KYBERNETIKA — VOLUME 9 (1973), NUMBER 6

On the Experience Utilization in Statistical Decisions

STANISLAV JÍLOVEC

Experience utilization for the independent repetition of a finite statistical decision problem is considered. For a special type of decision processes the uniform convergence rate of the corresponding losses and risks to the Bayes risk is established.

1. INTRODUCTION

Let us consider a statistical decision problem that is independently repeated with a fixed but unknown a priori distribution μ of parameters. If μ were known then statistician could use a Bayes decision function relative to μ in every step and the average losses would converge to a minimum that is equal to the Bayes risk $\varrho(\mu)$. Špaček's experience theory (see e.g. [7], [10], [1]) as well as Robbins's empirical Bayes approach (see e.g. [5], [6]) enable to construct decision processes with the same quality under a far weaker assumption than that of the knowledge of the a priori distribution. The principal idea consists in replacing the unknown a priori distribution by estimates obtained on the basis of experience concerning the preceding parameters and/or observations.

Although the primary Špaček's model has been considerably generalized, the actually suggested estimates can be almost always expressed as the arithmetic mean of independent random variables. In this paper we shall show that for decision processes based on such estimates, the assertion concerning the convergence to the Bayes risk can be essentially strengthened in comparison with [10]. Our improvement consists in showing that both average losses and risks converge uniformly in the set of all a priori distributions. Moreover, uniform bounds for convergence rates will be given.

2. NOTATION

Throughout this paper $(\Omega, \mathscr{S}, P)$ will be a basic probability space on which all random variables and random vectors will be defined and $\mathfrak{D} = (A, D, w, (X, \mathscr{X}), v)$ will be a given finite statistical decision problem. It means that A and D are finite non-empty sets, w is a finite real function on $A \times D$, $(X, \mathscr{X})$ is a measurable space (which is not assumed to be finite), and $v = \{v_a : a \in A\}$ is a set of probability measures on $\mathscr{X}$. The elements of A will be referred to as parameter values, the elements of D as decisions, $(X, \mathscr{X})$ as a sample space and w as a weight or loss function. The letters $\mathscr{A}$ and $\mathscr{D}$ will denote the classes of all subsets of A and D, respectively. The symbol M_0 will denote the set of all a priori distributions, i.e. probability measures on $\mathscr{A}$, the letter M will denote the set of all finite signed measures on $\mathscr{A}$ and the symbol $\mathscr{M}$ will denote the σ-field of subsets of the set M generated by the class of all sets of the form

$$\{\mu : \mu(\{a\}) < c\}$$

where c is a real number and $a \in A$. The set of all decision functions, i.e. measurable mappings from $(X, \mathscr{X})$ into $(D, \mathscr{D})$, will be denoted by Δ.

The symbols E and D will denote the operations of expectation and dispersion, respectively. Finally, the letter N will denote the set of all positive integers.

3. BAYESIAN MODEL OF STATISTICAL DECISION

The statistical decision problem $\mathfrak{D}$ can be verbally described in the following way. A statistician observes a random variable ξ (taking its values in X) knowing that its distribution, depending on an unknown parameter, is equal to v_a if the parameter value is a. On the basis of the observation the statistician has to choose a decision d incurring the loss $w(a, d)$ if the parameter value is a. Statistician's goal is to minimize the expected loss.

If the parameter takes the value a with probability $\mu(\{a\})$, i.e. the parameter is a random variable α and μ is its distribution, then the overall expected loss $\mathsf{E}w(\alpha, \delta(\xi))$ corresponding to a decision function $\delta \in \Delta$ is equal to

$$r(\mu, \delta) = \sum_{a \in A} \mu(\{a\}) \int w(a, \delta(x)) \, \mathrm{d}v_a(x) \tag{1}$$

and will be called the risk.

It is convenient, for our further purpose, to extend the definition range of the function r from $M_0 \times \Delta$ on $M \times \Delta$ defining $r(\mu, \delta)$ by (1) for every $\mu \in M$ and $\delta \in \Delta$.

The infimum

$$\varrho(\mu) = \inf_{\delta \in \Delta} r(\mu, \delta)$$

is called the Bayes risk relative to μ, and any decision function β satisfying

$$r(\mu, \beta) = \varrho(\mu)$$

is called a Bayes decision function.

Therefore, under the assumption that the parameter is a random variable with an a priori distribution μ known to the statistician (the so-called Bayes approach), the statistician can solve the given decision problem in the best possible way by using a Bayes decision function relative to μ.

