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# Optimal Control of a Linear Discrete System

JAN ŠTECHA, ALENA KOZÁČIKOVÁ, JAROSLAV KOZÁČIK, JIŘÍ LIDICKÝ

Optimal control of linear discrete dynamic systems with quadratic performance index is discussed. The optimal controller uses all states of the system. An additional constraint of using only the measurable output of the system is imposed. A suboptimal controller using only the measurable output of the system is derived. The tracking problem for a discrete system is also solved.

## 1. INTRODUCTION

Recently, increasing attention has been paid to the theory of systems; this is manifested by the great number of articles published all over the world. Application of this theory to the problems of control has led to the construction of new algorithms for the synthesis of dynamic systems which could not be obtained by classical methods in control engineering. Moreover, implementation of these algorithms necessitates utilization of digital computer.

In the well known literature [1], [2], [15], discussion is given of the synthesis of a continuous linear dynamic system (CLDS), i.e. determination of a feedback such that the control circuit fulfils our demands given by the specific performance criterion.

In recent literature [4], [5], [16], [13], a great deal of attention has been paid to the problems of synthesis of a CLDS using performance criterion formed by the integral of quadratic form of state and control of the system.

The present paper deals with the synthesis of a discrete linear dynamical system (DLDS). For determination of optimal discrete control we can use the dynamic programming approach [19] or the discrete maximum or minimum principle [20]. For determination of dynamic properties of a DLDS we shall use a state model which has state equation in the form

$$\mathbf{x}_{k+1} = \mathbf{M}\mathbf{x}_k + \mathbf{N}\mathbf{u}_k \,, \tag{1.1}$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k \tag{1.2}$$

where

$\mathbf{x}_k$, $\mathbf{u}_k$, $\mathbf{y}_k$ are $n$, $r$, $m$ vectors of state, control, and output respectively at time $k$,
$\mathbf{M}$, $\mathbf{N}$, $\mathbf{C}$ are $(n \times n)$, $(n \times r)$ and $(m \times n)$ matrices of system, control, and output respectively the elements of which may be functions of time, and
$k$ is discrete time, 0, 1, 2 ...

For finite time $K$ of control we assume that the performance criterion has the form

$$J_K = \tfrac{1}{2}\mathbf{x}_K^{\mathrm{T}}\mathbf{S}\mathbf{x}_K + \tfrac{1}{2}\sum_{k=0}^{K-1}(\mathbf{x}_k^{\mathrm{T}}\mathbf{Q}\mathbf{x}_k + \mathbf{u}_k^{\mathrm{T}}\mathbf{R}\mathbf{u}_k) \,. \tag{1.3}$$

where $\mathbf{Q}$, $\mathbf{S}$, $\mathbf{R}$ are symmetric positive semidefinite matrices of dimension $(n \times n)$, $(n \times n)$, $(r \times r)$ respectively ($\mathbf{S}$ is a constant matrix). For infinite time of control, the performance criterion has the form

$$J = \tfrac{1}{2}\sum_{k=0}^{\infty}(\mathbf{x}_k^{\mathrm{T}}\mathbf{Q}\mathbf{x}_k + \mathbf{u}_k^{\mathrm{T}}\mathbf{R}\mathbf{u}_k) \,. \tag{1.4}$$

where $\mathbf{Q}$, $\mathbf{R}$ are constant symmetric positive semidefinite matrices.

In the second part of the present paper we shall derive relations for optimal feedback assuming that all states of the system are obtainable — we have so-called complete information about the state of the system. In the third part we discuss the tracking problem in the case of complete information.

In practical applications we cannot measure all states of the system and offen it is not even economic to do so. In such case the designer can choose from various possibilities:

1. use reconstruction of state by observer - see [6], [7],
2. use reconstruction of state by Kalmann-Bucy filter - see [2],
3. use only the output of system for control and thus control the system in a suboptimal way.

The first and second cases utilizing reconstruction of the state of system have the disadvantage in increasing the order of the whole system.

The fourth part of this paper deals with the problem mentioned under item 3. A new algorithm is derived for solution of optimal feedback from the output of the system. In the fifth section, the tracking problem is solved in the case of incomplete information about the state of the system.

## 2. SYNTHESIS OF A DLDS WITH COMPLETE INFORMATION ABOUT THE STATE OF THE SYSTEM-CONTROL OF STATE

Given a DLDS described by state equations (1.1) and (1.2) and performance criterion (1.3). Using dynamic programming or the discrete minimum principe [8] we can derive the optimal control law

$$u_k^* = -G_{K-k} x_k^* \tag{2.1}$$

where

$$G_{K-k} = (R + N^T K_{K-k-1} N)^{-1} N^T K_{K-k-1} M \tag{2.2}$$

is time dependent matrix of linear feedback and matrix $K_{K-k}$ satisfies the discrete form of Riccati equation

$$\begin{aligned} K_{K-k} = Q + M^T K_{K-k} M - \\ - M^T K_{K-k-1} N (R + N^T K_{K-k-1} N)^{-1} N^T K_{K-k-1} M . \end{aligned} \tag{2.3}$$

Initial conditions for relations (2.2) and (2.3) are

$$G_0 = 0 , \quad K_0 = S . \tag{2.4}$$

Detailed derivation of these relations can be found for instance in [9].

In the case of infinite time of control with performance criterion in the form (1.4) and for a time invariant DLDS (1.1) and (1.2) relations (2.1) to (2.3) can be used in the limit for $K \to \infty$.

As shown in [8], matrices $G_{K-k}$, $K_{K-k}$ may converge to finite values $G$ and $K$. They really converge if the system is stabilizable. Then we have the following relation for optimal feedback

$$u_k^* = -G x_k^* \tag{2.5}$$

where

$$G = (R + N^T K N)^{-1} N^T K M \tag{2.6}$$

and $K$ is a symemtric positive definite matrix satisfying the discrete version of algebraic Riccati equation

$$K = Q + M^T K M - M^T K N (R + N^T K N)^{-1} N^T K M . \tag{2.7}$$

Equation (2.7) has only one positive semidefinite solution if the pair $(M, Q^{1/2})$ is observable.

