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Weighting Function and State Equations of Linear Discrete-Time-Varying System

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A linear nonstationary discrete-time system is considered in this work. The ways are presented for determination of system state equations if the weighting function is known.

State equations of a linear discrete-time system can be obtained from its input-output difference equation in stationary [1] as well as in nonstationary case [2]. It is the purpose of this paper to show the direct transformation of the system weighting function into the state space description.

I. FUNDAMENTAL RELATIONS

A single-input, single-output, linear discrete-time system can be described on definite time interval N by the state equations

$$\mathbf{x}(n+1) = \mathbf{A}(n)\,\mathbf{x}(n) + \mathbf{b}(n)\,u(n)\,, \tag{1a}$$

$$y(n) = \mathbf{c}(n)\,\mathbf{x}(n) + d(n)\,u(n) \tag{1b}$$

where a system input and output are denoted by $u(n)$ and $y(n)$ respectively, $\mathbf{x}(n)$ is an s-vector of state variables; $\mathbf{A}(n)$, $\mathbf{b}(n)$, $\mathbf{c}(n)$ and $d(n)$ are parameters of proper dimensions. The action period is assumed here to be $T = 1$ for simplicity, i.e., discrete values of time ranges over integers $n \in N \equiv [n_0, n_1]$.

Solving the equation (1a) we get [1]

$$\mathbf{x}(n) = \boldsymbol{\Phi}(n, n_0)\,\mathbf{x}(n_0) + \sum_{k=n_0}^{n-1} \boldsymbol{\Phi}(n, k+1)\,\mathbf{b}(k)\,u(k) \tag{2}$$

where the system transition (fundamental) matrix

$$\boldsymbol{\Phi}(n, k) = \mathbf{A}(n-1)\,\mathbf{A}(n-2)\ldots\mathbf{A}(k) \quad (n > k) \tag{3}$$

 satisfies the equation

$$\Phi(n + 1, k) - \mathbf{A}(n)\, \Phi(n, k) = \mathbf{0} \tag{4}$$

under the initial condition

$$\Phi(k, k) = \mathbf{I} \tag{5}$$

(identity matrix).

The transition matrix possesses the following properties:

$$\text{a)} \quad \Phi(n, n) = \mathbf{I}\,; \tag{6}$$

$$\text{b)} \quad \Phi(n, k) = \Phi(n, l)\, \Phi(l, k)\,; \quad n \geqq l \geqq k\,; \tag{7}$$

$$\text{c)} \quad \Phi(k, n) = \Phi^{-1}(n, k) = \mathbf{A}^{-1}(k)\, \mathbf{A}^{-1}(k + 1) \ldots \mathbf{A}^{-1}(n - 1) \tag{8}$$

provided the inverses of $\mathbf{A}(n)$ exist.

Using the equations (2) and (1b) the output can be expressed as

$$y(n) = \mathbf{c}(n)\, \Phi(n, n_0)\, \mathbf{x}(n_0) + \mathbf{c}(n) \sum_{k=n_0}^{n-1} \Phi(n, k + 1)\, \mathbf{b}(k)\, u(k) + d(n)\, u(n)\,. \tag{9}$$

Let us now consider the system weighting function (weighting sequence, impulse response) $g(n, k)$. It can be defined as the response of initially relaxed system (1) to the discrete-time equivalent of Dirac impulse signal determined [3] by the relation

$$\sigma(n - k) = \begin{cases} 0\,; & n \neq k\,, \\ 1\,; & n = k\,. \end{cases} \tag{10}$$