4. OPTIMUM PROCEDURES

If p is a measurable mapping from $(X \times M, \mathscr{X} \times \mathscr{M})$ into $(D, \mathscr{D})$, then, for every $\mu \in M$, the μ-section $p_\mu = p(\cdot, \mu)$ is a decision function, i.e. $p_\mu \in \Delta$. A measurable mapping p from $(X \times M, \mathscr{X} \times \mathscr{M})$ into $(D, \mathscr{D})$ will be called an optimum procedure if, for every $\mu \in M$, p_μ is a Bayes decision function relative to μ, i.e.

$$r(\mu, p_\mu) = \varrho(\mu), \quad \mu \in M .$$

Let us remark that this definition does not agree with that given in [2]. In order to verify the existence of an optimum procedure for the statistical decision problem $\mathfrak{D}$ let us choose a total ordering $\prec$ on the set D and a σ-finite measure λ on $\mathscr{X}$ with respect to which ν_a, $a \in A$, are absolutely continuous. (It is always possible to take $\lambda = \sum_{a \in A} \nu_a$.) Let

$$f_a = \frac{\mathrm{d}\nu_a}{\mathrm{d}\lambda}, \quad a \in A ,$$

be the Radon-Nikodym derivatives and let

$$h_d(x, \mu) = \sum_{a \in A} \mu(\{a\})\, w(a, d) f_a(x), \quad d \in D, \quad x \in X, \quad \mu \in M .$$

Obviously, h_d, $d \in D$, are $(\mathscr{X} \times \mathscr{M})$-measurable functions. Now, we shall define the mapping p^* from $X \times M$ into D by

$$p^*(x, \mu) = d$$

if and only if

$$h_d(x, \mu) < h_{d'}(x, \mu) \text{ for all } d' \text{ such that } d' \prec d,\ d' \neq d ,$$

and simultaneously

$$h_d(x, \mu) \leqq h_{d'}(x, \mu) \text{ for all } d' \text{ such that } d \prec d' .$$

The relation

$$\{(x,\mu) : p^*(x,\mu) = d\} =$$

$$= \bigcap_{\substack{d' \prec d \\ d' \neq d}} \{(x,\mu) : h_d(x,\mu) < h_{d'}(x,\mu)\} \cap \bigcap_{d' \prec d} \{(x,\mu) : h_d(x,\mu) \leqq h_{d'}(x,\mu)\}$$

shows that p^* is a measurable mapping from $(X \times M, \mathscr{X} \times \mathscr{M})$ into $(D, \mathscr{D})$. Further, for every $\delta \in \Delta$,

$$r(\mu,\delta) = \int h_{\delta(x)}(x,\mu)\, \mathrm{d}\lambda \geqq \int h_{p^*(x,\mu)}(x,\mu)\, \mathrm{d}\lambda = r(\mu, p_\mu^*)$$

and therefore p^* is an optimum procedure.

We close this section with giving some trivial inequalities and a useful lemma.

It follows immediately from (1) that

$$|r(\mu,\delta)| \leqq \max_{\substack{a\in A \\ d\in D}} |w(a,d)| \sum_{a\in A} |\mu(\{a\})| \,, \quad \mu \in M \,, \quad \delta \in \Delta \,. \tag{2}$$

Since both A and D are finite,

$$\max_{\substack{a\in A \\ d\in D}} |w(a,d)| < \infty \,. \tag{3}$$

Therefore

$$\sup_{\substack{\mu\in M_0 \\ \delta\in\Delta}} |r(\mu,\delta)| < \infty \tag{4}$$

and

$$\sup_{\mu\in M_0} \varrho(\mu) < \infty \,. \tag{5}$$

Lemma 1. *If p is an optimum procedure, $p_\mu = p(\cdot,\mu)$ denotes the μ-section of p and*

$$m = 2 \max \{|w(a,d)| : a \in A,\ d \in D\} \,,$$

then

$$0 \leqq r(\pi, p_\mu) - \varrho(\pi) \leqq m \sum_{a\in A} |\mu(\{a\}) - \pi(\{a\})|$$

for all $\pi, \mu \in M$.

Proof. It follows from the definitions of ϱ and p that

$$r(\pi, p_\mu) \geqq r(\pi,\pi) = \varrho(\pi) \,, \tag{6}$$

$$r\Big(\sum_{i=1}^{k} c_i \pi_i, p_\mu\Big) = \sum_{i=1}^{k} c_i r(\pi_i, p_\mu) \,, \tag{7}$$

wnere $\mu, \pi, \pi_i \in M$ and c_i are real numbers.

The lower bound in our lemma follows from (6). Adding the non-negative term $r(\mu, p_\pi) - \varrho(\mu)$ to $r(\pi, p_\mu) - \varrho(\pi)$ and using (6) and (7), we get

$$r(\pi, p_\mu) - \varrho(\pi) \leqq r(\mu - \pi, p_\pi) - r(\mu - \pi, p_\mu).$$

Hence, the desired result follows by (2).

