we shall formulate a theorem.

Theorem 2. Let $\Phi = [F, \Omega_F, \mathscr{P}_F]$ be a LP-expression.

1) Let F contain only variables contained in an LP-form corresponding to *int* (φ_i, F) for a probabilistic connective φ_i from F. Then there exist two sets Ω_0, Ω_1

for which: a) $\Omega_F = \Omega_0 \cup \Omega_1$, $\Omega_0 \cap \Omega_1 = \emptyset$; b) $func_F(\sigma, \omega) = 1$ iff $\omega \in \Omega_1$, and there exists an unique system of probabilities $\mathscr{P}'_F = \{P_\sigma(.)\}_\sigma$ on $\Omega^* = \{\Omega_0, \Omega_1\}$, for which c) $P(func_F(\sigma, \omega) = 1) = P_\sigma(\Omega_1)$.

2) Let $x_{j_1}, \ldots, x_{j_s}$ be variables which occur in F and which are not contained in any LP-form corresponding to $int(\varphi_i, F)$ for any φ_i from F. Let $\Sigma = \{0, 1\}^s$ be the space of values of these variables. Let us denote $\Omega'_F = \Sigma' \times \Omega_F$. Then there exist two sets Ω_0, Ω_1 for which: a) $\Omega'_F = \Omega_0 \cup \Omega_1$, $\Omega_0 \cap \Omega_1 = \emptyset$, b) $func_F(\sigma, \omega) = 1$ iff $(\sigma', \omega) \in \Omega_1$, where $\sigma' = (\sigma_{j_1}, \ldots, \sigma_{j_s})$, and a unique system of probabilities $\mathscr{P}'_F = \{P_\sigma(.)\}_\sigma$ on $\Omega^* = \{\Omega_1, \Omega_0\}$ for which: c) $P(func_F(\sigma, \omega) = 1) = P_\sigma(\Omega_1)$ and d) for $\sigma \in \{0, 1\}^n$, for which $(\sigma', \omega) \notin \Omega_1$ for every $\omega \in \Omega_F$, is $P_\sigma(\Omega_1) = 0$.

Note. The proof is given in Part II of this paper. As a consequence of this theorem we can describe the probabilistic properties of LP-expressions by the vector

$$\boldsymbol{p}_\Phi = (P_{0,\ldots,0}(\Omega_1), P_{0,\ldots,0,1}(\Omega_1), \ldots, P_{1,\ldots,1}(\Omega_1)) .$$

This vector will be called the characteristic vector of the given LP-expression. A general method of calculation of $\boldsymbol{p}_\Phi$ will be explained in the part II. Computation of these characteristic vectors in particular cases will be discussed in [11].

Example 3. This example is a continuation of Example 2. For the LP-expression (LP-net) decribed in Example 2 we obtain:

i	σ	p_Φ	ξ
1	000	0·2514	0
2	001	0·2514	0
3	010	0·7744	1
4	011	0·8872	1
5	100	0·2000	0
6	101	0·3117	0
7	110	0·3304	0
8	111	0·8872	1

where ξ are values of usual evaluation of LP-expression ($\bigwedge$ substituted for φ_j).

A convenient method for computation of $\boldsymbol{p}_\Phi$ in this case (stochastical independence) will be described in [11].

The connection of the LP-expressions (and LP-nets) to the probabilistic operators (automata) is very important. We define the probabilistic operator in accordance with definition of the probabilistic operator given by Rabin in [8].

A triplet $[A, \mathscr{P}_A, B]$, where $A = \{a_1, \ldots, a_n\}$ is the input alphabet, $B = \{b_1, b_2\}$ is the output alphabet and $\mathscr{P}_A$ is a system of probabilities on B, $\mathscr{P}_A = \{P_a\}_{a \in A}$, is called a *probabilistic operator* with binary output.

We denote $p_i = P_{a_i}(b_2)$ and we call $\boldsymbol{p} = (p_1, \ldots, p_n)$ the characteristic vector of a probabilistic operator with binary output. Let us define $|\Omega_i| = i$. Now if we define $A = (a_1, \ldots, a_{2^m})$, where $a_i = \sigma_i$, if σ_i is the binary form of the number $i - 1$, and if $B = (|\Omega_0|, |\Omega_1|)$, then $\Phi = [F, \Omega_F, \mathscr{P}_F]$ will determine a probabilistic operator with binary output.

For further considerations we must define two kinds of equivalence between LP-expressions.

Definition 9. Let $\Pi_1 = [A_1, \mathscr{P}_{A_1}, B_1]$ and $\Pi_2 = [A_2, \mathscr{P}_{A_2}, B_2]$ be two probabilistic operators with binary outputs. If we can find such a one-to-one mapping ψ of A_1 onto A_2 for which $P_a(b_1^1) = P_{\psi(a)}(b_1^2)$ or every $a \in A_1$, we say that Π_1 and Π_2 are equivalent.

If Φ_1 and Φ_2 are two LP-expressions and if $p_{\Phi_1} = p_{\Phi_2}$ we say that they are *probabilistically equivalent* ($\Phi_1 \equiv_p \Phi_2$).

Let us note that the problem of the construction of a probabilistic operator (or automaton) is the same as of finding to a given probabilistic operator an equivalent operator with a given structure, that is in particular a probabilistically equivalent LP-expression with a given LP-form. For the problem of construction of stochastical automata see, e.g., R. Knast [4]. The author intends to deal with the construction of these automata in [12].

Before formulating the next lemma we must define some auxiliary concepts: 1) We shall call a logical form to which probabilistic connective could be substituted (see Def. 8, point 1), 2)) a *LP-form.* 2) Two LP-forms F_1, F_2 will be called equivalent (shortly $F_1 \equiv F_2$) iff they differ in subscript of probabilistic connectives only. 3) Let T be a transformation of LP-form F, let the following steps be used consequently in this transformation: a) changing the names of variables (different variables must have different names), b) using the commutativity of some logical connectives and c) the associativity of some logical connectives (associativity and commutativity are the same as that used in L-expressions), d) ommiting brackets (we can write $F_1 \vee F_2 \vee F_3$ for $F_1 \vee (F_2 \vee F_3)$). Two LP-forms F_1 and F_2 will be called *slightly equivalent* (shortly $F_1 \equiv_s F_2$) iff there is a transformation for which $T(F_1) \equiv F_2$.

Lemma 1. Let $\Phi = [F, \Omega_F, \mathscr{P}_F]$ and $\Phi' = [F', \Omega_{F'}, \mathscr{P}_{F'}]$ be two LP-expressions, let $F \equiv_s F'$, let for every $\gamma \in \{0, 1\}^m$, $P_F(\gamma, \omega) = P_{F'}(\gamma, \omega)$, and let the probabilistic connectives in F have such indices so that $T(F) = F'$. Then $\Phi \equiv_p \Phi'$.

Proof. The lemma is a consequence of the proof of Theorem 2 (cf. Part II).

Note. If we have two probability spaces $[\Omega_1, P_1]$, $[\Omega_1, P_2]$ we shall call every probability on $\Omega_1 \times \Omega_2$, for which a) $\sum_{\omega_1} P(\omega_1, \omega_2) = P_2(\omega_2)$ and b) $\sum_{\omega_2} P(\omega_1, \omega_2) = P_1(\omega_1)$, as a probability associated to P_1, P_2.

Now we define the second kind of equivalence.

Definition 10. Let Φ_1, Φ_2 be two LP-expressions, let $\mathscr{P}^s = \{P^s_\sigma\}$ be a system of probabilities on $\Omega_1^* \times \Omega_2^*$ associated to $\mathscr{P}'_{F_1}$, $\mathscr{P}'_{F_2}$. We say that Φ_1 and Φ_2 are *functionally equivalent* with respect to $\mathscr{P}^s$ (shortly $\Phi_1 \equiv_f \Phi_2$) iff for every $\sigma \in \{0, 1\}$ (we assume equal numbers of variables)

$$P^s_\sigma(\Omega^1_i, \Omega^2_j) = \delta_{ij}\, P^1_\sigma(\Omega^1_i)$$

(i.e., $\sum_{i \neq j} P^s_\sigma(\Omega^1_i, \Omega^2_j) = 0$, $\sum_{i=j} P^s_\sigma(\Omega^1_i, \Omega^2_j) = 1$, $\Omega^1_i \in \Omega^*_1$, $\Omega^2_j \in \Omega^*_2$).

Let $\Phi = [F, \Omega_F, \mathscr{P}_F]$ be a LP-expression, let Φ_1, Φ_2 be its two subexpressions, and let Φ_1, Φ_2 have no common subexpression. We will call these subexpressions functionally equivalent, if for every γ', ω'

$$P'(\gamma', \sigma; \omega^*_1, \omega^*_2, \omega) = \delta_{|\omega_1^*|, |\omega_2^*|} \sum_{\omega_2^*} P'(\gamma', \sigma; \omega^*_1, \omega^*_2, \omega')\,,$$

where $\{i_1, \ldots, i_k\} = \{1, \ldots, n\} - I_{1,2}$, $\omega' = (\omega_{i_1}, \ldots, \omega_{i_k})$, $\gamma' = (\gamma_{i_1}, \ldots, \gamma_{i_k})$, if F_1 and F_2 contains probabilistic connectives φ_i, $i \in I_{1,2}$, and ω^*_1, ω^*_2, $P'(\gamma', \sigma; \omega^*_1, \omega^*_2, \omega')$ have the same meaning for Φ_1, Φ_2 as $\omega^* \in \Omega^*$ nad P'_σ for the LP-expression Φ (for more details see the method of calculation in Part II).

The LP-subexpression of an LP-expression is defined in accordance with the definition of an LP-subnet of an LP-net (Def. 6). Every subform of the LP-form determines then an LP-subexpression. If we write $F_1 \prec F$ we mean that the LP-subexpression determined by F_1 is an LP-subexpression of the LP-expression $[F, \Omega_F, \mathscr{P}_F]$.