These relations can be solved only using a digital computer especially in the case of systems of a higher order. Relations (2.2) and (2.3) can be programmed without difficulties if initial conditions (2.4) are known. If $S = 0$, we can use the initial condition in the form

$$\begin{aligned} G_1 = 0 , \\ K_1 = Q . \end{aligned} \tag{2.8}$$

From the point of view of practical use it is more convenient to consider infinite time of control. In that case we must solve the nonlinear matrix algebraic equations (2.6) and (2.7); there are two ways of solving them:

1. We can use relations (2.2) and (2.3) for the initial condition $\mathbf{K}_0 = 0$ and $\mathbf{G}_0 = 0$ and solve these relations for time $K \to \infty$. As a terminal condition we can use the relation

$$\|\mathbf{K}_{K-k} - \mathbf{K}_{K-k-1}\| \leqq \varepsilon \tag{2.9}$$

where $\varepsilon$ is a preset number of the order of $10^{-3}$ to $10^{-6}$. Normally 30 to 40 iterations are sufficient to satisfy relation (2.9).

2. Relations (2.6) and (2.7) can be solved using the following algorithm mentioned in [10].

**Theorem 1.** *Let $\mathbf{V}_k$, $k = 0, 1, 2, \ldots$, be the solution of the equation*

$$\mathbf{V}_k = \mathbf{M}_k^{\mathrm{T}} \mathbf{V}_k \mathbf{M}_k + \mathbf{L}_k^{\mathrm{T}} \mathbf{R} \mathbf{L}_k + \mathbf{Q} \tag{2.10}$$

*where*

$$\mathbf{L}_k = (\mathbf{R} + \mathbf{N}^{\mathrm{T}} \mathbf{V}_{k-1} \mathbf{N})^{-1}\, \mathbf{N}^{\mathrm{T}} \mathbf{V}_{k-1} \mathbf{M}\,, \quad k = 1, 2, \ldots, \tag{2.11}$$

$$\mathbf{M}_k = \mathbf{M} - \mathbf{N} \mathbf{L}_k\,. \tag{2.12}$$

*The matrix $\mathbf{L}_0$ must be chosen such that $\mathbf{M}_0$ be stable. This implies that the pair $(\mathbf{M}, \mathbf{N})$ must be stabilizable. Then*

$$\mathbf{K} \leqq \mathbf{V}_{k+1} \leqq \mathbf{V}_k \tag{2.13}$$

*and*

$$\lim_{k \to \infty} \mathbf{V}_k = \mathbf{K}\,. \tag{2.14}$$

The proof of this algorithm is given in [10] where it is shown that the convergence rate is quadratic, i.e.

$$\|\mathbf{K} - \mathbf{V}_{k+1}\| \leqq C \|\mathbf{K} - \mathbf{V}_k\|^2 \tag{2.15}$$

where $C$ is a constant which does not depend on the iteration index $k$.

This algorithm can be easily implemented on digital computer; for solving the linear matrix equation (2.10) the algorithm derived in [11] can be used. Solution of many examples on a digitial computer has shown rapid convergence of this algorithm. Usually 4 or 5 iterations are sufficient for the difference $\|\mathbf{V}_k - \mathbf{V}_{k+1}\|$ to be less then $10^{-3}$.

The only remaining problem is that of choosing the stabilization matrix $\mathbf{L}_0$. Evidently for a stable system $\mathbf{L}_0 = 0$ can be chosen. Matrix $\mathbf{L}_0$ can also be chosen on the basis of physical interpretation of the problem. In general we can use algorithms for computation of the feedback matrix $\mathbf{L}_0$ such that eingenvalues of the matrix $\mathbf{M}_0$ be arbitrary. Practical algorithms and programmes are described in [12].

## 3. TRACKING PROBLEM FOR A DLDS WITH COMPLETE INFORMATION ABOUT THE STATE OF THE SYSTEM

In section 2 we assumed that the aim of control is the state vector $\mathbf{x} = 0$. In practical cases we often have to solve a problem with the target of control not in the origin of coordinates and it is demanded that the output $\mathbf{y}_k$ of the system should track a given vector $\mathbf{z}_k$ which may be either constant in time or time-dependent. For a CLDS this problem is discussed in [1], [12], [13]. For a DLDS we give here analogous algorithms, the derivation of which can be found in [9].

Let us define the error vector

$$\mathbf{e}_k = \mathbf{z}_k - \mathbf{y}_k \,. \tag{3.1}$$

We shall try to find a control vector $\mathbf{u}_k$ giving the minimal value of performance index in the form

$$J_K = \tfrac{1}{2}\mathbf{e}_K^{\mathrm{T}}\mathbf{S}\mathbf{e}_K + \tfrac{1}{2}\sum_{k=0}^{K-1}(\mathbf{e}_k^{\mathrm{T}}\mathbf{Q}\mathbf{e}_k + \mathbf{u}_k^{\mathrm{T}}\mathbf{R}\mathbf{u}_k) \tag{3.2}$$

where $\mathbf{S}$, $\mathbf{Q}$, $\mathbf{R}$ are symmetric, positive semidefinite real matrices $(m \times m)$, $(m \times m)$ and $(r \times r)$ respectively, $\mathbf{S}$ is a constant matrix, and $K$ is the time of control.

Using the discrete minimum principle or dynamic programming we can derive the optimal control law

$$\mathbf{u}_k^* = -(\mathbf{R} + \mathbf{N}^{\mathrm{T}}\mathbf{K}_{k+1}\mathbf{N})^{-1}(\mathbf{N}^{\mathrm{T}}\mathbf{K}_{k+1}\mathbf{M}\mathbf{x}_k^* + \mathbf{N}^{\mathrm{T}}\mathbf{g}_{k+1}) \tag{3.3}$$

where matrix $\mathbf{K}_k$ satisfies the discrete version of Riccati equation

$$\mathbf{K}_k = \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C} + \mathbf{M}^{\mathrm{T}}\mathbf{K}_{k+1}\mathbf{M} - (\mathbf{N}^{\mathrm{T}}\mathbf{K}_{k+1}\mathbf{M})^{\mathrm{T}}(\mathbf{R} + \mathbf{N}^{\mathrm{T}}\mathbf{K}_{k+1}\mathbf{N})^{-1}\,\mathbf{N}^{\mathrm{T}}\mathbf{K}_{k+1}\mathbf{M} \tag{3.4}$$

and vector $\mathbf{g}_k$ satisfies equation

$$\mathbf{g}_k = \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{z}_k + [\mathbf{M}^{\mathrm{T}} - \mathbf{M}^{\mathrm{T}}\mathbf{K}_{k+1}\mathbf{N}(\mathbf{R} + \mathbf{N}^{\mathrm{T}}\mathbf{K}_{k+1}\mathbf{N})^{-1}\,\mathbf{N}^{\mathrm{T}}]\,\mathbf{g}_{k+1} \tag{3.5}$$

with boundary conditions

$$\mathbf{K}_K = \mathbf{C}^{\mathrm{T}}\mathbf{S}\mathbf{C}\,, \tag{3.6a}$$

$$\mathbf{g}_K = \mathbf{C}^{\mathrm{T}}\mathbf{S}\mathbf{z}_K\,. \tag{3.6b}$$

It is obvious from the form of the equations (3.4) to (3.6) that they can be solved starting from $k = K$ to $k = 0$. It means that this procedure can be used only if we know the values of the desired vector $\mathbf{z}_k$ in the whole interval in advance. Moreover we are limited to finite time of control only.