5. FURTHER AUXILIARY RESULTS

Now, we shall present uniform stability theorems for dependent random variables on which our main theorem will be based. First of all we shall recall Parzen's definition of uniform almost sure convergence.

Let T be a non-empty set, let $\{\xi_{t,n} : t \in T, n \in N\}$ and $\{\xi_t : t \in T\}$ be families of random variables. We shall say that $\xi_{t,n}$ converge to ξ_t almost surely uniformly in $t \in T$, and write

$$\xi_{t,n} \to \xi_t \quad \text{a.s. uniformly in} \quad t \in T,$$

if, for every $\varepsilon > 0$,

$$\lim_{k \to \infty} P(\bigcup_{n=k}^{\infty} \{|\xi_{t,n} - \xi_t| \geqq \varepsilon\} = 0$$

uniformly in $t \in T$.

The following lemma is a slightly modified version of Theorem 29.1.C in [3]. Let us remark that we shall use the same terminology and notation concerning conditional expectations as in [3].

Lemma 2. *Let $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \ldots \subset \mathscr{F}_n$ be sub-σ-fields of $\mathscr{S}$ and let $\xi_1, \xi_2, \ldots, \xi_n$ be integrable random variables. If ξ_i is $\mathscr{F}_{i+1}$-measurable, $i = 1, 2, \ldots, n-1$, then, for every $\varepsilon > 0$,*

$$P\{\max_{1 \leqq k \leqq n} |\sum_{i=1}^{k} (\xi_i - \mathsf{E}(\xi_i \mid \mathscr{F}_i))| \geqq \varepsilon\} \leqq \varepsilon^{-2} \sum_{i=1}^{n} \mathsf{D}(\xi_i).$$

Proof. Let us denote $\eta_i = \xi_i - \mathsf{E}(\xi_i \mid \mathscr{F}_i)$, $i = 1, 2, \ldots, n$. Since the random vector $(\eta_1, \eta_2, \ldots, \eta_{i-1})$ is $\mathscr{F}_i$-measurable,

$$\mathsf{E}(\mathsf{E}(\xi_i \mid \mathscr{F}_i) \mid \eta_1, \eta_2, \ldots, \eta_{i-1}) = \mathsf{E}(\xi_i \mid \eta_1, \eta_2, \ldots, \eta_{i-1})$$

and therefore

$$\mathsf{E}(\eta_i \mid \eta_1, \eta_2, \ldots, \eta_{i-1}) = 0,$$

so that the random variables $\eta_1, \eta_2, \ldots, \eta_n$ are centred at conditional expectations given the predecessors and, according to Theorem 29.1.C in [3], it holds

$$P\{\max_{1 \leqq k \leqq n} |\sum_{i=1}^{k} \eta_i| \geqq \varepsilon\} \leqq \varepsilon^{-2} \sum_{i=1}^{n} \mathsf{D}(\eta_i).$$

Since

$$D(\eta_i) \leqq D(\eta_i) + D(E(\xi_i \mid \mathscr{F}_i)) = D(\xi_i),$$

the lemma is proved.

Using Lemma 2 instead of Kolmogorov's inequality in the proof of theorem 16.A in [4] we shall prove the following uniform strong law of large numbers for random variables centred at conditional expectations.

Proposition 1. *Let $\{b_n\}_{n=1}^{\infty}$ be a monotonous sequence of positive numbers converging to zero, T be a non-empty set, $\{\{\xi_{t,n}\}_{n=1}^{\infty} : t \in T\}$ be a family of sequences of random variables and $\{\{\mathscr{F}_{t,n}\}_{n=1}^{\infty} : t \in T\}$ be a family of non-decreasing sequences of sub-σ-fields of $\mathscr{S}$. If, for every $t \in T$ and $n \in N$, $\xi_{t,n}$ is $\mathscr{F}_{t,n+1}$-measurable, and if the series*

$$\sum_{n=1}^{\infty} b_n^2 \, D(\xi_{t,n})$$

is uniformly convergent and uniformly bounded in $t \in T$, then

$$b_n \sum^{n} (\xi_{t,i} - E(\xi_{t,i} \mid \mathscr{F}_{t,i})) \to 0 \quad a.s. \text{ uniformly in } \quad t \in T.$$

The next proposition is also a form of the uniform strong law of large numbers for dependent random variables.