Before formulating the next lemma we must mention, in addition, a further type of equivalence of LP-forms. We say that two LP-forms F_1 and F_2 are *strongly equivalent* (shortly $F_1 \simeq F_2$) if there exists a transformation T such that $T(F_1) = F_2$. In the previous lemma we do not need $F_1 \simeq F_2$, we need only the corresponding ranking of probabilistic connectives in $T(F_1)$ and F_2.

Lemma 2. Let $\Phi = [F, \Omega_F, \mathscr{P}_F]$ be a LP-expression with subexpressions Φ_1 and Φ_2, determined by subforms F_1 and F_2 of F respectively, let F contain probabilistic connectives with indices $\{i_1, \ldots, i_k\} = I$ and F_2 with indices $\{j_1, \ldots, j_l\} = J$, and: 1) let F be a one-to-one mapping from I onto J such that $F_1 \simeq F'_2$, where F'_2 is obtained from F_2 by the substitution of $\varphi_{\psi^{-1}(j)}$ for φ_j, $j = 1, \ldots, l$, 2) let $P(\gamma, \omega) = 0$ if $\omega_i \neq \omega_{\psi(i)}$ and $\gamma_i = \gamma_{\psi(i)}$ for some $i \in I$. Then $\Phi_1 \equiv_f \Phi_2$.

Proof. The lemma is a consequence of the method of calculation of $P'(\gamma', \sigma; \omega^*_1, \omega^*_2, \omega')$ (see Part II, calculation od p_Φ).

Now we shall define two kinds of normal forms of LP-expressions.

Definition 11. We say that a LP-expression $\Phi = [F, \Omega_F, \mathscr{P}_F]$ is in *probabilistic disjunctive normal form* (*PDNF*) if its LP-form is expressed as follows: for every $\varphi_i(F') \prec F$ is $F \simeq F_1 \vee \ldots \vee F_k$, where $F_i = \varepsilon_1 F_{i_1} \& \ldots \& \varepsilon_l F_{i_l}$ $(i = 1, \ldots, k)$, where F_{ij} is either a variable or an LP-subform of the type $\varphi_j(F'')$, and ε_j is either $\sim$ or $\wedge$.

We can now formulate an interesting theorem about the PDNF.

Theorem 3. Let $\Phi = [F, \Omega_F, \mathscr{P}_F]$ be a LP-expression containing probabilistic connectives $\varphi_1, \ldots, \varphi_n$. We can construct a LP-expression Φ' in the PDNF containing probabilistic connectives $\varphi_1, \ldots, \varphi_n, \varphi_{11}, \ldots, \varphi_{1n_1}, \ldots, \varphi_{nn_n}$ such that

(1)

$$P_{F'}(\gamma'; \omega') = \begin{cases} P_F(\gamma; \omega) \text{ if } \omega_1 = \omega_{11} = \ldots = \omega_{1n_1}, \ldots, \omega_n = \omega_{n1} = \omega_{n2} = \ldots = \omega_{nn_n} \\ 0 \text{ otherwise}, \end{cases}$$

and which is functionally equivalent to Φ with respect to the system of probabilities associated with $\mathscr{P}'_F$, $\mathscr{P}'_{F'}$ and defined by the system (1).

The proof together with some other theorems is included in the second part of this paper.

Note. It is important that the PDNF can be constructed without knowledge of p_Φ.

It is useful to define the degree of a probabilistic connective in the given LP-expression. Let $[F, \Omega_F, \mathscr{P}_F]$ be an LP-expression. We say that a probabilistic connective φ occuring in F has degree 0 if no other probabilistic connectives occur in $int(\varphi, F)$. We say that φ has the degree n ($d(\varphi) = n$ or $d(\varphi, F) = n$) if in $int(\varphi, F)$ a probabilistic connective of $d(\varphi, F) = n - 1$ occurs, but no probabilistic connective occurs with a higher degree.

Let $\{\varphi_{j_1}, \ldots, \varphi_{j_k}\}$ be probabilistic connectives occuring in F. Let us denote $r(\varphi_j)$ the rank of φ_j in $(\varphi_{j_1}, \ldots, \varphi_{j_k})$. Then it holds that: 1) $r(\varphi_j) \leqq r(\varphi_1)$ iff $d(\varphi_j) \leqq d(\varphi_i)$, 2) $r(\varphi_j) < r(\varphi_i)$ implies $d(\varphi_j) < d(\varphi_i)$, 3) if $d(\varphi_j) = d(\varphi_i)$, then for every $l \in \mathscr{N}$ such that $r(\varphi_i) \leqq r(\varphi_l) \leqq r(\varphi_j)$ is $d(\varphi_l) = d(\varphi_i)$.

Note. When keeping the notation from Theorem 3, it holds that:

$$d(\varphi_i, F) = d(\varphi_{i1}, F') = \ldots = d(\varphi_{in_i}, F') = d(\varphi_i, F') \quad (i = 1, \ldots, n).$$

Thus we can say that a transformation to the PDNF preserves the position of the probabilistic connectives in this manner: 1) it preserves the degree of a connective, 2) the interior of every probabilistic connective in Φ' (as an LP-subexpression) is functionally equivalent to the interior of the probabilistic connective in Φ. This fact is very important in view of the aplication to nets with probabilistic elmeents. It makes it possible for us to concern ourselves, in structural considerations, with LP-expression in the PDNF only. Other applications of the Theorem 3 are useful in the case of computation of characteristic vectors. We can make the transformation without considering the probabilistic characteristic. Thus we can again concern ourselves, in computation characteristic vectors, with this special case making possible a considerable simplifications, especially in the case of stochastical independence.

Now, there is a problem. Let us consider a stochastically independent LP-expression Φ and the corresponding LP-expression in the PDNF Φ'. As we shall see in the proof of Theorem 3, Φ' does not need to be stochastically independent. Φ' can contain stochastically independent groups of functionally equivalent connectives, i.e.

$$P_{F'}(\gamma'; \omega) = P(\gamma_1, \gamma_{11}, \ldots, \gamma_{1n_1}; \omega_1, \omega_{11}, \ldots, \omega_{1n_1}) \times$$
$$\times \ldots \times P(\gamma_n, \gamma_{n_1}, \ldots, \gamma_{n_n}; \omega_n, \omega_{n_1}, \ldots, \omega_{n_n})$$

for every $\gamma' \in \{0, 1\}^{n+\Sigma n_j}$.

The class of stochastically independent LP-expressions is not closed to the transformation to LP-expressions in PDNF. This problem as well as the problem of the computation of $\boldsymbol{p}_\Phi$ and the problems connected with the usage of elements of the fork-junction in nets shall be pursued in greather detail in another papers [11] and [12].

It is possible to define another kind of normal form.

Definition 12. We say that a LP-expression $\Phi = [F, \Omega_F, \mathscr{P}_F]$ is in *simple probabilistic disjunctive normal form (SPDNF)* if its LP-form

$$F \simeq \bigvee_{i=1}^{k} (\varepsilon_{i_1} F_{i_1} \,\&\, \ldots \,\&\, \varepsilon_{i_{k_i}} (F_{i_{k_i}})\,,$$

where each F_{i_j} is either a variable or of the form $\varphi_l(x_s)$ for some l, s, and

$$P_F(\gamma; \omega) = \prod_{i=1}^{n} P_i(\gamma_i; \omega_i)$$

for every $\gamma \in \{0, 1\}^n$.

The following theorem holds.

Theorem 4. To every LP-expression Φ we can construct a probabilistically equivalent LP-expression Φ', which is in the SPDNF.

Proof. Let the characteristic vector of Φ be $\boldsymbol{p}_\Phi = (p_1, \ldots, p_{2^m})$ let

$$x_i^{\varepsilon_i} = \begin{cases} x_i & \text{if} \quad \varepsilon_i = 1\,, \\ \sim x_i & \text{if} \quad \varepsilon_i = 0\,, \end{cases} \quad (i = 1, \ldots, m)\,.$$

We can denote $\boldsymbol{\varepsilon}_i = (\varepsilon_i^1, \ldots, \varepsilon_m^i)$ the binary form of the number $i - 1$. Let $\Phi' = [F', \Omega_{F'}, \mathscr{P}_{F'}]$ be a LP-expression for which

$$F'(x_1, \ldots, x_m) \simeq \bigvee_{i=1}^{2^m} \varphi_i(x_1^{\varepsilon_1{}^i}) \,\&\, x_2^{\varepsilon_2{}^i} \,\&\, \ldots \,\&\, x_m^{\varepsilon_m{}^i}$$

and

$$P_{F'}(\gamma'; \omega') = \prod_{i=1}^{2^m} P_i(\gamma_i; \omega_i)$$

for every $\gamma \in \{0, 1\}^m$ and let $P_i(0; 1) = 0$, $P_i(1; 1) = p_i$ $(i = 1, \ldots, 2^m)$.
This LP-expression is in the SPDNF and $\boldsymbol{p}_{\Phi'} = \boldsymbol{p}_\Phi$. □

For the construction of SPDNF, in contrast with the PDNF, we need to know the characteristic vector of LP-expression. Then we have the advantage of a considerably simplified form. This normal form is useful in the application to the realization of the probabilistic operators with binary output, where the characteristic vector is known and we need the form of the resulting expression (and of corresponding net) to be as simple as possible. By the resulting expression we mean an LP-expression having the same characteristic vector as the given probabilistic operator. The realization of LP-net corresponding to SPDNF may be simplified, using elements of the fork-junction. Realization of probabilistic operators with multiple output can also be based on SPDNF, but the whole realization is much more complicated. The author will pursue this problem, and the problem of minimization, in a special paper [12]. Probabilistic operators with multiple output are connected with vectors of LP-expressions, which we will now define.