In some cases, the tracking problem can be transformed into the problem of control of state. Let us suppose that the desired vector $\mathbf{z}_k$ equals the solution of the difference equation

$$\mathbf{z}_{k+1} = \mathbf{F}\mathbf{z}_k \,. \tag{3.7}$$

Let us define the extended vector $\mathbf{w}_k$ of dimension $n + m$

$$\mathbf{w}_k = \begin{bmatrix} \mathbf{x}_k \\ \mathbf{z}_k \end{bmatrix} \tag{3.8}$$

which will be a state vector of the system described by state equations

$$\begin{aligned} \mathbf{w}_{k+1} &= \mathbf{M}\mathbf{w}_k + \mathbf{N}\mathbf{u}_k \,, \\ \mathbf{y}_k &= \mathbf{C}\mathbf{w}_k \end{aligned} \tag{3.9}$$

where matrices $\mathbf{M}$, $\mathbf{N}$, $\mathbf{C}$ of dimensions $(n + m) \times (n + m)$, $(n + m) \times r$, $m \times \times (n + m)$ respectively have the form

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} \mathbf{N} \\ \mathbf{0} \end{bmatrix}, \quad \hat{\mathbf{C}} = [\mathbf{C} \;\, \mathbf{0}] \,. \tag{3.10}$$

Tracking problem with performance criterion (3.2) can be transformed into a problem of control of state of the system (3.9) with the performance criterion

$$J_K = \tfrac{1}{2}\mathbf{w}_K^{\mathrm{T}}\hat{\mathbf{S}}\mathbf{w}_K + \tfrac{1}{2}\sum_{k=0}^{K-1}(\mathbf{w}_k^{\mathrm{T}}\hat{\mathbf{Q}}\mathbf{w}_k + \mathbf{u}_k^{\mathrm{T}}\hat{\mathbf{R}}\mathbf{u}_k) \tag{3.11}$$

where matrices $\mathbf{S}$, $\mathbf{Q}$, $\mathbf{R}$ are defined by

$$\hat{\mathbf{S}} = \begin{bmatrix} \mathbf{C}^{\mathrm{T}}\mathbf{S}\mathbf{C} & -\mathbf{C}^{\mathrm{T}}\mathbf{S} \\ -\mathbf{S}\mathbf{C} & \mathbf{S} \end{bmatrix}, \quad \hat{\mathbf{Q}} = \begin{bmatrix} \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C} & -\mathbf{C}^{\mathrm{T}}\mathbf{Q} \\ -\mathbf{Q}\mathbf{C} & \mathbf{Q} \end{bmatrix}, \quad \hat{\mathbf{R}} = \mathbf{R} \,. \tag{3.12}$$

If system (3.9) is not stabilizable, functional (3.11) does not converge for $K \to \infty$ and this procedure cannot be used for infinite time of control. When $\mathbf{F} = \mathbf{M}$ and rank $(\mathbf{C}) = m = n$, the tracking problem can always be transformed into a control-of-state problem even for infinite time of control.

## 4. SYNTHESIS OF A DLDS WITH INCOMPLETE INFORMATION ABOUT THE STATE OF THE SYSTEM

In sections 2 and 3 we assumed that all states of the system are measurable. It is often not possible in praxis and it is not even economic. In such a case one of the three possibilities mentioned in the introduction can be used.

In this and the following section we derive some new algorithms utilizing, for linear feedback, only the measurable output of the system.

Let us have a DLDS described by equations (1.1) and (1.2) and the performance criterion in the form

$$J_K = \tfrac{1}{2} \sum_{k=0}^{K-1} (\mathbf{x}_k^T \mathbf{Q} \mathbf{x}_k + \mathbf{u}_k^T \mathbf{R} \mathbf{u}_k) \tag{4.1}$$

where matrices $\mathbf{Q}$, $\mathbf{R}$ are positive semidefinite.

Optimal control is assumed in the form

$$\mathbf{u}_k = -\mathbf{F}_k \mathbf{y}_k = -\mathbf{F}_k \mathbf{C} \mathbf{x}_k \tag{4.2}$$

where $\mathbf{F}_k$ is the feedback matrix in the $k$-th step of control. Performance criterion (4.1) can then be written in the form

$$J_K = \tfrac{1}{2} \sum_{k=0}^{K-1} \mathbf{x}_k (\mathbf{Q} + \mathbf{C}^T \mathbf{F}_k \mathbf{R} \mathbf{F}_k \mathbf{C})\, \mathbf{x}_k\,. \tag{4.3}$$

Solution of the state equation (1.1) with control (4.2) can be written in the form

$$\mathbf{x}_k = \boldsymbol{\Phi}_k \mathbf{x}_0 \tag{4.4}$$

where the state transition matrix has the form

$$\boldsymbol{\Phi}_k = \prod_{i=0}^{k-1} (\mathbf{M} - \mathbf{N} \mathbf{F}_i \mathbf{C})\,, \quad k = 1, 2, \ldots, K \tag{4.5}$$

and $\mathbf{x}_0$ is the initial state of the system.