Proposition 2. *Let $\{b_n\}_{n=1}^{\infty}$ be a monotonous sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} b_n^2 < \infty.$$

Let T be a non-empty set and $\{\xi_{t,n} : t \in T,\ n \in N\}$ be a family of random variables. If there exists a constant $c < \infty$ such that

$$E\Big(\sum_{i=1}^{n} |\xi_{t,i}|\Big)^2 \leqq cn \quad \text{for every} \quad t \in T \quad \text{and} \quad n \in N,$$

then

$$b_n \sum_{i=1}^{n} \xi_{t,i} \to 0 \quad a.s. \text{ uniformly in } \quad t \in T.$$

Proof. See Theorem 2 in [2].

6. INDEPENDENT REPETITION OF $\mathfrak{D}$

Let us consider that the statistical decision problem $\mathfrak{D}$ is independently repeated with a fixed but unknown a priori distribution of the parameters and assume that an estimate of the unknown a priori distribution is known to the statistician before he decides at the n-th step, $n = 1, 2, \ldots$

Let $\mathscr{K}_\nu$ be the class of all sequences $\{(\alpha_n, \xi_n)\}_{n=1}^{\infty}$ of independent random vectors (with values in $A \times X$) satisfying

$$P(\xi_n \in E \mid \alpha_n = a) = \nu_a(E) .$$

Random variables α_n and ξ_n will be interpreted as the random parameter and observation in the n-th step, respectively. Let us remark that the a priori distribution of α_n will be denoted by $P\alpha_1^{-1}$,

$$P\alpha_1^{-1}(E) = P\{\alpha_1 \in E\} = P\{\alpha_n \in E\} ,$$

and under the notion „estimate of $P\alpha_1^{-1}$" we shall understand any measurable mapping from $(\Omega, \mathscr{S})$ into $(M, \mathscr{M})$.

If p is an optimum procedure and the estimates μ_n in some sense well-approximate the unknown a priori distribution $P\alpha_1^{-1}$, then, as is proved in [10],

$$\text{(8)} \qquad \frac{1}{n} \sum_{i=1}^{n} \left(w(\alpha_i, p(\xi_i, \mu_i)) - \varrho(P\alpha_1^{-1})\right) \to 0 \quad \text{almost surely}$$

for every $\{(\alpha_n, \xi_n)\}_{n=1}^{\infty} \in \mathscr{K}_\nu$. It means that the decision process $(p(\cdot, \mu_1), p(\cdot, \mu_2), \ldots)$, constructed without a knowledge of the true a priori distribution $P\alpha_1^{-1}$, is asymptotically as good as the best possible decision in the case that the a priori distribution is known.

In this paper the estimates μ_n are assumed to be expressible in the form

$$\text{(C)} \qquad (\mu_n(\omega))(E) = \mu_n(\omega, E) = \sum_{j \in E} \frac{1}{n - l} \sum_{i=1}^{n-l} g_j(\alpha_i(\omega), \xi_i(\omega)) ,$$

$$E \subset A , \quad \omega \in \Omega , \quad n = l + 1, \; l + 2, \ldots ,$$

where l is a positive integer (delay) and g_j, $j \in A$, are Borel functions on $(A \times X, \mathscr{A} \times \mathscr{X})$ satisfying

$$\text{(C}_1\text{)} \qquad \int g_j(a, x) \, \mathrm{d}\nu_a(x) = \begin{cases} 1 & \text{if} \quad a = j , \\ 0 & \text{if} \quad a \neq j , \end{cases} \qquad a, j \in A ,$$

and

$$\text{(C}_2\text{)} \qquad \max_{\substack{a \in A \\ j \in A}} g_j^2(a, x) \, \mathrm{d}\nu_a(x) < \infty .$$

This assumption together with the finiteness of the sets A and D enable us to stregthen assertion (8). Our improvement consists in showing that (8) holds uniformly in $\mathscr{K}_\nu$, the fraction $1/n$ in (8) being allowed to be replaced by b_n, where $\{b_n\}_{n=1}^{\infty}$ is any monotonous sequence of positive numbers satisfying $\sum_{n=1}^{\infty} b_n^2 < \infty$. Moreover, uniform bounds for convergence of risks will be given.

Let us remark that (C_1) is equivalent to

$$(C_1') \qquad Eg_j(\alpha_1, \xi_1) = P\alpha_1^{-1}(\{j\}) \quad \text{for every} \quad j \in A \quad \text{and every} \quad \{(\alpha_i, \xi_i)\}_{i=1}^{\infty} \in \mathscr{K}_\nu$$

and (C_2) is equivalent to

$$(C_2') \qquad \sup\{Dg_j(\alpha_1, \xi_1) : j \in A,\ \{(\alpha_i, \xi_i)\}_{i=1}^{\infty} \in \mathscr{K}_\nu\} < \infty .$$

The assumption (C) covers the usual cases. For example, the standard estimate based on the knowledge of the preceding parameters (see [7], [10]) defined by

$$\mu_n(\omega, E) = \frac{1}{n-l} \sum_{i=1}^{n-l} \chi_E(\alpha_i(\omega)) , \tag{9}$$

where χ_E denotes the indicator of the set E, can be transformed into form (C) putting

$$g_j(a, x) = \begin{cases} 0 & \text{if} \quad a \neq j , \\ 1 & \text{if} \quad a = j , \end{cases} \tag{10}$$

and both (C_1) and (C_2) are evidently satisfied.