Definition 13. Let us consider $\boldsymbol{F} = [F_1, \ldots, F_k]$ the k-tuple of LP-forms. This k-tuple will be called the *vector of LP-forms* if the following condition holds:

Let $\{\varphi_{i_1}, \ldots, \varphi_{i_n}\}$ be the probabilistic connectives occuring in $\boldsymbol{F}$, let $r(\varphi_i)$ denote the rank of φ_i in $(\varphi_{i_1}, \ldots, \varphi_{i_n})$, and let φ_i occur in F_r and φ_j in F_s. Then $r < s$ implies $r(\varphi_i) < r(\varphi_j)$.

Note. It follows that for no $r, s = 1, \ldots, k$, $r \neq s$, there exist φ_j which occurs in F_r and in F_s.

Definition 14. Let us consider a vector of LP-forms $\boldsymbol{F} = [F_1, \ldots, F_k]$; for every F_i let Ω_{F_i} be as in Def. 8. We define $\Omega_{\boldsymbol{F}} = \mathsf{X}_{i=1}^k \Omega_{F_i}$ (we assume that $r(\varphi_i, F_s) < r(\varphi_j. F_s)$ implies $r(\varphi_i) < r(\varphi_j)$). For the pair $[\boldsymbol{F}, \Omega_{\boldsymbol{F}}]$ we define a mapping $func_{\boldsymbol{F}}$ from $\Sigma \times \Omega_{\boldsymbol{F}}$ to $\{0, 1\}^k$ (where $\Sigma = \{0, 1\}^m$ if $\boldsymbol{F}$ contains, at most, the variables $x_1, \ldots, x_m$) in this way:

$$\begin{aligned} func_{\boldsymbol{F}}(\boldsymbol{\sigma}, \omega) = (&func_{F_1}(\boldsymbol{\sigma}, \omega_1, \ldots, \omega_{n_1}), \ldots, \\ &func_{F_l}(\boldsymbol{\sigma}, \omega_{\sum_{i=1}^{l-1} n_i + 1}, \ldots, \omega_{\sum_{i=1}^{l} n_i}), \ldots, \\ &func_{F_k}(\boldsymbol{\sigma}, \omega_{\sum_{i=1}^{k-1} n_i + 1}, \ldots, \omega_n)) \,. \end{aligned}$$

Let $\boldsymbol{F}$ be a vector of LP-forms, let $\mathscr{P}_{\boldsymbol{F}}$ be a system of probabilities satisfying conditions 3) and 4) from Def. 5. Then the triplet $[\boldsymbol{F}, \Omega_{\boldsymbol{F}}, \mathscr{P}_{\boldsymbol{F}}]$ will be called a *vector of LP-expressions.*

It is possible to formulate a theorem analogical to Theorem 2.

Theorem 5. Let $\Phi = [\boldsymbol{F}, \Omega_{\boldsymbol{F}}, \mathscr{P}_{\boldsymbol{F}}]$ be a vector of LP-expressions. 1) Let $\boldsymbol{F}$ contain only variables contained in an LP-form corresponding to $int(\varphi_i)$ for some φ_i from $\boldsymbol{F}$. Then there exist the sets $\Omega_1, \ldots, \Omega_{2^k}$ for which: a) $\Omega_1, \ldots, \Omega_{2^k}$ are disjoint, $\Omega_\xi \subset \Omega_{\boldsymbol{F}}$ $(\xi = 1, \ldots, 2^k)$, $\bigcup_\xi \Omega_\xi = \Omega_{\boldsymbol{F}}$; b) $func_{\boldsymbol{F}}(\boldsymbol{\sigma}, \omega) = \boldsymbol{\xi}$ iff $(\omega \in \Omega_\xi$ where $\boldsymbol{\xi}$ is a binary form of the number $\xi - 1$, and a system of probabilities $\mathscr{P}' = \{P_{\boldsymbol{\sigma}}(.)\}_{\boldsymbol{\sigma}}$ on $\Omega^* =$

$= \{\Omega_1, \ldots, \Omega_{2^k}\}$ for which: c) $P(func_{\boldsymbol{F}}(\boldsymbol{\sigma}, \boldsymbol{\omega}) = \xi) = P_{\boldsymbol{\sigma}}(\Omega_\xi)$ ($\xi = 1, \ldots, 2^k$). 2) Let $x_{j_1}, \ldots, x_{j_l}$ be variables occuring in the vector of LP-forms $\boldsymbol{F}$ and not contained in any form corresponding to $int(\varphi_i)$ for any φ_i from $\boldsymbol{F}$. Let $\Omega'_{\boldsymbol{F}} = \mathsf{X}_{i=1}^{l} \Sigma_{i_j} \times \Omega_{\boldsymbol{F}}$. Then there exist the sets $\Omega_1, \ldots, \Omega_k$ for which: a) $\Omega_1, \ldots, \Omega_{2^k}$ are disjoint, $\Omega_\xi \subseteq \Omega'_{\boldsymbol{F}}$, $\bigcup_\xi \Omega_\xi = \Omega'_{\boldsymbol{F}}$ ($\xi = 1, \ldots, 2^k$); b) $func_{\boldsymbol{F}}(\boldsymbol{\sigma}, \boldsymbol{\omega}) = \xi$ iff $(\boldsymbol{\sigma_F}, \boldsymbol{\omega}) \in \Omega_\xi$ ($\boldsymbol{\sigma_F} = (\sigma_{j_1}, \ldots, \sigma_{j_l})$, and a system of propabilities $\mathscr{P}' = \{P_{\boldsymbol{\sigma}}(.)\}_{\boldsymbol{\sigma}}$ on $\Omega^* = \{\Omega_1, \ldots, \Omega_{2^k}\}$ for which: c) $P(func_{\boldsymbol{F}}(\boldsymbol{\sigma}, \boldsymbol{\omega}) = \xi) = P_{\boldsymbol{\sigma}}(\Omega_\xi)$.

Note. We can define a probabilistic operator as a triplet $\Pi = [A, \mathscr{P}_A, B]$, where $A = (a_1, \ldots, a_n)$ is an input alphabet, $B = (b_1, \ldots, b_m)$ is an output alphabet and $\mathscr{P}_A = \{P_a\}_{a \in A}$ is a system of probabilities on B. As in the case of Theorem 2 we can define $|\Omega_\xi| = \xi$ and then a vector of LP-expressions determines a probabilistic operator with the output alphabet $B = (|\Omega_1|, \ldots, |\Omega_{2^k}|)$.

II. PROOFS

In this part we will present proofs of Theorems 2, 3 and 5 from Part I. Also two auxiliary theorems (6 and 7) will be formulated and proved. These theorems have an importance of their own for the computation of the characteristic vectors of LP-expressions and for proving assertions concerning the functional and probabilistic equivalence of some actual LP-expressions. First we will prove Theorem 2.

Proof of Theorem 2: 1) Using the definition of the mapping $func_F$ we obtain $func_F(\boldsymbol{\sigma}, \boldsymbol{\omega}) = func_F(\boldsymbol{\sigma}', \boldsymbol{\omega})$ for every $\boldsymbol{\sigma}, \boldsymbol{\sigma}' \in \Sigma$. Then we can define a mapping $func_F^*$ (from Ω to $\{0, 1\}$): $func_F^*(\boldsymbol{\omega}) = func_F(\boldsymbol{\sigma}, \boldsymbol{\omega})$ (for some $\boldsymbol{\sigma} \in \Sigma$). Then $\Omega_0 = (func_F^*)^{-1}(0)$ and $\Omega_1 = (func_F^*)^{-1}(1)$, where $(func_F^*)^{-1}(\xi)$ is the inverse image of ξ. This mapping preserves all set operations and so $\Omega_0 \cap \Omega_1 = \emptyset$ and $\Omega_0 \cup \Omega_1 = \Omega_F$.

Let $\varphi_{k+1}, \ldots, \varphi_n$ be probabilistic connectives contained in $int(\varphi_j)$ for no other probabilistic connective. We denote $\boldsymbol{\omega}_l = (\omega_1, \ldots, \omega_l)$, $\boldsymbol{\omega}'_l = (\omega_{l+1}, \ldots, \omega_n)$ and define $\Omega'_F = T(\Omega_F) = \{\boldsymbol{\omega}'_k; \boldsymbol{\omega} \in \Omega_F\}$, $\Omega'_1 = T(\Omega_1) = \{\boldsymbol{\omega}'_k; \boldsymbol{\omega} \in \Omega_1\}$, $\Omega'_0 = T(\Omega_0) = \{\boldsymbol{\omega}'_k; \boldsymbol{\omega} \in \Omega_0\}$. Then $func_F^*(\boldsymbol{\omega}^1_k, \boldsymbol{\omega}'_k) = func_F^*(\boldsymbol{\omega}^2_k, \boldsymbol{\omega}'_k)$ holds for every $\boldsymbol{\omega}^1_k, \boldsymbol{\omega}^2_k \in \{0, 1\}^k$ (it follows from Def. 5, point e)). From Def. 5 we know that $\gamma_i = func_F \,.\, int(\varphi_i, F)(\boldsymbol{\sigma}, \boldsymbol{\omega})$ for $i = 1, \ldots, n$. So we have a mapping $\boldsymbol{f}(=\boldsymbol{\gamma})$ from $\Sigma \times \Omega^k$ to Γ, where $\Omega^k = \{\boldsymbol{\omega}_k; \boldsymbol{\omega} \in \Omega_F\}$ and $\Gamma = \{0, 1\}^n$. The value of the mapping $\boldsymbol{f}$ can be written as $\boldsymbol{f}(\boldsymbol{\sigma}, \boldsymbol{\omega}_k) = (f_1(\boldsymbol{\sigma}, \boldsymbol{\omega}_k), \ldots, f_n(\boldsymbol{\sigma}, \boldsymbol{\omega}_k))$. By the Def. 5 we can see that a given f_i depends on $(\boldsymbol{\sigma}, \boldsymbol{\omega}_{i-1})$ only. It follows that

$$\boldsymbol{f}(\boldsymbol{\sigma}, \boldsymbol{\omega}_k) = (f_1(\boldsymbol{\sigma}), f_2(\boldsymbol{\sigma}, \boldsymbol{\omega}_1), \ldots, f_n(\boldsymbol{\sigma}, \boldsymbol{\omega}_{n-1}))\,.$$