Obviously with respect to physical realizability of the system

$$g(n, k) \equiv 0 \quad \text{for} \quad n < k\,. \tag{11}$$

Assuming a system that is fully relaxed at $n < n_0$, the output $y(n)$ resulting from any input $u(n)$, $n \geqq n_0$, is determined by the summation

$$y(n) = \sum_{k=n_0}^{n} g(n, k)\, u(k)\,. \tag{12}$$

Comparing the relations (12) and (9) with $\mathbf{x}(n_0) = \mathbf{0}$ we have

$$g(n, k) = \mathbf{c}(n)\, \Phi(n, k + 1)\, \mathbf{b}(k)\,; \quad n_0 \leqq k < n \tag{13}$$

and

$$g(n, n) = d(n)\,. \tag{14}$$

The weighting function of linear, finite-dimensional, discrete-time system can always be written in the form

$$g(n, k) = \mathbf{q}(n)\,\mathbf{h}(k) \tag{15}$$

where

$$\mathbf{q}(n) = [q_1(n)\; q_2(n) \ldots q_s(n)] \tag{16}$$

is an $(1 \times s)$ row vector and

$$\mathbf{h}(k) = [h_1(k)\; h_2(k) \ldots h_s(k)]^T \tag{17}$$

is an $(s \times 1)$ column vector.

The order s of minimal system realization results directly from the form (15) of the weighting function.

If we compare the equations (13) and (15) and respect the above properties of $\boldsymbol{\Phi}(n, k)$, the following relations are valid:

$$\mathbf{q}(n) = \mathbf{c}(n)\,\boldsymbol{\Phi}(n, 0) \tag{18}$$

and

$$\mathbf{h}(k) = \boldsymbol{\Phi}(0, k + 1)\,\mathbf{b}(k)\,. \tag{19}$$

II. DETERMINATION OF THE STATE EQUATIONS

Using the relations (14)–(19) the weighting function can be found from the state equations (1) provided that the transition matrix $\boldsymbol{\Phi}(n, k)$ is available.

We shall investigate the converted problem here, i.e., the formulation of state equations from given weighting function. A system described by its weighting function can be represented, of course, in variety of equivalent state-space forms. Only $d(n)$ is given unambiguously by the equation (14) while we must always choose s^2 elements to determine all other parameters.

Now several convenient ways will be given for obtaining the state equations of linear discrete-time-varying system represented by its weighting function $g(n, k)$.

1. Writing $g(n, k)$ in the form (15) and choosing

$$\boldsymbol{\Phi}(n, 0) = \begin{bmatrix} q_1(n) & 0 & 0 & \ldots & 0 \\ 0 & q_2(n) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & q_s(n) \end{bmatrix} \tag{20}$$

we have from the equation (18)

$$\mathbf{c}(n) = \mathbf{c} = [1\; 1\; \ldots\; 1]\,. \tag{21}$$

According to (4) we determine

$$A(n) = \Phi(n+1, 0)\,\Phi^{-1}(n, 0) \tag{22}$$

having the diagonal structure

$$A(n) = \begin{bmatrix} \frac{q_1(n+1)}{q_1(n)} & 0 & 0 \ldots & 0 \\ 0 & \frac{q_2(n+1)}{q_2(n)} & 0 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots & \frac{q_s(n+1)}{q_s(n)} \end{bmatrix} \tag{23}$$

and at last from the equation (19) we get

$$b(n) = \Phi^{-1}(0, n+1)\,h(n) = \Phi(n+1, 0)\,h(n) = \begin{bmatrix} q_1(n+1)\,h_1(n) \\ q_2(n+1)\,h_2(n) \\ \vdots \\ q_s(n+1)\,h_s(n) \end{bmatrix}. \tag{24}$$

2. The matrix $A(n)$ can be taken in the diagonal form as

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_s \end{bmatrix} \tag{25}$$

where all λ_i are arbitrary real constants.