It is also possible, on the basis of the preceding observations (see [8], [6]), to construct an estimate satisfying (C), (C_1) and (C_2), provided that the class of all mixtures of ν_a's, $a \in A$, is identifiable*. Since the identifiability guarantees the existence of disjoint sets E_i, $i \in A$, such that det $\|\nu_a(E_i)\| \neq 0$ (in fact, both these conditions are equivalent), it is sufficient to put

$$g_j(a, x) = \begin{cases} h_{i,j} & \text{if} \quad x \in E_i , \\ 0 & \text{if} \quad x \notin \bigcup_{i \in A} E_i , \end{cases}$$

where $(h_{i,j} : i \in A, j \in A)$ is the solution of the system of linear equations

$$\sum_{i \in A} h_{i,j} \nu_a(E_i) = \begin{cases} 1 & \text{if} \quad a = j , \\ 0 & \text{if} \quad a \neq j , \end{cases} \qquad a, j \in A .$$

Another construction of g_j is described in [8].

Now, we shall formulate and prove our result concerning the utilization of experience in the statistical decision problem $\mathfrak{D}$.

* A mixture of ν_a's, $a \in A$, is a probability measure P_μ on $\mathscr{A} \times \mathscr{X}$ defined by $P_\mu(E \times F) = \sum_{a \in E} \mu(\{a\})\, \nu_a(F)$, where $\mu \in M_0$. The identifiability means: $\mu \neq \mu' \Rightarrow P_\mu \neq P_{\mu'}$.

Theorem. *Let p be an optimum procedure, let g_j, $j \in A$, be Borel measurable functions on $(A \times X, \mathscr{A} \times \mathscr{X})$ satisfying the conditions (C_1) and (C_2), let l be a positive integer, let μ_n, $n = l+1, l+2, \ldots$ be mappings from Ω into M satisfying (C) and let $\mu_1, \mu_2, \ldots, \mu_l$ be arbitrary measurable mappings from $(\Omega, \mathscr{S})$ into $(M, \mathscr{M})$. Then the following assertions hold:*

(i) *There is a finite constant c_1 such that for every n and every $\{(\alpha_i, \xi_i)\}_{i=1}^{\infty} \in \mathscr{K}_v$*

$$0 \leqq \mathsf{E}w(\alpha_n, p(\xi_n, \mu_n)) - \varrho(P\alpha_1^{-1}) \leqq \frac{c_1}{\sqrt{n}}.$$

(ii) *There is a finite constant c_2 such that for every n and every $\{(\alpha_i, \xi_i)\}_{i=1}^{\infty} \in \mathscr{K}_v$*

$$0 \leqq \mathsf{E}\frac{1}{n}\sum_{i=1}^{n} w(\alpha_i, p(\xi_i, \mu_i)) - \varrho(P\alpha_1^{-1}) \leqq \frac{c_2}{\sqrt{n}}.$$

(iii) *If $\{b_n\}_{n=1}^{\infty}$ is a monotonous sequence of positive numbers satisfying $\sum_{n=1}^{\infty} b_n^2 < < \infty$, then*

$$b_n \sum_{i=1}^{n} (w(\alpha_i, p(\xi_i, \mu_i)) - \varrho(P\alpha_1^{-1})) \to 0$$

a.s. uniformly in $\{(\alpha_i, \xi_i)\}_{i=1}^{\infty} \in \mathscr{K}_v$.

Proof. Let us denote

$$t = \{(\alpha_i, \xi_i)\}_{i=1}^{\infty},$$

$$\varrho_t = \varrho(P\alpha_1^{-1}),$$

and let $w_{t,i}$, $r_{t,i}$ and $\mu_{t,i}^{(a)}$ be functions on Ω defined by

$$w_{t,i}(\omega) = w(\alpha_i(\omega), p(\xi_i(\omega), \mu_i(\omega))),$$

$$r_{t,i}(\omega) = r(P\alpha_1^{-1}, p(\cdot, \mu_i(\omega))),$$

$$\mu_{t,i}^{(a)}(\omega) = |(\mu_i(\omega))(\{a\}) - P\alpha_1^{-1}(\{a\})|,$$

the function r being the risk defined by (1). Further, let $\mathscr{F}_{t,i+1}$ denote the σ-field induced by $\alpha_1, \xi_1, \alpha_2, \xi_2, \ldots, \alpha_i, \xi_i$ and let $\mathscr{F}_{t,1} = \{\emptyset, \Omega\}$. First of all let us remark that μ_i are measurable mappings from $(\Omega, \mathscr{S})$ into $(M, \mathscr{M})$ and therefore, by the well-known theorem about compound mappings, $w_{t,i}$, $r_{t,i}$ and $\mu_{t,i}^{(a)}$ are random variables.