Now we can define

$$(1) \qquad P'_{\boldsymbol{\sigma}}(\Omega_1) = \sum_{\boldsymbol{\omega}_k' \in \Omega_1'} \sum_{\omega_1} \cdots \sum_k P(\boldsymbol{f}(\boldsymbol{\sigma}, \boldsymbol{\omega}_k); \boldsymbol{\omega})$$

and

$$P'_\sigma(\Omega_0) = \sum_{\omega_k' \in \Omega_0'} \sum_{\omega_1} \dots \sum_{\omega_k} P(f(\sigma, \omega_k); \omega) .$$

We must show that for a given σ, $P'_\sigma(.)$ is a probability on Ω^*. It is clear that $P'_\sigma(.)$ is a non negative and additive function. We try to show that $P'_\sigma(\Omega_F) = 1$. We have

$$(2) \qquad P'_\sigma(\Omega_F) = P'_\sigma(\Omega'_F) = = \sum_{\omega_k'} \sum_{\omega_1} \dots \sum_{\omega_k} P(f(\sigma, \omega_k); \omega) = = \sum_{\omega_1} \sum_{\omega_2} \dots \sum_{\omega_k} \sum_{\omega_k'} P(f(\sigma, \omega_k); \omega) .$$

By the definition 5, point 4), we see that $\sum_{\omega_k'} P((f_1, \dots, f_k, f_{k+1}, \dots, f_n); \omega)$ does not depend on $f_{k+1}, \dots, f_n$ and we obtain

$$(3) \qquad (2) = \sum_{\omega_1} \dots \sum_{\omega_k} P'((f_1(\sigma), \dots, f_k(\sigma, \omega_{k-1})); \omega_k)$$

From the same condition it follows that

$$P''(f_1, \dots, f_k; \omega_{k-1}) = \sum_{\omega_k} P'(f_1, \dots, f_k; \omega_k) = P''(f_1, \dots, f_{k-1}, \omega_{k-1}) ,$$

and so

$$(3) = \sum_{\omega_1} \dots \sum_{\omega_{k-1}} P''(f_1, \dots, f_{k-1}; \omega_{k-1}) = \dots = \sum_{\omega_1} P^{(k)}(f(\sigma); \omega_1) = 1 .$$

The uniquenes follows clearly from (1).

2) Applying Def. 5, point e), once more, we obtain that

$$func_F(\sigma, \omega) = func_F(\sigma', \omega)$$

for

$$\sigma, \sigma' \in \Sigma , \quad (\sigma_{j_1}, \dots, \sigma_{j_s}) = (\sigma'_{j_1}, \dots, \sigma'_{j_s}) ,$$

and the second assertion is a consequence of analogical considerations as the first assertion. □

Now we try to explain a general method of computation of characteristic vectors of LP-expressions, i.e., a method of computation of output probabilities for some LP-net, conditioned by its input values. The given metod has two advantages: 1) we do not need to find Ω_0, Ω_1, $func_F$ and f, 2) we can find probabilities like P'_σ (output probabilities) for every LP-subexpression of the given LP-expression.

Method. The computation is carried out by recursion on the degree of probabilistic connectives. Let Φ be a given LP-expression. Let l be a given degree, let $\Phi_1, \dots, \Phi_k$ be a sequence of subexpressions, the LP-forms of which are the maximal subforms containing no probabilistic connectives with $d(\varphi, F) = l$. Let γ', ω' be parameters and events of probabilistic connectives with $d(\varphi, F) \geqq l$. Then if ω_i^* are sets determining the value of $func_F F_i$ we calculate probabilities

$$P(\sigma, \gamma'; \omega_1^*, \dots, \omega_k^*, \omega') .$$

1) Let us define $\Omega'_F = \Sigma \times \Omega_F$ and denote $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\xi = (\xi_1, \ldots, \xi_m)$, $\omega = (\omega_1, \ldots, \omega_n)$. We consider Σ in the form $\Sigma = \mathsf{X}_i\, \Sigma_i$. We introduce on Ω'_F a system of probabilities

$$\mathscr{P}'_F = \{P(\gamma, \boldsymbol{\sigma}; \xi, \omega)\}_{\gamma,\boldsymbol{\sigma}},$$

$$P(\gamma, \boldsymbol{\sigma}; \xi, \omega) = \prod_{i=1}^{m} P_{\sigma_i}(\xi_i)\, P(\gamma; \omega),$$

where

$$P_{\sigma_i}(\xi_i) = \begin{cases} 1 & \text{if} \quad \sigma_i = \xi_i, \\ 0 & \text{if} \quad \sigma_i \neq \xi_i. \end{cases}$$

2) Now we proceed, as in the first step, with the calculation of probabilities on the maximal subforms (subexpressions) containing no probabilistic connective. The calculation must be proceeded by recursion on the length of subforms $F' \prec F_i$ where F_i is some maximal subform.

a) Let x_1 be a variable. We define $\Omega'_1 = \Sigma_1 \times \Omega_F$ and denote $\xi' = (\xi_2, \ldots, \xi_m)$. Now we have $\Omega_1^* = \{\{1\} \times \Omega_F, \{0\} \times \Omega_F\}$ and we obtain the induced system of probabilities

$$P'(\boldsymbol{\sigma}, \gamma; \xi_1, \xi', \omega) = P''(\boldsymbol{\sigma}, \gamma; \omega_1^*, \xi', \omega)$$

for $\omega_1^* = \{\xi_1\} \times \Omega_F$.

Analogically, we define Ω_i^* and Ω'_i for the variables $x_2, \ldots, x_m$ and succesively we obtain the system of probabilities $\{P(\boldsymbol{\sigma}, \gamma; \omega_1^*, \ldots, \omega_m^*, \omega)\}_{\gamma,\boldsymbol{\sigma}}$ on $\Omega_1^* \times \ldots \times \Omega_m^* \times \times\, \Omega_F$.

b) Let us consider subforms $F_1, \ldots, F_r$, $F_i \prec F'$, corresponding spaces $\Omega_1^*, \ldots, \Omega_r^*$ and, on $\Omega_1^* \times \ldots \times \Omega_r^* \times \Omega_F$, the system of probabilities

$$\{P(\boldsymbol{\sigma}, \gamma; \omega_1^*, \ldots, \omega_r^*, \omega\}_{\boldsymbol{\sigma},\gamma}.$$

To simplify the matter we shall assume that the following step will be carried out with F_1 or F_1, F_2.

There are two cases:

case ba) The LP-form is of the form $\sim F_1$; then we substitute Ω_1^1 for Ω_1^0 in Ω_1^* (i.e., $\Omega_1^{1'} = \Omega_1^0$) and we have

$$P'(\boldsymbol{\sigma}, \gamma; \Omega_1^{1'}, \omega_2^*, \ldots, \omega_r^*, \omega) = P(\boldsymbol{\sigma}, \gamma; \Omega_1^0, \omega_2^*, \ldots, \omega_r^*, \omega).$$

case bb) The LP-form is of the form $\alpha_j(F_1, F_2)$: Let $x_{j_1}, \ldots, x_{j_s}$ be variables occuring in F_1 or F_2, then $\Omega'_{F_1,F_2} = \mathsf{X}_{i=1}^{s}\, \Sigma_{j_i} \times \Omega_F$. Let us define $|\omega_i^*| = 1$ if $\omega_i^* = \Omega_i^1$ and $=0$ if $\omega_i^* = \Omega_i^0$. Then $\Omega^1 = \{(\sigma_{j_1}, \ldots, \sigma_{j_s}) \times \Omega_F;\ (\boldsymbol{\sigma}_{F_1}, \omega) \in \omega_1^*,\ (\boldsymbol{\sigma}_{F_2}, \omega) \in \omega_2^*$ and $\alpha_j^*(|\omega_1^*|, |\omega_2^*|) = 1\}$ and

$$P'(\boldsymbol{\sigma}, \gamma; \Omega^1, \omega_3^*, \ldots, \omega_r^*, \omega) = \sum_{\{\omega_1^*, \omega_2^*; \alpha_j^*(|\omega_1^*|, |\omega_2^*|) = 1\}} P(\boldsymbol{\sigma}, \gamma; \omega_1^*, \ldots, \omega_r^*, \omega).$$

No other case can occure.

In this way we proceed till $F_1, \ldots, F_s$ are the maximal subforms containing no φ_i.
Then we obtain a system of probabilities on $\Omega_1^* \times \ldots \times \Omega_s^* \times \Omega_F$ namely

$$P'(\boldsymbol{\sigma}, \gamma; \omega_1^*, \ldots, \omega_s^*, \boldsymbol{\omega}) = \prod_{i=1}^s P(\boldsymbol{\sigma}_i, \omega_i^*) \, P(\gamma; \omega)\,,$$

where $\boldsymbol{\sigma}_i = (\sigma_{j_1}, \ldots, \sigma_{j_l})$ if variables $x_{j_1}, \ldots, x_{j_l}$ occur in F_i and

$$\begin{aligned} P(\boldsymbol{\sigma}_i, \omega_i^*) &= 1 \quad \text{if} \quad \boldsymbol{\sigma}_i \times \Omega_F \subset \omega_1^*\,, \\ &= 0 \quad \text{in other cases}\,. \end{aligned}$$

3) To make the second part of the first step of recursion on the degree of probabilistic connectives, we proceed in the calculation of probabilities on the subexpressions having the form of the type $\varphi_i(F_j)$, where φ_i is a probabilistic connective with $d(\varphi_i, F) = 0$.