From (4.5) it is obvious that matrix $\boldsymbol{\Phi}_k$ satisfies the equation

$$\boldsymbol{\Phi}_{k+1} = (\mathbf{M} - \mathbf{N} \mathbf{F}_k \mathbf{C})\, \boldsymbol{\Phi}_k\,, \quad \boldsymbol{\Phi}_0 = \mathbf{I}\,. \tag{4.6}$$

Substituting relation (4.5) into (4.3) we obtain

$$J_K = \tfrac{1}{2} \sum_{k=0}^{K-1} \mathbf{x}_0^T \boldsymbol{\Phi}_k^T (\mathbf{Q} + \mathbf{C}^T \mathbf{F}_k \mathbf{R} \mathbf{F}_k \mathbf{C})\, \boldsymbol{\Phi}_k \mathbf{x}_0$$

and utilizing relation $(\mathbf{x}^T \mathbf{A} \mathbf{x}) = \operatorname{tr}(\mathbf{x} \mathbf{x}^T \mathbf{A})$

$$J_K = \tfrac{1}{2} \operatorname{tr} \sum_{k=0}^{K-1} \mathbf{x}_0 \mathbf{x}_0^T \boldsymbol{\Phi}_k (\mathbf{Q} + \mathbf{C}^T \mathbf{F}_k \mathbf{R} \mathbf{F}_k \mathbf{C})\, \boldsymbol{\Phi}_k\,. \tag{4.7}$$

Functional (4.7) depends on state transition matrix $\boldsymbol{\Phi}_k$, feedback matrix $\mathbf{F}_k$ and also on initial state $\mathbf{x}_0$.

Let us assume that the vector of initial state $x_0$ is a random variable which satisfies the relations

$$E\{\mathbf{x}_0\} = 0\,, \quad E\{\mathbf{x}_0 \mathbf{x}_0^T\} = \mathbf{I}\,. \tag{4.8}$$

Criterion (4.7) is now a random variable. Evaluating the expected value of the performance criterion we obtain the modified performance measure in the form

$$\hat{J}_K = \tfrac{1}{2}E \sum_{k=0}^{K-1} \boldsymbol{\Phi}_k^{\mathrm{T}}(\mathbf{Q} + \mathbf{C}^{\mathrm{T}}\mathbf{F}_k^{\mathrm{T}}\mathbf{R}\mathbf{F}_k\mathbf{C})\,\boldsymbol{\Phi}_k\,. \tag{4.9}$$

In this way we have transformed the problem of minimizing criterion (4.1) from the space of states to the matrix space where matrix $\boldsymbol{\Phi}_k$ corresponds to the state of the system, matrix $\mathbf{F}_k$ corresponds to the control of the system and the system is described by equation (4.5). Instead of performance criterion (4.1) we shall minimize the expected value in (4.9).

This problem can be solved using dynamic programming [13]. Here we use the matrix minimum principle [17]. In the following derivation we use the identities [17]

$$\begin{aligned}
&\operatorname{tr}[\mathbf{ABC}] = \operatorname{tr}[\mathbf{CAB}] = \operatorname{tr}[\mathbf{BCA}]\,,\\
&\frac{\partial}{\partial \mathbf{x}}\operatorname{tr}[\mathbf{AX}] = \mathbf{A}^{\mathrm{T}}\,,\\
&\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}[\mathbf{AX}^{\mathrm{T}}] = \mathbf{A}\,,\\
&\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}[\mathbf{X}] = \mathbf{I}\,,\\
&\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}[\mathbf{X}^{\mathrm{T}}] = \mathbf{I}\,,\\
&\frac{\partial}{\partial \mathbf{X}}\operatorname{tr}[\mathbf{AXBX}^{\mathrm{T}}] = \mathbf{AXB} + \mathbf{A}^{\mathrm{T}}\mathbf{XB}^{\mathrm{T}}\,.
\end{aligned} \tag{4.10}$$

Let us define the Hamiltonian function by

$$H = \tfrac{1}{2}\operatorname{tr}[\boldsymbol{\Phi}_k^{\mathrm{T}}(\mathbf{Q} + \mathbf{C}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{R}\mathbf{F}_k\mathbf{C})\,\boldsymbol{\Phi}_k] + \operatorname{tr}[\mathbf{P}_{k+1}^{\mathrm{T}}(\mathbf{M} - \mathbf{N}\mathbf{F}_k\mathbf{C})\,\boldsymbol{\Phi}_k]\,. \tag{4.11}$$

The minimum of the Hamiltonian satisfies the condition

$$\frac{\partial H}{\partial \mathbf{F}_k} = 0\,. \tag{4.12}$$

From here it follows

$$\mathbf{0} = \mathbf{R}\mathbf{F}_k\mathbf{C}\boldsymbol{\Phi}_k\boldsymbol{\Phi}_k^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} - \mathbf{N}^{\mathrm{T}}\mathbf{P}_{k+1}\boldsymbol{\Phi}_k^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\,. \tag{4.13}$$

Matrix $\mathbf{P}_k$ is solution of the adjoint system

$$\mathbf{P}_k = \frac{\partial H}{\partial \boldsymbol{\Phi}_k} = \mathbf{Q}\boldsymbol{\Phi}_k + \mathbf{C}^{\mathrm{T}}\mathbf{F}_k^{\mathrm{T}}\mathbf{R}\mathbf{F}_k\mathbf{C}\boldsymbol{\Phi}_k + (\mathbf{M} - \mathbf{N}\mathbf{F}_k\mathbf{C})\,\mathbf{P}_{k+1}\,. \tag{4.14}$$

Let us assume that the constate matrix $\boldsymbol{P}_k$ has the form

(4.15) $$\boldsymbol{P}_k = \boldsymbol{K}_k \boldsymbol{\Phi}_k$$

and also

$$\boldsymbol{P}_{k+1} = \boldsymbol{K}_{k+1} \boldsymbol{\Phi}_{k+1} .$$

Equaling the expressions (4.15) and (4.14) we get a difference equation for $\boldsymbol{K}_k$

(4.16) $$\boldsymbol{K}_k = \boldsymbol{Q} + \boldsymbol{C}^{\mathrm{T}} \boldsymbol{F}_k^{\mathrm{T}} \boldsymbol{R} \boldsymbol{F}_k \boldsymbol{C} + (\boldsymbol{M} - \boldsymbol{N} \boldsymbol{F}_k \boldsymbol{C})^{\mathrm{T}} \boldsymbol{K}_{k+1} (\boldsymbol{M} - \boldsymbol{N} \boldsymbol{F}_k \boldsymbol{C}) .$$