(i) Since the families of random variables $\{\alpha_1, \xi_1, \alpha_2, \xi_2, \ldots, \alpha_{n-1}, \xi_{n-1}\}$ and $\{\alpha_n, \xi_n\}$ are independent and μ_n is $\mathscr{F}_{t,n}$-measurable, it holds

$$\mathsf{E}\{w_{t,n} \mid \mathscr{F}_{t,n}\} = r_{t,n} \tag{11}$$

 and hence

$$Ew_{t,n} = Er_{t,n} . \tag{12}$$

By Lemma 1,

$$0 \leqq r_{t,n} - \varrho_t \leqq m \sum_{a \in A} \mu_{t,n}^{(a)} , \tag{13}$$

so that, by Schwartz's inequality,

$$0 \leqq Er_{t,n} - \varrho_t \leqq m \sum_{a \in A} \sqrt{[E(\mu_{t,n}^{(a)})^2]} . \tag{14}$$

Observe that for $n > l$

$$\mu_{t,n}^{(a)} = \left| \frac{1}{n - l} \sum_{j=1}^{n-l} (g_a(\alpha_j, \xi_j) - P\alpha_1^{-1}(\{a\})) \right| .$$

Since $g_a(\alpha_j, \xi_j)$, $j = 1, 2, \ldots$, are independent and equally distributed random variables with $Eg_a(\alpha_j, \xi_j) = P\alpha_1^{-1}(\{a\})$, it holds

$$E(\mu_{t,n}^{(a)})^2 = \sum_{i=1}^{n-l} \frac{Dg_a(\alpha_i, \xi_i)}{(n - l)^2} ,$$

where $Dg_a(\alpha_i, \xi_i)$ denotes the dispersion of $g_a(\alpha_i, \xi_i)$. The assumption (C_2) implies that all these dispersions are uniformly bounded by a finite constant c. Therefore

$$E(\mu_{t,n}^{(a)})^2 \leqq \frac{c}{n - l} \quad \text{for all} \quad a \in A,\ t \in \mathscr{K}_v \quad \text{and} \quad n > l . \tag{15}$$

From (12), (14) and (15) it follows that

$$0 \leqq Ew_{t,n} - \varrho_t \leqq \frac{m\bar{A}\sqrt{c}}{\sqrt{(n - l)}}$$

for all $t \in \mathscr{K}_v$ and all $n > l$, the symbol $\bar{A}$ denoting the number of elements of A. Therefore, with regard to (3) and (5), the assertion (i) is proved.

(ii) The assertion (ii) is an immediate consequence of (i) and of the inequality

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leqq 1 + \int_1^n \frac{dx}{\sqrt{x}} = 2\sqrt{n} - 1 . \tag{16}$$

(iii) Since $b_n \downarrow 0$, it follows from (3) and (5)

$$b_n \sum_{i=1}^{\min(n,l)} (w_{t,i} - \varrho_t) \to 0 \quad \text{a.s. uniformly in } \mathscr{K}_v . \tag{17}$$

Since $\sum_{n=1}^{\infty} b_n^2 < \infty$, the assumption ($C_2$) implies that the series

$$\sum_{i=1}^{\infty} \frac{\mathsf{D}w_{t,i+l}}{b_{i+l}^2}$$

is uniformly convergent and uniformly bounded in $\mathscr{K}_v$. Moreover, (11) holds. Therefore, applying Proposition 1 with $\xi_{t,n}$ replaced by $w_{t,i+l}$ and b_n replaced by b_{n+l}, we obtain

$$b_n \sum_{i=l+1}^{n} (w_{t,i} - r_{t,i}) \to 0 \quad \text{a.s. uniformly in } \mathscr{K}_v . \tag{18}$$

From Schwartz's inequality and from (15) we get

$$\mathsf{E}\mu_{t,i}^{(a)}\mu_{t,j}^{(a)} \leqq \sqrt{[\mathsf{E}(\mu_{t,i}^{(a)})^2 \, \mathsf{E}(\mu_{t,j}^{(a)})^2]} \leqq \frac{c}{\sqrt{[(i-l)(j-l)]}}, \quad i > l, \quad j > l .$$