Let us consider a subexpression corresponding to subform $\varphi_1(F_1)$ and let us denote $\gamma' = (\gamma_2, \ldots, \gamma_n)$, $\boldsymbol{\omega}' = (\omega_2, \ldots, \omega_n)$. Then there is

$$\Omega^1 = \{1\} \times \mathsf{X}_{i=1}^n \, \Omega_i\,, \quad \Omega^0 = \{0\} \times \mathsf{X}_{i=1}^n \, \Omega_i$$

and

$$P'(\boldsymbol{\sigma}, \gamma'; \omega_2^*, \ldots, \omega_s^*, \Omega^1, \boldsymbol{\omega}') =$$
$$= \prod_{i=2}^s P(\boldsymbol{\sigma}_i, \omega_i^*) \left[P(\boldsymbol{\sigma}_1, \Omega_1^1) \, P(1, \gamma'; 1, \boldsymbol{\omega}') + P(\boldsymbol{\sigma}_1, \Omega_1^0) \, P(0, \gamma'; 1, \boldsymbol{\omega}')\right].$$

We proceed in this operation for all connectives of degree 0 step by step. (*Note.* If we denote a variable for events Ω^0, Ω^1 corresponding to $\varphi_1(F_1)$ as ω_1^{**}, we obtain

$$P'(\boldsymbol{\sigma}, \gamma'; \omega_2^*, \ldots, \omega_s^*, \omega_1^{**}, \boldsymbol{\omega}') = \prod_{i=2}^s P(\boldsymbol{\sigma}_i, \omega_1^*) \, P''(\boldsymbol{\sigma}_1, \gamma'; \omega_1^{**}, \boldsymbol{\omega}')\,.$$

(Then for $\varphi_2(F_2)$ we obtain

$$\begin{aligned} P'(\boldsymbol{\sigma}, \gamma''; \omega_3^*, \ldots, \omega_s^*, \omega_1^{**}, \Omega^{1\prime}, \boldsymbol{\omega}'') = \big[P(\boldsymbol{\sigma}_2, \Omega_2^1) \, P''(\boldsymbol{\sigma}_1, 1, \gamma''; \omega_1^{**}, 1, \boldsymbol{\omega}'') + \\ + P(\boldsymbol{\sigma}_2, \Omega_2^0) \, P''(\boldsymbol{\sigma}_1, 0, \gamma''; \omega_1^{**}, 1, \boldsymbol{\omega}'')\big] \prod_{i=3}^s P(\boldsymbol{\sigma}_i, \omega_i^*)\,, \end{aligned}$$

where $\boldsymbol{\omega}'' = (\omega_3, \ldots, \omega_n)$, $\gamma'' = (\gamma_3, \ldots, \gamma_n)$ etc.)

Let $\varphi_1, \ldots, \varphi_r$ be the probabilistic connectives with $d(\varphi) = 0$. We denote $\gamma' = (\gamma_{r+1}, \ldots, \gamma_n)$. Then we have $F_1, \ldots, F_r$ the subforms of the type $\varphi_i(F_i')$ and $F_{r+1}, \ldots, F_s$ subforms which contains no probabilistic connectives. We have $\Omega_i \subset \Omega_F$ $(i = 1, \ldots, r)$ and $\Omega_i^1 \subset \mathsf{X}_j \, \Sigma_{i_j} \times \Omega_F$ $(i = 1, \ldots, r)$. We obtain a system of probabilities on $\mathsf{X}_{i=1}^s \, \Omega_i^* \times \mathsf{X}_{i=r+1}^s \, \Omega_i$ namely

$$(1) \qquad P(\boldsymbol{\sigma}, \gamma; \omega_1^*, \ldots, \omega_s^*, \boldsymbol{\omega}') = \prod_{i=r+1}^s P_i(\boldsymbol{\sigma}_1, \omega_i^*) \, P(\boldsymbol{\sigma}, \gamma'; \omega_1^*, \ldots, \omega_s^*, \boldsymbol{\omega}')\,.$$

We can see that if $n \neq 0$ then $r \neq 0$ and then the dimension of $\boldsymbol{\omega}'$ is smaller than then dimension of $\boldsymbol{\omega}$.

4) Let us have performed the calculation of probabilities on maximal subexpressions not contanting any probabilistic connective of degree k and for subexpressions

of the form $\varphi_i(F')$, where φ_i is a probabilistic connective of degree k. Now we try to make the $(k + 2)$-th step of recursion. We are trying to find the maximal subexpressions (probabilities on subexpressions) not containing any probabilistic connective of degree $k + 1$. We proceed by recursion on the length of subforms. (It is important to note that we are performing two kinds of recursion: the first on the degree of probabilistic connectives and the second – for every step of the first one – on the length of subforms.) Let us assume that we have the following groups of subforms: a) $\{F_{r_2+1}, \ldots, F_s\}$ – subforms containing no probabilistic connectives, b) $\{F_{r_1+1}, \ldots, F_{r_2}\}$ – maximal subforms containing probabilistic connectives with $d(\varphi) < k$ and some subforms containing probabilistic connectives with $d(\varphi) = k$ and in which variables not contained in some $inf(\varphi, F)$ occur, c) $\{F_1, \ldots, F_{r_1}\}$ – the other subforms (i.e., subforms in which no variable occurs outside $inf(\varphi)$; some of these subforms can contain a probabilistic connective of degree k and need not be in form of the type $\varphi_i(F')$.

We have $\Omega_i^* = \{\Omega_i^0, \Omega_i^1\}$, where $\Omega_i^0, \Omega_i^1 \subset \Sigma_i \times \Omega_F$ for $i = r_1 + 1, \ldots, s$ and $\Omega_i^0, \Omega_i^1 \subset \Omega_F$ for $i = 1, \ldots, r_1$; $\Sigma_i = \mathsf{X}_{j=1}^k \Sigma_{i_j}$, if variables $x_{i_1}, \ldots, x_{i_k}$ occur in $F_{r_1+1}, \ldots, F_s$. If $\varphi_1, \ldots, \varphi_r$ are connectives with $d(\varphi) \leqq k$ then we consider a system of probabilities $\{P(\boldsymbol{\sigma}, \gamma); \omega_1^*, \ldots, \omega_s^*, \omega')\}_{\boldsymbol{\sigma},\gamma'}$ on $\mathsf{X}_{i=1}^s \Omega_i^* \times \mathsf{X}_{i=r+1}^n \Omega_i$, where $\gamma' = (\gamma_{r+1}, \ldots, \gamma_n)$, $\boldsymbol{\omega}' = (\omega_{r+1}, \ldots, \omega_n)$. Let us assume that $P(\boldsymbol{\sigma}, \gamma'; \omega_1^*, \ldots, \omega_s^*, \boldsymbol{\omega}') =$ $= 0$ if $\boldsymbol{\sigma}_i \times \Omega_F \cap \omega_i^* = \emptyset$ for some

$$(3) \qquad i = r_1 + 1, \ldots, s$$

(resp. $\Omega_F \cap \omega_i^* = \emptyset$ for some $i = s, \ldots, r_1$) and

$$(4) \qquad P(\boldsymbol{\sigma}, \gamma'; \omega_1^*, \ldots, \omega_s^*, \boldsymbol{\omega}') = 1 \quad \text{if} \quad \boldsymbol{\sigma}_i \times \Omega_F \subset \omega_i^*$$

for every $i = r_1 + 1, \ldots, s$ and $\omega_i^* = \Omega_F$ for every $i = 1, \ldots, r_1$.

In the calculation, we proceed in this way:

There are two cases:

Case a) The LP-form is of the type $\sim F_i$. Then we substitute Ω_i^1 for Ω_i^0 analogically as in 2), case ba). If F_i is from the group a), is $\sim F_i$ from the same group, analogically for other groups. With regards to

$$P'(\boldsymbol{\sigma}, \gamma'; \omega_1^*, \ldots, \omega_{i-1}^* \Omega_i^{1'}, \ldots, \omega_s^*, \boldsymbol{\omega}') = P(\boldsymbol{\sigma}, \gamma'; \omega_i^*, \ldots, \Omega_i^0, \ldots, \boldsymbol{\omega}')$$

the properties (3) and (4) are preserved.

Case b) The LP-form is of the type $\alpha_k(F_i, F_j)$. Then we can distinguish three subcases:

Subcase ba) F_i and F_j are from the group a) or b). We define

$$\Omega_{ij}^1 = \{(\boldsymbol{\sigma}_{ij}, \boldsymbol{\omega}); (\boldsymbol{\sigma}_i, \boldsymbol{\omega}) \in \omega_i^*, (\boldsymbol{\sigma}_j, \boldsymbol{\omega}) \in \omega_j^*, \alpha_k^*(|\omega_i^*|, |\omega_j^*|) = \},$$

where $\boldsymbol{\sigma}_{ij} = (\sigma_{j_1}, \ldots, \sigma_{j_t})$ if the variables $x_{j_1}, \ldots, x_{j_t}$ occur in F_i or F_j.

Subcase bb) If F_i is from the group c), F_j from a) or b), we define

$$\Omega^1_{ij} = \{(\sigma_j, \omega);\ \omega \in \omega^*_i,\ (\sigma_j, \omega) \in \omega^*_j,\ \alpha^*_k(|\omega^*_i|, |\omega^*_j|) = 1\}\,.$$

Subcase bc) F_i and F_j are both from c). We define

$$\Omega^1_{ij} = \{\omega;\ \omega \in \omega^*_i,\ \omega \in \omega^*_j,\ \alpha^*_k(|\omega^*_i|, |\omega^*_j|) = 1\}\,.$$

In all the subcases ba), bb) and bc) we obtain

$$P'(\sigma, \gamma'; \omega^*_1, \ldots, \omega^*_{i-1}, \Omega^1_{ij}, \omega^*_{i+1}, \ldots, \omega^*_{j-1}, \omega^*_{j+1}, \ldots, \omega^*_s, \omega') =$$
$$= \sum\nolimits_{\{(\omega_i^*, \omega_j^*); \alpha_k^*(|\omega_i^*|, |\omega_j^*|) = 1\}} P(\sigma, \gamma; \omega^*_1, \ldots, \omega^*_s, \omega')\,.$$

The properties (3) and (4) are then preserved.