Substituting (4.15) into (4.13) we can write

(4.17) $$0 = \boldsymbol{R} \boldsymbol{F}_k \boldsymbol{C} \boldsymbol{\Phi}_k \boldsymbol{\Phi}_k^{\mathrm{T}} \boldsymbol{C}^{\mathrm{T}} - \boldsymbol{N}^{\mathrm{T}} \boldsymbol{K}_{k+1} (\boldsymbol{M} - \boldsymbol{N} \boldsymbol{F}_k \boldsymbol{C}) \, \boldsymbol{\Phi}_k \boldsymbol{\Phi}_k^{\mathrm{T}} \boldsymbol{C} .$$

Let us introduce a matrix $\boldsymbol{L}_k$ defined by the equation

(4.18) $$\boldsymbol{L}_k = \boldsymbol{\Phi}_k \boldsymbol{\Phi}_k^{\mathrm{T}} .$$

From this (considering 4.6) the difference equation for $\boldsymbol{L}_k$ can be obtained as

(4.19) $$\boldsymbol{L}_k = (\boldsymbol{M} - \boldsymbol{N} \boldsymbol{F}_{k-1} \boldsymbol{C}) \, \boldsymbol{L}_{k-1} (\boldsymbol{M} - \boldsymbol{N} \boldsymbol{F}_{k-1} \boldsymbol{C})^{\mathrm{T}} .$$

Substituting relation (4.18) into (4.17) the equation for optimal feedback matrix $\boldsymbol{F}_k$ is derived $\boldsymbol{F}_k = (\boldsymbol{R} + \boldsymbol{N}^{\mathrm{T}} \boldsymbol{K}_{k+1} \boldsymbol{N})^{-1} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{K}_{k+1} \boldsymbol{M} \boldsymbol{L}_k \boldsymbol{C}^{\mathrm{T}} (\boldsymbol{C} \boldsymbol{L}_k \boldsymbol{C}^{\mathrm{T}})^{-1}$. From here follows a theorem giving a necessary condition for optimality.

**Theorem 2.** *Let $\boldsymbol{F}_k^*$ minimize the functional* (4.9). *Then*

(4.20) $$\boldsymbol{F}_k^* = (\boldsymbol{R} + \boldsymbol{N}^{\mathrm{T}} \boldsymbol{K}_{k+1} \boldsymbol{N})^{-1} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{K}_{k+1} \boldsymbol{M} \boldsymbol{L}_k \boldsymbol{C}^{\mathrm{T}} (\boldsymbol{C} \boldsymbol{L}_k \boldsymbol{C}^{\mathrm{T}})^{-1}$$

*where*

(4.21) $$\boldsymbol{K}_k = \boldsymbol{Q} + \boldsymbol{C}^{\mathrm{T}} \boldsymbol{F}_k^{*\mathrm{T}} \boldsymbol{R} \boldsymbol{F}_k^* \boldsymbol{C} + (\boldsymbol{M} - \boldsymbol{N} \boldsymbol{F}_k^* \boldsymbol{C})^{\mathrm{T}} \boldsymbol{K}_{k+1} (\boldsymbol{M} - \boldsymbol{N} \boldsymbol{F}_k^* \boldsymbol{C})$$

*and*

(4.22) $$\boldsymbol{L}_k = (\boldsymbol{M} - \boldsymbol{N} \boldsymbol{F}_{k-1}^* \boldsymbol{C}) \, \boldsymbol{L}_{k-1} (\boldsymbol{M} - \boldsymbol{N} \boldsymbol{F}_{k-1}^* \boldsymbol{C})^{\mathrm{T}} .$$

*Initial conditions for equations* (4.21) *and* (4.22) *are*

(4.23) $$\boldsymbol{L}_0 = \boldsymbol{I} , \quad \boldsymbol{K}_{K-1} = \boldsymbol{Q} , \quad \boldsymbol{F}_{K-1} = \boldsymbol{0} .$$

Equations (4.21), (4.22) and (4.20) are nonlinear matrix difference equations whose solution is difficult. Convenient algorithm for digitial computer has not been found yet.

Let us derive simple algorithms which would enable us to solve approximately equations (4.20) and (4.21). Equation (4.17) for the feedback matrix $\boldsymbol{F}_k$ has the form 

(4.24) $$\boldsymbol{A} \,.\, \boldsymbol{B} = \boldsymbol{0}$$

where

(4.25) $$\boldsymbol{A} = \boldsymbol{R}\boldsymbol{F}_k\boldsymbol{C} - \boldsymbol{N}^{\mathsf{T}}\boldsymbol{K}_{k+1}\boldsymbol{M} + \boldsymbol{N}^{\mathsf{T}}\boldsymbol{K}_{k+1}\boldsymbol{N}\boldsymbol{F}_k\boldsymbol{C}\,,$$
$$\boldsymbol{B} = \boldsymbol{\Phi}_k\boldsymbol{\Phi}_k^{\mathsf{T}}\boldsymbol{C}^{\mathsf{T}}\,.$$

Equation (4.24) has dimension $(r \times m)$ and has the solution for $(r \times m)$ elements of the feedback matrix $\boldsymbol{F}_k$. Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are divisors of zero but relation (4.24) is satisfied whencver $\boldsymbol{A} = \boldsymbol{0}$. This is a special solution of relation (4.24). According to [14] we can write a matrix equation

(4.26) $$\boldsymbol{A} = \boldsymbol{\Delta}$$

where $\boldsymbol{\Delta}$ is the error matrix. We want the error matrix to be minimal. Let us choose the criterion in the form

(4.27) $$J_1 = \operatorname{tr}(\boldsymbol{\Delta}\boldsymbol{\Delta}^{\mathsf{T}})\,.$$

Now we derive the equation for feedback matrix $\boldsymbol{F}_k$ giving minimal value of criterion (4.27). Substituting (4.25) and (4.26) into relation (4.27) we obtain

(4.28) $$J_1 = \operatorname{tr}\{[(\boldsymbol{R} + \boldsymbol{N}^{\mathsf{T}}\boldsymbol{K}_{k+1}\boldsymbol{N})\,\boldsymbol{F}_k\boldsymbol{C} - \boldsymbol{N}^{\mathsf{T}}\boldsymbol{K}_{k+1}\boldsymbol{M}]\,.$$
$$.\,[\boldsymbol{C}^{\mathsf{T}}\boldsymbol{F}_k(\boldsymbol{R} + \boldsymbol{N}^{\mathsf{T}}\boldsymbol{K}_{k+1}\boldsymbol{N}) - \boldsymbol{M}^{\mathsf{T}}\boldsymbol{K}_{k+1}\boldsymbol{N}]\}\,.$$