The according to the relations (3) and (6)

$$\Phi(n, k) = A^{n-k} = \begin{bmatrix} \lambda_1^{n-k} & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2^{n-k} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_s^{n-k} \end{bmatrix}. \tag{26}$$

Using the equations (18) and (19) we obtain

$$c(n) = q(n)\,\Phi^{-1}(n, 0) = [q_1(n)\,\lambda_1^{-n};\ q_2(n)\,\lambda_2^{-n};\ \ldots;\ q_s(n)\,\lambda_s^{-n}] \tag{27}$$

and

$$b(n) = \Phi(n+1, 0)\,h(n) = \begin{bmatrix} \lambda_1^{n+1}\,h_1(n) \\ \lambda_2^{n+1}\,h_2(n) \\ \vdots \\ \lambda_s^{n+1}\,h_s(n) \end{bmatrix} \tag{28}$$

respectively.

3. The special case of the previous way may be formed by putting

$$A = I . \tag{29}$$

Then obviously

$$\Phi(n, k) = I , \tag{30}$$

$$c(n) = q(n) \tag{31}$$

and

$$b(n) = h(n) . \tag{32}$$

The following theorem summarizes these results.

Theorem. *Every linear, finite-dimensional, discrete-time system, given by the weighting function $g(n, k) = \mathbf{q}(n)\,\mathbf{h}(k)$, can be described in the state-space form*

$$\begin{aligned} \mathbf{x}(n + 1) &= \mathbf{x}(n) + \mathbf{h}(n)\, u(n) , \\ y(n) &= \mathbf{q}(n)\, \mathbf{x}(n) + g(n, n)\, u(n) . \end{aligned} \tag{33}$$

The system matrix $\mathbf{A}(n)$ and the transition matrix $\Phi(n, k)$ are unit matrices.

Note. The analogous form with $\mathbf{A}(t) = \mathbf{0}$ and $\Phi(t, \xi) = \mathbf{I}$ given for continuous-time system by Kalman [4] is called the *normalized canonical form*.

4. The widely used canonical form of state equations possesses the parameters

$$\mathbf{A}(n) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0(n) & a_1(n) & a_2(n) & \dots & a_{s-1}(n) \end{bmatrix} \tag{34}$$

and

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} . \tag{35}$$

Obviously just s^2 elements are fixed in advance by the relations (34) and (35) provided a vector

$$\mathbf{a}(n) = [a_0(n)\, a_1(n) \dots a_{s-1}(n)] \tag{36}$$

is required to be stated.

Substituting (34) into the equation (4) we get

$$\Phi(n+1,0)=\begin{bmatrix}\varphi_1(n+1)\\ \varphi_2(n+1)\\ \vdots\\ \varphi_{s-1}(n+1)\\ \varphi_s(n+1)\end{bmatrix}=\begin{bmatrix}\varphi_2(n)\\ \varphi_3(n)\\ \vdots\\ \varphi_s(n)\\ \mathbf{a}(n)\,\Phi(n,0)\end{bmatrix} \tag{37}$$

where $\varphi_j(n)$ is the j-th row of $\Phi(n, 0)$.

Simply writing $\varphi(n)$ instead of $\varphi_1(n)$ it follows from (37)

$$\Phi(n,0)=\begin{bmatrix}\varphi(n)\\ \varphi(n+1)\\ \vdots\\ \varphi(n+s-1)\end{bmatrix} \tag{38}$$

and

$$\varphi(n+s)=\mathbf{a}(n)\,\Phi(n,0)\,. \tag{39}$$

In accordance with (19) we can write

$$\Phi(n+1+i,0)\,\mathbf{h}(n+i)=\mathbf{b}(n+i)=\mathbf{b} \tag{40}$$

where $i = 0, 1, \ldots, s-1$ and $\mathbf{b}$ stands in (35).

Then substituting (38) into (40) the following equations are valid for the rows of $\Phi(n, 0)$:

$$\varphi(n+i)=[1\;0\;\ldots\;0]\,\mathbf{H}^{-1}(n-s+i)\,,\qquad i=0,1,\ldots,s\,. \tag{41}$$

The $(s \times s)$ matrix

$$\mathbf{H}(n)=[\mathbf{h}(n);\mathbf{h}(n+1);\ldots;\mathbf{h}(n+s-1)] \tag{42}$$

is always nonsingular if minimal s in (16) and (17) is taken.