Clearly no other subcases can occur. The properties (3) and (4) are required so that the second assertion of Theorem 2 holds.

5) Assume that the calculation of probabilities for the maximal subexpressions not containing any probabilistic connectives of degree $k + 1$ has been performed. To complete the $(k + 2)$-th step of recursion, we can calculate probabilities on subexpressions of the form $\varphi_i(F'_i)$, where $d(\varphi_i, F) = k + 1$. We define $\Omega^1_i = \{\omega;\ \omega_i = 1\}$ and we have

$$P(\sigma, \gamma_{r+1}, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_n; \omega^*_1, \ldots, \Omega^1_i, \ldots, \omega^*_s, \omega_{r+1}, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_n) =$$
$$\sum\nolimits_{\alpha \in \{0,1\}} P(\sigma, \gamma_{r+1}, \ldots, \gamma_{i-1}, \alpha, \gamma_{i+1}, \ldots, \gamma_n; \omega^*_1, \ldots, \Omega^\alpha_i, \ldots, \omega^*_s,$$
$$\omega_{r+1}, \ldots, \omega_{i-1}, 1, \omega_{i+1}, \ldots, \omega_n)\,.$$

6) Clearly, because the LP-form F is of the finite length (and so the number of probabilistic connectives is finite), the recursion must be finite. If $\max d(\varphi, F) = k$ then we need $k + 1$ steps. Each step has a finite number of substeps. The number of γ_i in $P(\sigma, \gamma'; \omega^*_1, \ldots, \omega^*_{s'}, \omega')$ is decreasing. After the $k + 1$ step we obtain a system of probabilities $P(\sigma; \omega^*_1, \ldots, \omega^*_s)$. If $s = 1$, then the calculation is finished. If $s > 1$, we must proceed in our calculation in the same way as in the point 4). There is now only one maximal subform and it is F. So we obtain $P''(\sigma, \omega^*)$ again. If we compare our calculation and the definition of the mapping $func_F$, we can see that $\Omega'_1 = \Omega''_1$ and $\Omega'_0 = \Omega''_0$, where Ω''_1 and Ω''_0 are the possible values of ω^*. Following the calculation from the point 5) and the uniquenes of $P'_\sigma(\Omega_i)$ (from Theorem 2), we can see that

$$P'_\sigma(\Omega_1) = P''(\sigma, \Omega''_1) \quad \text{and} \quad P'_\sigma(\Omega_0) = P''(\sigma, \Omega''_0)\,.$$

Note. The probabilities $P(\sigma, \gamma'; \omega^*_1, \ldots, \omega')$ and the condition 4) from Definition 5 make it possible to find

$$P(\sigma, \Omega^1_i) = \sum\nolimits_{\{(\omega_1^*, \ldots, \omega_s^*), \omega_i^* = \Omega_i^1\}} \sum\nolimits_{\omega'} P(\sigma, \gamma'; \omega^*_1, \ldots, \omega')$$

for every subexpression Φ_i. 2) Theorem 2 will hold if, instead of condition 4) from Definition 5, we give the following condition: for every

$$\gamma \in \{0, 1\}^n, \quad \sum_{\omega'' \in \{0,1\}^k} P(\gamma, \gamma''; \omega', \omega'')$$

does not depend on γ'' $(\gamma' = (\gamma_1, \ldots, \gamma_k), \gamma'' = (\gamma_{k+1}, \ldots, \gamma_n))$.

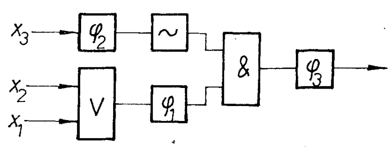

Fig. 8.

Example 4. Let N be a labeled net from the Fig. 8. Then $F(N) = \varphi_3(\varphi_1(x_1 \vee x_2) \,\&\, \sim\varphi_2(x_3))$. There are three probabilistic connectives, then $\Omega_F = \{0,1\}^3$. The values of probabilities from the system $\mathscr{P}_F$ are given in the following table:

γ \ ω	000	001	010	011	100	101	110	111
000	0·9	0·1	0	0	0	0	0	0
001	0·1	0·9	0	0	0	0	0	0
010	0·09	0·01	0·81	0·09	0	0	0	0
011	0·01	0·09	0·09	0·81	0	0	0	0
100	0·09	0·01	0	0	0·81	0·09	0	0
101	0·01	0·09	0	0	0·09	0·81	0	0
110	0·09	0·01	0	0	0	0	0·81	0·09
111	0·01	0·09	0	0	0	0	0·09	0·81

The reader can see that this system of probabilities fulfils the condition 4) from Def. 5. Now we will proceed in the calculation of the characteristic vectors following the previously described method:

1) $\Omega'_F = \{0,1\}^3 \times \{0,1\}^3$, $m = 3$, $n = 3$,

$$P(\gamma, \sigma; \xi, \omega) = \prod_{i=1}^{3} P_{\sigma_i}(\xi_i)\, P(\gamma; \omega);$$

2a) $\Omega_i^* = \{\{0, \omega\}, \{1, \omega\}\}$ for $i = 1, 2, 3$

and we obtain a system of probabilities given in the following matrix:

$$P(\sigma, \gamma; \omega_1^*, \omega_2^*, \omega_3^*, \omega))_{(\sigma,\gamma');(\omega^*,\omega)} = \begin{pmatrix} \boldsymbol{P} & \boldsymbol{0} & \ldots\ldots\ldots\ldots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P} & \boldsymbol{0} \ldots\ldots\ldots & \boldsymbol{0} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots \\ \boldsymbol{0} & \ldots\ldots & \ldots\ldots\ldots\ldots \boldsymbol{0} & \boldsymbol{P} \end{pmatrix}$$

where $\boldsymbol{P}$ is the matrix from the previous table. The usual lexicografical order has been used. Now we proceed following the point 2b): The first maximal subform if $F_1 = x_1 \vee x_2$, the second $F_2 = x_3$ we obtain system of probabilities

$$P'(\boldsymbol{\sigma}, \gamma; \omega_1^*, \omega_2^*, \boldsymbol{\omega}) = \prod_{i=1}^{2} P(\boldsymbol{\sigma}_i, \omega_i^*)\, P(\gamma; \omega)\,,$$

where the values of $\prod_{i=1}^{2} P(\sigma_i, \omega_i^*)$ are given in the following table:

σ \ ω_1^*, ω_2^*	00	01	10	11
000	1	0	0	0
001	0	1	0	0
010	0	0	1	0
011	0	0	0	1
100	1	0	0	0
101	0	1	0	0
110	0	0	1	0
111	0	0	0	1

Now we complete the first step of recursion following the point 3): We have two subforms $F_1 = \varphi_1(x_1 \vee x_2)$ and $F_2 = \varphi_2(x_3)$.

We obtain

$$P'(\boldsymbol{\sigma}, \gamma'; \omega_2^*, \Omega^1, \boldsymbol{\omega}')$$

in the form

$$P(\sigma_2, \omega_2^*)\left[P(\sigma_1, \Omega_1^1)\, P(1, \gamma'; 1, \omega') + P(\sigma_1, \Omega_1^0)\, P(1, \gamma'; 1, \boldsymbol{\omega})\right].$$

For example for $\boldsymbol{\sigma} = 101$ is

$$P(\sigma_2, \omega_2^*) = 0 \quad \text{if} \quad \omega_2^* = \Omega_2^0\,, \quad = 1 \quad \text{if} \quad \omega_2^* = \Omega_2^1$$

and

$$P(\sigma_1, \omega_1^*) = 0 \quad \text{if} \quad \omega_1^* = \Omega_1^0\,, \quad = 1 \quad \text{if} \quad \omega_1^* = \Omega_1^1$$

and then $P''(\sigma, \gamma'; \omega_1^{**}, \omega')$ have the values given in the following table:

γ' \ ω_1^{**}, ω'	000	001	010	011	100	101	110	111
00	0·09	0·01	0	0	0·81	0·09	0	0
01	0·01	0·09	0	0	0·09	0·81	0	0
10	0·09	0·01	0	0	0	0	0·81	0·09
11	0·01	0·09	0	0	0	0	0·09	0·81

In the next step we obtain (for $\varphi_3(x_3)$) the values of $P'(\sigma, \gamma'; \omega_1^*, \omega_2^*, \omega')$ (for $\boldsymbol{\sigma} = 101$):

	γ' \ $\omega_1^*, \omega_2^*, \omega'$	000	001	010	011	100	101	110	111
(2)	0	0·09	0·01	0	0	0	0	0·81	0·09
	1	0·01	0·09	0	0	0	0	0·09	0·81

Now we must proceed in our calculation for the maximal subform contained in φ_3. This subform is $\varphi_1(x_1 \vee x_2) \mathbin{\&} \sim\varphi_2(x_3) = F_1 \mathbin{\&} \sim F_2$. For $\sim F_2$ we obtain values of $P'(\sigma, \gamma'; \omega_1^*, \omega_2^*, \omega')$ by permutation of columns in (2): 000 → 010, 001 → 011, 100 → 110, 101 → 111.

Now for $F_1 \mathbin{\&} \sim F_2$ we obtain

γ' \ ω^*, ω'	00	01	10	11
0	0·9	0·1	0	0
1	0·1	0·9	0	0

and according to point 5) we obtain

$$p_\sigma = P(\sigma, 0; 0, 1) + P(\sigma, 1; 1, 1) = 0{,}1 .$$

To prove Theorem 3 from Part I we must formulate and prove two assertions.