Denote

(4.29) $$\boldsymbol{D} = \boldsymbol{R} + \boldsymbol{N}^{\mathsf{T}}\boldsymbol{K}_{k+1}\boldsymbol{N}\,,$$
$$\boldsymbol{E} = \boldsymbol{N}^{\mathsf{T}}\boldsymbol{K}_{k+1}\boldsymbol{M}\,.$$

Equation (4.28) now has the form

$$J_1 = \operatorname{tr}(\boldsymbol{D}\boldsymbol{F}_k\boldsymbol{C}\boldsymbol{C}^{\mathsf{T}}\boldsymbol{F}_k^{\mathsf{T}}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{F}_k\boldsymbol{C}\boldsymbol{E}^{\mathsf{T}} - \boldsymbol{E}\boldsymbol{C}^{\mathsf{T}}\boldsymbol{F}_k^{\mathsf{T}}\boldsymbol{D} + \boldsymbol{E}\boldsymbol{E}^{\mathsf{T}})\,,$$

or

$$J_1 = \operatorname{tr}(\boldsymbol{D}\boldsymbol{F}_k\boldsymbol{C}\boldsymbol{C}^{\mathsf{T}}\boldsymbol{F}_k\boldsymbol{D}) - 2\operatorname{tr}(\boldsymbol{D}\boldsymbol{F}_k\boldsymbol{C}\boldsymbol{E}^{\mathsf{T}}) + \operatorname{tr}(\boldsymbol{E}\boldsymbol{E}^{\mathsf{T}})\,.$$

From the condition for $J_1$ to be minimal in every step of control it follows

(4.30) $$\frac{\partial J_1}{\partial \boldsymbol{F}_k} = 0\,.$$

Using relations (4.10) we obtain $\boldsymbol{D}\boldsymbol{F}_k\boldsymbol{C}\boldsymbol{C}^{\mathsf{T}} - \boldsymbol{E}\boldsymbol{C}^{\mathsf{T}} = \boldsymbol{0}$.

From here it follows

(4.31) $$\boldsymbol{F}_k = (\boldsymbol{R} + \boldsymbol{N}^{\mathsf{T}}\boldsymbol{K}_{k+1}\boldsymbol{N})^{-1}\,\boldsymbol{N}^{\mathsf{T}}\boldsymbol{K}_{k+1}\boldsymbol{M}\boldsymbol{C}^{\mathsf{T}}(\boldsymbol{C}\boldsymbol{C}^{\mathsf{T}})^{-1}\,.$$

 Utilizing criterion (4.27) in the form

$$J_1 = \operatorname{tr}(\Delta \boldsymbol{V} \Delta^{\mathrm{T}}) \tag{4.32}$$

where $\boldsymbol{V}$ is a symmetric positive definite weighting matrix of dimension $(n \times n)$ we obtain the optimal value of the feedback matrix in the form

$$\boldsymbol{F}_k = (\boldsymbol{R} + \boldsymbol{N}^{\mathrm{T}}\boldsymbol{K}_{k+1}\boldsymbol{N})^{-1}\, \boldsymbol{N}^{\mathrm{T}}\boldsymbol{K}_{k+1}\boldsymbol{M}\boldsymbol{V}\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{C}\boldsymbol{V}\boldsymbol{C}^{\mathrm{T}})^{-1}\,. \tag{4.33}$$

Relations (4.31) ro (4.33) for feedback matrix $\boldsymbol{F}_k$ with relation (4.21) for matrix $\boldsymbol{K}_k$ are easy to solve.

Theorem 2 gives only a necessary condition and so the uniqueness of the solution of equations (4.20)–(4.22) remains an open problem just as in the continuous version of this problem [18]. Results obtained using relations (4.31) or (4.33) do not guarantee the minimal value of criterion (4.10). Equation (4.31) is satisfied as nearly as possible in accordance with criterion (4.27) or (4.32).

## 5. TRACKING PROBLEM FOR A DLDS WITH INCOMPTLETE INFORMATION ABOUT THE STATE OF THE SYSTEM

As in section 3 we shall now discuss the problem of tracking a given vector $\mathbf{z}_k$. Let us define the error vector

$$\mathbf{e}_k = \mathbf{z}_k - \mathbf{y}_k\,. \tag{5.1}$$

The control law has the form

$$\mathbf{u}_k = -\boldsymbol{F}_k\mathbf{y}_k = -\boldsymbol{F}_k\boldsymbol{C}_k\mathbf{x}_k\,. \tag{5.2}$$

The performance criterion for this problem has the form

$$J_K = \tfrac{1}{2}\mathbf{e}_K^{\mathrm{T}}\boldsymbol{S}\mathbf{e}_K + \tfrac{1}{2}\sum_{k=0}^{K-1}(\mathbf{e}_k^{\mathrm{T}}\boldsymbol{Q}\mathbf{e}_k + \mathbf{u}_k^{\mathrm{T}}\boldsymbol{R}\mathbf{u}_k) \tag{5.3}$$

where matrices $\boldsymbol{Q}$, $\boldsymbol{R}$ satisfy the same conditions as in section 3. After substitution and modification criterion (5.3) has the form

$$\begin{aligned} J_k = {}& \tfrac{1}{2}(\mathbf{z}_K^{\mathrm{T}}\boldsymbol{S}\mathbf{z}_K - 2\mathbf{x}_0^{\mathrm{T}}\boldsymbol{\Phi}_K^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{S}\mathbf{z}_K + \mathbf{x}_0^{\mathrm{T}}\boldsymbol{\Phi}_K^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{C}\boldsymbol{\Phi}_K\mathbf{x}_0) + \\ &+ \tfrac{1}{2}\sum_{k=0}^{K-1}(\mathbf{z}_k\boldsymbol{Q}\mathbf{z}_k - 2\mathbf{x}_0^{\mathrm{T}}\boldsymbol{\Phi}_k^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\mathbf{z}_k + \mathbf{x}_0^{\mathrm{T}}\boldsymbol{\Phi}_k^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{C}\boldsymbol{\Phi}_k\mathbf{x}_0 + \mathbf{x}_0^{\mathrm{T}}\boldsymbol{\Phi}_k^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}_k^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F}_k\boldsymbol{C}\boldsymbol{\Phi}_k\mathbf{x}_0) \end{aligned} \tag{5.4}$$

where matrix $\boldsymbol{\Phi}_k$ satisfies relation (4.6),

$$\boldsymbol{\Phi}_{k+1} = (\boldsymbol{M} - \boldsymbol{N}\boldsymbol{F}_k)\,\boldsymbol{\Phi}_k\,, \quad \boldsymbol{\Phi}_0 = \boldsymbol{I}\,. \tag{5.5}$$