Having summed over all $l < i \leqq n$ and $l < j \leqq n$ we obtain by (16)

$$\mathsf{E}\Big(\sum_{i=l+1}^{n} \mu_{t,i}^{(a)}\Big)^2 \leqq \sum_{i=1}^{n-l} \sum_{j=1}^{n-l} \frac{1}{\sqrt{i}} \cdot \frac{1}{\sqrt{j}} < 4n .$$

Therefore, applying Proposition 2 with $\xi_{t,n}$ replaced by $\mu_{t,n+l}^{(a)}$ and b_n replaced by b_{n+l}, we get

$$b_n \sum_{i=l+1}^{n} \mu_{t,i}^{(a)} \to 0 \qquad \text{a.s. uniformly in } \mathscr{K}_v$$

so that, by (13),

$$b_n \sum_{i=l+1}^{n} (r_{t,i} - \varrho_t) \to 0 \quad \text{a.s. uniformly in } \mathscr{K}_v . \tag{19}$$

Since

$$b_n \sum_{i=1}^{n} (w_{t,i} - \varrho_t) = b_n \sum_{i=1}^{\min(n,l)} (w_{t,i} - \varrho_t) +$$

$$+ \, b_n \sum_{i=l+1}^{n} (w_{t,i} - r_{t,i}) + b_n \sum_{i=l+1}^{n} (r_{t,i} - \varrho_t) ,$$

the assertion (iii) follows from (17), (18) and (19). This concludes the proof of Theorem.

8. COMMENTS

I. Main theorem remains valid for $l = 0$. This can be proved by combining the methods of the proofs of our Theorem and Theorem 4 in [2]. Decision processes corresponding to $l = 0$ were proposed by E. Samuel as quite natural decisions in the case that there is no knowledge of the sequence of parameter values and the decision in the n-th step, $n = 1, 2, \ldots$, may be a function of $\xi_1, \xi_2, \ldots, \xi_n$ only.

II. The assumptions (C), (C_1) and (C_2) concerning the structure of estimates μ_n could be weakened. In the proof of our Theorem we use only the fact that μ_n, $n = l + 1, l + 2, \ldots$, have the form

$$\mu_n(\omega, E) = \sum_{i \in E} \frac{1}{n - l} \sum_{j=1}^{n-l} f_j^{(i)}(\omega),$$

where, for every $i \in A$, $\{f_n^{(i)}\}_{n=1}^{\infty}$ is a sequence of independent (not necessarily equally distributed) random variables satisfying

a) the families $\{f_1^{(i)}, f_2^{(i)}, \ldots, f_n^{(i)}\}$ and $\{\alpha_{n+1}, \xi_{n+1}\}$ are independent,
b) $Ef_n^{(i)} = P\alpha_1^{-1}(\{i\})$,
c) $\sup Df_n^{(i)} < \infty$.

However, I do not know any reasonable application of this more general form that cannot be covered by the assumptions (C), (C_1) and (C_2).

III. Under some supplementary assumptions the assertion (iii) of our Theorem can be proved on the basis of Theorem 3 in [2], concerning convergence of average losses in the case when the parameter is not assumed to be a random one. To verify it, observe that

$$b_n \sum_{i=1}^{n} (w(\alpha_i, p(\xi_i, \mu_i)) - \varrho(P\alpha_1^{-1})) = U_n + V_n,$$

where

$$U_n = b_n \sum_{i=1}^{n} (w(\alpha_i, p(\xi_i, \mu_i)) - \varrho(\bar{\xi}_n)),$$
$$V_n = nb_n(\varrho(\bar{\xi}_n) - \varrho(P\alpha_1^{-1}))$$

and $\bar{\xi}_n$ denotes the empirical distribution of the parameter in the n-th step. Writing

$$\varrho(\bar{\xi}_n) - \varrho(P\alpha_1^{-1}) = (r(\bar{\xi}_n, p(\cdot, \bar{\xi}_n) - r(\bar{\xi}_n, p(\cdot, P\alpha_1^{-1}))),$$
$$+ r(\bar{\xi}_n - P\alpha_1^{-1}, p(\cdot, P\alpha_1^{-1}))$$

and using Lemma 2, (2) and Proposition 1 we shall find that

$$V_n \to 0 \quad \text{a.s. uniformly in } \mathscr{K}_v .$$

Further

$$P(\bigcup_{n=k}^{\infty} \{|U_n| \geqq \varepsilon\}) = EP(\bigcup_{n=k}^{\infty} \{|U_n| \geqq \varepsilon\} \mid \alpha_1, \alpha_2, \ldots).$$

If we neglect the unessential difference between the definitions of optimal procedures in this paper and in [2] and if we assume that