Note. Let $F_1, \ldots, F_k$ are subforms of some LP-form F ($\Phi = [F, \Omega_F, \mathscr{P}_F]$). Let $F \simeq G(F_1, \ldots, F_k)$. By substitution x_{m+i} for F_i we obtain LP-form $G(x_{m+i}, \ldots, x_{m+k})$. Let $G(x_{m+i}, \ldots, x_{m+k})$ contain probabilistic connectives $\varphi_{j_1}, \ldots, \varphi_{j_{k_0}}$. Let variables $x_1, \ldots, x_m$ occur in F. If $\{j'_1, \ldots, j'_{n-k_0}\} = \{1, \ldots, n\} - \{j_1, \ldots, j_{k_0}\}$ then we can denote $\omega'' = (\omega_{j_1}, \ldots, \omega'_{j_{n-k_0'}})$, analog. γ'', $\boldsymbol{\omega} = (\omega_{j_1}, \ldots, \omega_{j_{k_0}})$, analog. γ'.

Now we can define a matrix

$$\boldsymbol{P} = (p_{\sigma,\zeta})_{\sigma,\zeta} ,$$

where

$$p_{\sigma,\zeta} = \sum_{\omega'} P'(\sigma, \gamma'; \Omega_{F_1}^{\zeta_1}, \ldots, \Omega_{F_k}^{\zeta_k}, \omega') .$$

Clearly $G(x_{m+1}, \ldots, x_{m+k})$ determines a LP-expression $[G, \Omega_G, \mathscr{P}_G]$, where $\Omega_G = \times_{i=1}^{k_0} \Omega_{j_i}$ and $\mathscr{P}_G = \{P^1(\gamma'; \omega')\}$; $\boldsymbol{p}_G$ is the characteristic vector of this LP-expression.

Theorem 6. Let $\Phi = [F, \Omega_F, \mathscr{P}_F]$ be a LP-expression, let $F \simeq G(F_1, \ldots, F_k)$. Let $P(\gamma; \boldsymbol{\omega}) = P^2(\gamma''; \omega'') \, P_1(\gamma'; \omega')$ for every $\gamma \in \{0, 1\}^n$. Then $\boldsymbol{p}_F = \boldsymbol{P} \,.\, \boldsymbol{p}_G$.

Proof. It is true that

$$P(\boldsymbol{\sigma}, \gamma'; \Omega_{F_1}^{\zeta_1}, \ldots, \Omega_{F_k}^{\zeta_k}, \boldsymbol{\omega}') = P^{2\prime}(\boldsymbol{\sigma}; \Omega_{F_1}^{\zeta_1}, \ldots, \Omega_{F_k}^{\zeta_k}) \, P^1(\gamma'; \boldsymbol{\omega}') ,$$

where $P^{2\prime}(\boldsymbol{\sigma}; .)$ was obtained by recursion from $P^2(\gamma''; \omega'')$. With given values of $\boldsymbol{\sigma}$ we compute $P_G(\zeta; \Omega_G^1)$ from $P^1(\gamma'; \boldsymbol{\omega}')$ following the previously described method, i.e.:
1) If the LP-subform is of the type $\alpha_j(F_1, F_2)$ then

$$P^{2\prime}(\boldsymbol{\sigma}; \Omega'_1, \Omega_3^{\zeta_3}, \ldots, \Omega_k^{\zeta_k}) \, P^1(\gamma'; \boldsymbol{\omega}') =$$
$$= \sum_{\{\zeta_1, \zeta_2; \alpha_j^*(\zeta_1, \zeta_2) = 1\}} P^{2\prime}(\boldsymbol{\sigma}; \Omega_1^{\zeta_1}, \ldots, \Omega_k^{\zeta_k}) \, P^1(\gamma'; \boldsymbol{\omega}') =$$
$$= \sum_{\zeta_1, \zeta_2} P^{2\prime}(\boldsymbol{\sigma}; \Omega_1^{\zeta_1}, \ldots, \Omega_k^{\zeta_k}) \sum_{\{\xi_1, \xi_2; \alpha_j^*(\xi_1, \xi_2) = 1\}} P^1(\gamma', \boldsymbol{\omega}') \, P_{\zeta_1}^*(\xi_1) \, P_{\zeta_2}^*(\xi_2) ,$$

where $P^*_{\zeta_1}(\xi_1) = 1$ if $\zeta_1 = \xi_1$, $= 0$ in other cases, analogically $P^*_{\zeta_2}(\xi_2)$ and

$$\sum_{\{\xi_1,\xi_2;\alpha_j^*(\xi_1,\xi_2)=1\}} P^1(\gamma', \omega')\, P^*_{\zeta_1}(\xi_1)\, P^*_{\zeta_2}(\xi_2)$$

corresponds to the first induction step for the computation of $P_G(\zeta, \Omega^1_G)$.

2) If the LP-subform is of the type $\sim F_1$, then

$$P'(\sigma; \Omega^{1\prime}, \Omega^{\zeta_2}_2, \ldots, \Omega^{\zeta_k}_k)\, P^1(\gamma'; \omega') =$$
$$= \sum_{\{\zeta_1;\zeta_1=0\}} P^{2\prime}(\sigma; \Omega^{\zeta_1}_1, \ldots, \Omega^{\zeta_k}_k)\, P^1(\gamma'; \omega') =$$
$$= \sum_{\zeta_1} P^{2\prime}(\sigma; \Omega^{\zeta_1}, \ldots, \Omega^{\zeta_k}) \sum_{\{\xi_1;\xi_1=0\}} P^1(\gamma'; \omega')\,(1 - P^*_{\zeta_1}(\xi_1))$$

and again $\sum_{\{\xi_1;\xi_1=0\}} P^1(\gamma'; \omega')\,(1 - P^*_{\zeta_1}(\xi_1))$ corresponds to the first step of recursion for $P_G(\zeta, \Omega^1_G)$.

3) For a subform of the type $\varphi_j(F_1)$, the assertion is evident.

Thus the probability (following the method of computation) can be expressed as follows

$$P'(\sigma; \Omega_1) = \sum_\zeta P^{2\prime}(\sigma; \Omega^{\zeta_1}_1, \ldots, \Omega^{\zeta_k}_k)\, P(\zeta, \Omega^1_G)\,. \qquad \square$$

The above theorem is important for the computation of the characteristic vectors in special cases and enables us to prove Theorem 7.

Note. To formulate the following theorem, we must explain some useful notions. For the following considerations let $\Phi = [F, \Omega_F, \mathscr{P}_F]$ be a LP-expression, in which $F \simeq G(F_1, \ldots, F_k)$, where $F_1, \ldots, F_k$ are subforms in which the variables $x_1, \ldots, x_m$ occur. Let $G(x_{m+1}, \ldots, x_{m+k})$ be a LP-form, and let $[G, \Omega_G, \mathscr{P}_G]$ be the same expression as in the above theorem.

1) We shall consider a LP-form G' in which variables $x_{m+1}, \ldots, x_{m+k}$ occur. 2) Let $n_i + 1$ be the number of occurences of the i-th variable in G'. Let us consider the subexpressions (their subforms)

$$F_{11}, \ldots, F_{1n_1}, \ldots, F_{k_1}, \ldots, F_{kn_k}\,,$$

containing different probabilistic connectives, for which:

$$P'(\sigma, \gamma'; \omega^*_1, \ldots, \omega^*_k, \omega^*_{11}, \ldots, \omega^*_{kn_k}, \omega') = \begin{cases} P(\sigma, \gamma'; \omega^*_1, \ldots, \omega^*_k, \omega') & \text{if} \\ \omega^*_i = \omega^*_{i1} = \ldots = \omega^*_{in_i} & \text{for} \quad i = 1, \ldots, k\,, \\ 0 \quad \text{in other cases} \end{cases}$$

(thus subexpressions corresponding to $F_{i1}, \ldots, F_{in}$ are functionally equivalent to subexpression corresponding to F_i). Then we denote F' the LP-form obtained from G' by substitution F_i into the first occurence of x_{m+i}, F_{i1} into the second occurence of x_{m+i} and so on.

3) Then F' and $\mathscr{P}' = \{P'(\sigma, \gamma'; .)\}$ determine a LP-expression. We denote it Φ'.

Theorem 7. Let Φ and Φ' be the LP-expressions described in the above note. If 1) $[G, \Omega_G, \mathscr{P}_G] \equiv_p [G', \Omega_{G'}, \mathscr{P}_{G'}]$ and 2a) the condition (1) from Theorem 6 holds for $[G, \Omega_G, \mathscr{P}_G]$ in Φ and for $[G', \Omega_{G'}, \mathscr{P}_{G'}]$ in Φ' or 2b) G and G' do not contain any probabilistic connective, then $\Phi \equiv_p \Phi'$.

If the condition 2b) holds, then moreover, $\Phi \equiv_f \Phi'$ with respect to the system of the probabilities on $\Omega^{*1} \times \Omega^{*2}$ induced by the system $\{P'(\sigma; \omega_1^*, \ldots, \omega_{kn_k}^*)\}$.

Proof. Let us denote $\zeta = (\zeta_1, \ldots, \zeta_k) \in \{0, 1\}^k$, $\zeta' = (\zeta_1, \ldots, \zeta_k, \zeta_{11}, \ldots, \zeta_{kn_k}) \in \{0, 1\}^{k+\Sigma n_i}$; ζ corresponds to values of variables $x_{m+1}, \ldots, x_{m+k}$, ζ' corresponds to values of variables $x_{m-1}, \ldots, x_{m+k}, x_{11}, \ldots, x_{kn_k}$ of the LP-form G'' which is obtained from G' in such a way that every time a new variable is substituted for the second and further occurence of the variable x_{m+i}, $i = 1, \ldots, k$. In the matrix

$$\boldsymbol{P}' = (P_{\sigma,\zeta'})_{\sigma,\zeta'} = (P'(\sigma; \Omega_1^{\zeta_1}, \ldots, \Omega_{kn_k}^{\zeta kn_k}))_{\sigma,\zeta},$$

are whole columns corresponding to ζ', in which for some i and some j ($j = 1, \ldots, n_i$) is $\zeta_i \neq \zeta_{ij}$, are equal to $\boldsymbol{0}$. We complete the vector $\boldsymbol{p}_{G'}$ to $\boldsymbol{p}'_G = P(\zeta', \Omega^1_{G''})$, which is the characteristic vector of G''.