We shall now evaluate the expected values for $n$ linear independent initial vectors $\mathbf{x}_0$

satisfying the relation 

$$\sum_{i=1}^{n} \mathbf{x}_{0i}\mathbf{x}_{0i}^{\mathrm{T}} = \mathbf{I}\,. \tag{5.6}$$

The modified performance criterion has now the form

$$\hat{J}_K = \frac{1}{2n}\left(n\mathbf{z}_K^{\mathrm{T}}\mathbf{S}\mathbf{z}_K - 2\sum_{i=1}^{n}\mathbf{x}_{0i}^{\mathrm{T}}\boldsymbol{\Phi}_K^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{S}\mathbf{z}_K + \sum_{i=1}^{n}\mathbf{x}_{0i}^{\mathrm{T}}\boldsymbol{\Phi}_K^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{S}\mathbf{C}\boldsymbol{\Phi}_K\sum_{i=1}^{n}\mathbf{x}_{0i}\right) + \tag{5.7}$$
$$+ \frac{1}{2n}\sum_{k=0}^{K-1}\left(n\mathbf{z}_k^{\mathrm{T}}\mathbf{Q}\mathbf{z}_k - 2\sum_{i=1}^{n}\mathbf{x}_{0i}^{\mathrm{T}}\boldsymbol{\Phi}_k^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{z}_k + \sum_{i=1}^{n}\mathbf{x}_{0i}^{\mathrm{T}}\boldsymbol{\Phi}_k^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}\boldsymbol{\Phi}_k\sum_{i=1}^{n}\mathbf{x}_{0i} + \right.$$
$$\left. + \sum_{i=1}^{n}\mathbf{x}_{0i}^{\mathrm{T}}\boldsymbol{\Phi}_k^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{F}_k^{\mathrm{T}}\mathbf{R}\mathbf{F}_k\mathbf{C}\boldsymbol{\Phi}_k\sum_{i=1}^{n}\mathbf{x}_{0i}\right).$$

Let us assume that

$$\sum_{i=1}^{n}\mathbf{x}_{0i}^{\mathrm{T}} = [1, 1, \ldots, 1]\,. \tag{5.8}$$

As in [13] let us define a symmetric matrix of dimension $(m \times m)$

$$\bar{\mathbf{Z}}_k = n\mathbf{z}_k\mathbf{z}_k^{\mathrm{T}} \tag{5.9}$$

and a matrix $(m \times n)$

$$\tilde{\mathbf{Z}}_k = \mathbf{z}_k\sum_{i=1}^{n}\mathbf{x}_{0i}^{\mathrm{T}}\,. \tag{5.10}$$

After substitution of (5.8), (5.9), (5.10) into (5.7) the performance criterion now has the form

$$J_K = \frac{1}{2n}\,\mathrm{tr}\,[\boldsymbol{\Phi}_K^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{S}\mathbf{C}\boldsymbol{\Phi}_K + \bar{\mathbf{Z}}_K\mathbf{S} - 2\tilde{\mathbf{Z}}_K\boldsymbol{\Phi}_K\mathbf{C}^{\mathrm{T}}\mathbf{S}] + \tag{5.11}$$
$$+ \frac{1}{2n}\,\mathrm{tr}\,[\sum_{k=0}^{K-1}(\boldsymbol{\Phi}_k^{\mathrm{T}}(\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C} + \mathbf{C}^{\mathrm{T}}\mathbf{F}_k\mathbf{R}\mathbf{F}_k\mathbf{C})\,\boldsymbol{\Phi}_k + \bar{\mathbf{Z}}_k\mathbf{Q} - 2\tilde{\mathbf{Z}}_k\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{Q})]\,.$$

This modified performance criterion (5.11) must be minimized by $\mathbf{F}_k$ subject to the constraint imposed by the system equation (5.5).

For the solution we shall again use the discrete matrix minimum principle. The constant $1/n$ in (5.11) has been left out for reasons of simplicity. The Hamiltonian for this problem is

$$H = \tfrac{1}{2}\,\mathrm{tr}\,[\boldsymbol{\Phi}_k^{\mathrm{T}}(\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C} + \mathbf{C}^{\mathrm{T}}\mathbf{F}_k^{\mathrm{T}}\mathbf{R}\mathbf{F}_k\mathbf{C})\,\boldsymbol{\Phi}_k + \bar{\mathbf{Z}}_k\mathbf{Q} - 2\tilde{\mathbf{Z}}_k\boldsymbol{\Phi}_k^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{Q}] + \tag{5.12}$$
$$+ \mathrm{tr}\,[(\mathbf{M} - \mathbf{N}\mathbf{F}_k\mathbf{C})\,\boldsymbol{\Phi}_k\mathbf{P}_{k+1}^{\mathrm{T}}]\,.$$

Necessary condition of optimality is

$$\frac{\partial H}{\partial \mathbf{F}_k} = \mathbf{R}\mathbf{F}_k\mathbf{C}\boldsymbol{\Phi}_k\boldsymbol{\Phi}_k^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} - \mathbf{N}^{\mathrm{T}}\mathbf{P}_{k+1}\boldsymbol{\Phi}_k^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} = \mathbf{0}\,. \tag{5.13}$$

 Matrix $\boldsymbol{P}_k$ is the solution of the adjoint systems

$$(5.14) \qquad \boldsymbol{P}_k = \frac{\partial H}{\partial \boldsymbol{\Phi}_k} = (\boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{C} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}_k^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F}_k\boldsymbol{C})\,\boldsymbol{\Phi}_k - 2\boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\tilde{\boldsymbol{Z}}_k + (\boldsymbol{M} - \boldsymbol{N}\boldsymbol{F}_k\boldsymbol{C})^{\mathrm{T}}\,\boldsymbol{P}_{k+1}\,.$$