(20) for every $a \in A$, the rank of the covariance matrix $C^{(a)} = (c_{i,j}^{(a)})_{i,j \in A}$ is equal to $\bar{A}$ or is equal to $\bar{A} - 1$ and $\sum_{j \in A} c_{i,j}^{(a)} = 0$, where $\bar{A}$ denotes the number of elements of the set A,

$$c_{i,j}^{(a)} = \int (g_i(a, x) - \delta_{i,a}) (g_j(a, x) - \delta_{j,a}) \, \mathrm{d}\nu_a(x) \, ,$$

$$\delta_{i,a} = \begin{cases} 1 & \text{if} \quad i = a \, , \\ 0 & \text{if} \quad i \neq a \, , \end{cases}$$

then Theorem 3 in [2] implies that

$$P(\bigcup_{n=k}^{\infty} \{|U_n| \geqq \varepsilon\} \mid \alpha_1, \alpha_2, \ldots\} \to 0$$

uniformly in all sequences $\{\alpha_i\}_{i=1}^{\infty}$. Therefore, under (20),

$$U_n \to 0 \quad \text{a.s. uniformly in } \mathscr{K}_\nu \, .$$

The quite analogous remark could be added to the assertion (ii).

The assumption (20) is rather restrictive, for example the functions g_i defined by (10) do not satisfy it and for μ_n defined by (9) U_n need not converge to zero (see Example on p. 267 in [2]). Our Theorem shows, among others, that the assumption (20) is superfluous if the parameter is a random one.

IV. It is interesting to compare the convergence rate of losses corresponding to the decision process $(p(\xi_1, \mu_1), p(\xi_2, \mu_2), \ldots)$ that is guaranteed by our Theorem with that corresponding to the optimal decision process $(p(\xi_1, P\alpha_1^{-1}), p(\xi_2, P\alpha_1^{-1}), \ldots,$ constructed on the full knowledge of the a priori distribution. We shall limit ourselves to the sequences $\{b_n\}_{n=1}^{\infty}$ having, for sufficiently large n, the form

$$\frac{1}{b_n^2} = c_n n \log \log n \, ,$$

where $c_n \uparrow \infty$.

As $w(\alpha_i, p(\xi_i, P\alpha_1^{-1}))$, $i = 1, 2, \ldots$, are bounded independent variables, Theorem 18.2.A in [3] implies that

$$b_n \sum_{i=1}^{n} (w(\alpha_i, p(\xi_i, P\alpha_1^{-1})) - \varrho(P_1^{-1}\alpha)) \to 0$$

for ever such a sequence $\{b_n\}_{n=1}^{\infty}$.

On the other hand, our Theorem guarantees that

$$b_n \sum_{i=1}^{n} (w(\alpha_i, p(\xi_i, \mu_i)) - \varrho(P\alpha_1^{-1})) \to 0$$

only if $\sum_{n=1}^{\infty} b_n^2 < \infty$, and for this inequality it is necessary that

$$\limsup_{n\to\infty} \frac{c_n}{\log n} = \infty .$$

(Received July 3, 1972.)

REFERENCES

[1] M. Driml, O. Hanš: On Experience Theory Problems. Transactions of the Second Prague Conference on Information Theory, Statistical Decision Functions and Random Processes. Prague 1959, 93—111.

[2] S. Jílovec: On the Uniform Almost Sure Convergence of Losses for the Repetition of a Statistical Decision Problem. Kybernetika *6* (1970), 4, 249—269.

[3] M. Loève: Probability Theory. New York 1955.

[4] E. Parzen: On Uniform Convergence of Families of Sequences of Random Variables. Univ. of California Publ. in Statist. 2 (1954), 23—54.

[5] H. Robbins: An Empirical Bayes Approach to Statistics. Proc. Third Berkeley Symp. Math. Statist. Prob. 1955, 1, 157—163.

[6] H. Robbins: The Empirical Bayes Approach to Statistical Decision Problem. Ann. Math. Statist. *35* (1964), 1—20.

[7] A. Špaček: An Elementary Experience Problem. Transactions of the First Prague Conference on Information Theory, Statistical Decision Functions, Random Processes. Prague 1956, 253—258.

[8] J. Van Ryzin: The Compound Decision Problem with $m \times n$ Finite Loss Matrix. Ann. Math. Statist. *37* (1966), 412—424.

[9] J. Van Ryzin: The Sequential Compound Decision Problem with $m \times n$ Finite Loss Matrix. Ann. Math. Statist. *37* (1966), 954—975.

[10] K. Winkelbauer: Experience in Games of Strategy and in Statistical Decision. Transactions of the First Prague Conference on Information Theory, Statistical Decision Functions, Random Processes. Prague 1956, 297—354.

RNDr. Stanislav Jílovec, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Vyšehradská 49, 128 48 Praha 2. Czechoslovakia.