Then $\Phi(n, 0)$ is stated and we obtain

$$\mathbf{a}(n)=\varphi(n+s)\,\Phi^{-1}(n,0) \tag{43}$$

according to (39) and using (18)

$$\mathbf{c}(n)=\mathbf{q}(n)\,\Phi^{-1}(n,0)\,. \tag{44}$$

EXAMPLE

We want to formulate state equations of a system characterized by the weighting function

$$g(n,k)=1-ne^{-(2n-k)}\,.$$

Using (15)—(17) we put at first

$$q_1(n) = 1\,, \quad q_2(n) = -n\mathrm{e}^{-2n}\,,$$

$$h_1(k) = 1\,, \quad h_2(k) = \mathrm{e}^k\,.$$

According to (14) we have

$$d(n) = 1 - n\mathrm{e}^{-n}\,.$$

The other parameters will be gradually found by applying all above derived equivalent ways.

1. Substituting determined q_i and h_i into the relations (20)—(24) we get

$$\boldsymbol{\Phi}(n, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -n\mathrm{e}^{-2n} \end{bmatrix},$$

$$\mathbf{c}(n) = [1 \;\; 1]\,,$$

$$\boldsymbol{A}(n) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n+1}{n}\,\mathrm{e}^{-2} \end{bmatrix}$$

and

$$\boldsymbol{b}(n) = \begin{bmatrix} 1 \\ -(n+1)\,\mathrm{e}^{-(n+2)} \end{bmatrix}.$$

2. If we choose

$$\boldsymbol{A} = \begin{bmatrix} \mathrm{e}^{-1} & 0 \\ 0 & \mathrm{e}^{-2} \end{bmatrix}$$

the remaining parameters are determined by (26)—(28) as

$$\boldsymbol{\Phi}(n, 0) = \begin{bmatrix} \mathrm{e}^{-n} & 0 \\ 0 & \mathrm{e}^{-2n} \end{bmatrix},$$

$$\mathbf{c}(n) = [\mathrm{e}^n; \; -n]$$

and

$$\boldsymbol{b}(n) = \begin{bmatrix} \mathrm{e}^{-(n+1)} \\ \mathrm{e}^{-(n+2)} \end{bmatrix}.$$

3. Choosing $\boldsymbol{A} = \boldsymbol{I}$ the normalized canonical form of state equations is given by (33):

$$\mathbf{x}(n+1) = \mathbf{x}(n) + \begin{bmatrix} 1 \\ \mathrm{e}^n \end{bmatrix} u(n)\,,$$

$$y(n) = [1; \; -n\mathrm{e}^{-2n}]\,\mathbf{x}(n) + (1 - n\mathrm{e}^{-n})\,u(n)\,.$$

4. In accordance with (42)

$$\boldsymbol{H}(n) = \begin{bmatrix} 1 & 1 \\ \mathrm{e}^n & \mathrm{e}^{n+1} \end{bmatrix}$$

and

$$H^{-1}(n) = \frac{1}{e-1}\begin{bmatrix} e & -e^{-n} \\ -1 & e^{-n} \end{bmatrix}.$$

Then using (41), (38), (43) and (44)

$$\varphi(n+2) = \frac{1}{e-1}\,[e;\ -e^{-n}],$$

$$\Phi(n, 0) = \frac{1}{e-1}\begin{bmatrix} e & -e^{-(n-2)} \\ e & -e^{-(n-1)} \end{bmatrix},$$

$$\mathbf{a}(n) = \mathbf{a} = [-e^{-1};\ 1 + e^{-1}]$$

and

$$\mathbf{c}(n) = [e^{-1}(ne^{-n} - 1);\ 1 - ne^{-(n+1)}]$$

respectively.

The results are completed by

$$\mathbf{A}(n) = \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -e^{-1} & 1 + e^{-1} \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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