From the transversality conditions it follows that

$$(5.15) \qquad \boldsymbol{P}_K = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{C}\boldsymbol{\Phi}_K - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{S}\tilde{\boldsymbol{Z}}_K\,.$$

Let us assume that the matrix $\boldsymbol{P}_k$ can be written in the form

$$(5.16) \qquad \boldsymbol{P}_k = \boldsymbol{K}_k\boldsymbol{\Phi}_k - \boldsymbol{C}_k$$

where the unknown matrix $\boldsymbol{G}_k$ is connected with the reference input $\boldsymbol{z}_k$. Equaling equations (5.16) and (5.14) we get difference equations for matrices $\boldsymbol{K}_k$ and $\boldsymbol{G}_k$.

$$(5.17) \qquad \boldsymbol{K}_k = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{C} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}_k^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F}_k\boldsymbol{C} + (\boldsymbol{M} - \boldsymbol{N}\boldsymbol{F}_k\boldsymbol{C})^{\mathrm{T}}\,\boldsymbol{K}_{k+1}(\boldsymbol{M} - \boldsymbol{N}\boldsymbol{F}_k\boldsymbol{C})\,,$$

$$\boldsymbol{G}_k = 2\boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\tilde{\boldsymbol{Z}}_k + (\boldsymbol{M} - \boldsymbol{N}\boldsymbol{F}_k\boldsymbol{C})^{\mathrm{T}}\,\boldsymbol{G}_{k+1}\,.$$

From (5.15) we obtain the boundary conditions

$$(5.18) \qquad \boldsymbol{K}_K = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{C}\,, \quad \boldsymbol{G}_K = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{S}\tilde{\boldsymbol{Z}}_K\,.$$

The optimal output feedback $\boldsymbol{F}_k$ follows from equation (5.13)

$$(5.19) \qquad \boldsymbol{R}\boldsymbol{F}_k\boldsymbol{C}\boldsymbol{\Phi}_k\boldsymbol{\Phi}_k^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}} - \boldsymbol{N}^{\mathrm{T}}[\boldsymbol{K}_{k+1}(\boldsymbol{M} - \boldsymbol{N}\boldsymbol{F}_k\boldsymbol{C})\,\boldsymbol{\Phi}_k - \boldsymbol{G}_k]\,\boldsymbol{\Phi}_k^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}} = \boldsymbol{0}\,.$$

From here it follows

$$(5.20) \qquad \boldsymbol{F}_k = (\boldsymbol{R} + \boldsymbol{N}^{\mathrm{T}}\boldsymbol{K}_{k+1}\boldsymbol{N})^{-1}\,\boldsymbol{N}^{\mathrm{T}}(\boldsymbol{K}_{k+1}\boldsymbol{M}\boldsymbol{\Phi}_k - \boldsymbol{G}_k)\,\boldsymbol{\Phi}_k^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{C}\boldsymbol{\Phi}_k\boldsymbol{\Phi}_k^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}})\,.$$

The optimal feedback gain can thus be divided into two parts. One is identical with the feedback gain designed for the state regulator problem-equation (4.20), the second part is determined by reference input $\boldsymbol{z}_k$ only.

In case where the initial value $\boldsymbol{z}_0$ of the vector $\boldsymbol{z}_k$ can be considered to be a random variable as well it can be shown that the tracking problem can be transformed into the state regulator problem. Let us assume that vector $\boldsymbol{z}_k$ is a solution of the difference equation

$$(5.21) \qquad \boldsymbol{z}_{k+1} = \boldsymbol{H}\boldsymbol{z}_k\,.$$

Let us define the extended state vector

$$(5.22) \qquad \boldsymbol{w}_k = \begin{bmatrix} \boldsymbol{x}_k \\ \boldsymbol{z}_k \end{bmatrix}.$$

Equation for the original system (1.1) and the equation (5.21) can be put together and using (5.22) we obtain 

(5.23) $$\mathbf{w}_{k+1} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \mathbf{w}_k + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u}_k .$$

Let the output of the system (5.23) be

(5.24) $$\bar{\mathbf{y}}_k = \bar{\mathbf{C}} \mathbf{w}_k ,$$

where

$$\bar{\mathbf{C}} = [\mathbf{C}, -\mathbf{I}] .$$

It follows that

(5.25) $$\bar{\mathbf{y}}_k = \mathbf{y}_k - \mathbf{z}_k = -\mathbf{e}_k .$$

The tracking problem for the original system is equivalent to the output regulartor problem for the system (5.23) and (5.24).

## 6. CONCLUSION

In the present paper the problem of optimal control of a DLDS with respect to quadratic performance criterion is solved.

When all state of the system are measurable, solution of this problem is well known and is discussed in section 2. Algorithms convenient for digital computer are also mentioned here.

When the additional constraint of using only the measurable output is imposed we have the so called problem of incomplete information about the state of the system. Optimal solution of this problem uses state reconstrution by an additional dynamic system (observer or Kalman-Bucy filter) whose dimension is specified.

In optimal solution we can use dynamic controller of a lower dimension that is necessary for state reconstruction. In this paper the limiting problem is solved where the regulator is only proportional and uses only the measurable output of the system. This suboptimal controller is derived assuming the initial state of the system has some statistical properties. Solution of this problem results in nonlinear matrix equations solving of which is difficult. A simplified but approximate solution of this problem is discussed at the end of section 4. Like in the continuous version [18] questions about the existence and the uniqueness of the solution of the suboptimal proportional feedback controller are yet to be answered.

The discrete version of the tracking problem both for complete and incomplete information about the state of the system is also solved in the present paper.

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*Ing. Jan Štecha, CSc.; Katedra řídicí techniky - fakulta elektrotechnická ČVUT (Technical University, Department of Automatic Control) Karlovo nám. 13, 121 35 Praha 2. Czechoslovakia.*
*Ing. Alena Kozáčiková; Ústředí výpočetní techniky Tesla (Computer Centre Tesla), Všehrdova 2, 110 00 Praha 1. Czechoslovakia.*
*Ing. Jaroslav Kozáčik; Katedra teorie obvodů - fakulta elektrotechnická ČVUT (Technical University, Department of Circuit Theory), Suchbatárova 2, 166 27 Praha 6. Czechoslovakia.*
*Ing. Jiří Lidický; Výzkumný ústav energetický, oddělení 244 (Power Research Institute, Department 244), Partyzánská 7a, 170 00 Praha 7. Czechoslovakia.*