The elements corresponding to ζ' in which $\zeta_i = \zeta_{i_1} = \ldots = \zeta_{in_i}$, $i = 1, \ldots, k$, are equal to $\boldsymbol{p}_\zeta$ from $\boldsymbol{p}_G$ and then

$$\boldsymbol{p}_{F'} = \boldsymbol{P}'\boldsymbol{p}'_G = \boldsymbol{P}\boldsymbol{p}_{G'} = \boldsymbol{P}\boldsymbol{p}_G = \boldsymbol{p}_F\,.$$

2) Let $L\text{-}func_G$ and $L\text{-}func_{G'}$ be the usual evaluating functions of logical expressions $G(x_{m+1}, \ldots, x_{m+k})$ and $G''(x_{m+1}, \ldots, x_{m+k}, x_{11}, \ldots, x_{kn_k})$.

We have

$$\Omega_i^1 = \bigcup_{L-func_G(|\omega_1{}^*|, \ldots, |\omega_k{}^*|) = i} \omega_1^* \cap \ldots \cap \omega_h^*\,,$$

$$\Omega_j^2 = \bigcup_{L-func_{G'}(|\omega_1{}^*|, \ldots, |\omega^*{}_{kn_k}|) = j} \omega_1^* \cap \ldots \cap \omega_{kn_k}^*$$

and thus

$$P(\sigma; \Omega_i^1) = \sum_{L-func_G(|\omega_1{}^*|, \ldots, |\omega_k{}^*|) = i} P(\sigma; \omega_1^*, \ldots, \omega_k^*)\,,$$

$$P(\sigma; \Omega_j^2) = \sum_{L-func_{G'}(|\omega_1{}^*|, \ldots, |\omega^*{}_{kn_k}|) = j} P(\sigma; \omega_1^*, \ldots, \omega_{kn_k}^*)\,.$$

If we define

$$P^{12}(\sigma; \Omega_i^1, \Omega_j^2) = P'(\sigma; \Omega_i^1 \cap \Omega_j^2)$$

then

$$P^{12}(\sigma; \Omega_i^1, \Omega_j^2) =$$

$$= \sum_{L-func_G(|\omega_1{}^*|, \ldots, |\omega_k{}^*|) = i} \sum_{L-func_{G'}(|\omega_1{}^*|, \ldots, |\omega^*{}_{kn_k}|) = j} P'(\sigma; \omega_1^*, \ldots, \omega_{kn_k}^*) =$$

$$= \sum_{L-func_G(|\omega_1{}^*|, \ldots, |\omega_k{}^*|) = i} \sum_{\substack{L-func_{G'}(|\omega_1{}^*|, \ldots, |\omega^*{}_{kn_k}|) = j,\\ \omega_i{}^* = \omega_{i_1}{}^* = \ldots = \omega_{in_i}{}^* \text{ for } i = 1, \ldots, k}} P(\sigma; \omega_1^*, \ldots, \omega_k^*) =$$

$$= \delta_{ij}\, P(\sigma; \Omega_i^1)\,. \qquad \square$$

Note. 1) Now we can show in what sense the assertions of propositional calculus are preserved. For example let Φ_1, Φ_2 be two LP-expressions, let $G \simeq y_1 \,\&\, y_2$ and $G_2 \simeq \sim(\sim y_1 \vee \sim y_2)$. From the propositional calculus it is known that $\boldsymbol{p}_{G_1} = \boldsymbol{p}_{G_2}$ and by Theorem 6 we see that $\Phi_1 \equiv_f \Phi_2$

$$(\Phi_1 = [G_1(F_1, F_2), \Omega_{F_1} \times \Omega_{F_2}, \mathscr{P}_{F_1} \times \mathscr{P}_{F_2}]\,,$$
$$\Phi_2 = [G_2(F_1, F_2), \Omega_{F_1} \times \Omega_{F_2}, \mathscr{P}_{F_1} \times \mathscr{P}_{F_2}])\,.$$

In the same way for $G_1 \simeq y_1 \to y_2$ and $G_2 \simeq \sim y_1 \vee y_2$ it is $\Phi_1 \equiv_f \Phi_2$.

2) Let us consider two LP-expressions

$$\Phi_1 = [\sim\varphi(x), \Omega_1, \mathscr{P}_1]\,, \quad \Phi_2 = [\varphi(\sim x), \Omega_2, \mathscr{P}_2]\,,$$

let $P_1^1 = P_0^2$, $P_0^1 = P_1^2$, then $\Phi_1 \equiv_p \Phi_2$ (in particular for $P_0^1 = 1 - P_1^1$ and $\mathscr{P}_2 = \mathscr{P}_2$).

Proof. We can denote $P^1(0; \Omega_1^0) = p_0$, $P^2(0; \Omega_2^0) = p_0'$, $P^1(1; \Omega_1^1) = p_1$ and $P^2(1; \Omega_2^1) = p_1'$, and we obtain

$$p_{\sim\varphi(x)} = \begin{pmatrix} p_0, & 1 - p_0 \\ 1 - p_1, & p_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_0 \\ 1 - p_1 \end{pmatrix},$$

$$p_{\varphi(\sim x)} = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} \begin{pmatrix} 1 - p_0' \\ p_1' \end{pmatrix} = \begin{pmatrix} p_1' \\ 1 - p_0' \end{pmatrix} = \begin{pmatrix} p_0 \\ 1 - p_1 \end{pmatrix}.$$

3) Certain nets with probabilistic elements were considered by other authors (e.g. [6] by V. I. Levin) for the purposes of reliability theory. These nets are usual logical nets with unreliable elements $\sim$, $\vee$, &. These elements work with errors of the following two kinds:

error:	correct output:	incorrect output:	probability:
first kind $(1 \to 0)$	1	0	p_2
second kind $(0 \to 1)$	0	1	p_1

We shall show that these elements can be replaced by standart logical elements together with our probabilistic elements:

a) For unreliable disjunction $x_1 \vee_p x_2$ the vector $(p_1, 1 - p_2, 1 - p_2, 1 - p_2)$ is the characteristic vector. If we have a connective φ with the characteristic vector $(p_1, 1 - p_2)$ then $\varphi(x_1 \vee x_2)$ has the same characteristic vector as $x_1 \vee_p x_2$:

$$\boldsymbol{P}\boldsymbol{p}_\varphi = \begin{pmatrix} 1, & 0 \\ 0, & 1 \\ 0, & 1 \\ 0, & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ 1 - p_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ 1 - p_2 \\ 1 - p_2 \\ 1 - p_2 \end{pmatrix}.$$

b) For unreliable conjunction, if φ is a probabilistic connective with the characteristic vector $(p_1, 1 - p_2)$ then for $\varphi(x_1 \,\&\, x_2)$ is

$$\boldsymbol{P}\boldsymbol{p}_{\varphi} = \begin{pmatrix} 1, & 0 \\ 1, & 0 \\ 1, & 0 \\ 0, & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ 1 - p_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_1 \\ p_1 \\ 1 - p_2 \end{pmatrix}$$

and so $\varphi(x_1 \,\&\, x_2) \equiv_p x_1 \,\&_p\, x_2$.

c) By the same method we can see, that if $\sim_p$ is an unreliable negation and if φ is a probabilistic connective with the characteristic vector $(p_1, 1 - p_2)$, then $\varphi(x) \equiv_p \sim_p x$.

Now we prove Theorem 3.

The construction of the equivalent expression is performed by induction in this way:

1) We transform, in a usual way, every maximal subform which contains no probabilistic connective to a logically equivalent disjunctive normal form F_i' (see [3]) and substitute F_i' instead of F_i into F. The obtained form will be denoted F^0.

2) For every subform of the type $\varphi_i(F_i)$, where φ_i is of degree 0, we substitute again $\varphi_i(F_i)$ into F^0 (i.e., we do not make any change).

3) Assume that the transformation has been done for all interiors of probabilistic connectives up to degree n (we denote the corresponding form as F^n). So the interior of every φ_i having degree $n + 1$ has the form $G(F_1, \ldots, F_k)$, where $F_1, \ldots, F_k$ are either variables or forms of the type $\varphi_j(F_j')$ where $d(\varphi_j) \leqq n$, and G is a L-expression. We transform $G(y_1, \ldots, y_k)$ to a disjunctive normal form $G'(y_1, \ldots, y_k)$. Let us consider the groups of functionally equivalent connectives

$$\boldsymbol{\varphi}_1 = (\varphi_1, \varphi_{11}, \ldots, \varphi_{1n_1}),$$
$$\boldsymbol{\varphi}_2 = (\varphi_2, \varphi_{21}, \ldots, \varphi_{2n_2}), \ldots$$

(for the functional equivalence of connectives cf. assumption 2) from Lemma 2 or the note before Theorem 7). We proceed now according to Theorem 7 and we substitute functionally equivalent subexpressions into the first and others occurences of a variable y_i in G': if we need a subexpression F_{ij} functionally equivalent to F_i, we substitute the connectives from the same group in place of the given probabilistic connectives in F_i (i.e., if there is somewhere in F_i a connective φ_r then some connective φ_{rj_i} will be in F_{ij}).

In accordance with the second assertion of Theorem 7, we obtain a LP-expression functionally equivalent to the original one.

4) Let n_0 be the greatest degree of a probabilistic connective in F. If F^{n_0} is of the type $\varphi(F')$ the procedure is finished. If it is not in this form we must repeat the point 3) once more. $\square$

The proof of Theorem 5 is clearly analogical to the proof of Theorem 2.

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