KYBERNETIKA — VOLUME 9 (1973), NUMBER 5

# On Generalized Credence Functions

PETR HÁJEK, DAGMAR HARMANCOVÁ

The aim of the present paper is to show that Carnap's conditions of rationality of credence functions can always be satisfied for a generalized notion of credence functions.

## INTRODUCTION

In recent years a great deal of interest in the area of artificial intelligence and robotics was directed towards a logical representation of facts about the external world as a basis for robot's reasoning and acting in it. In the development of computer programs capable of acting intelligently, sooner or later the need will arise to reckon with the degree of credibility of various facts about the world and of their consequences. One of the possible approaches is to introduce certain functions assigning a value, the credence, to each formula of a formal system. Carnap [1] introduced and analyzed the notion of rational credence functions (subjective probabilities) and demonstrated their existence under certain simplifying assumptions about the language (see [2]). However, there exist language systems (theories) for which Carnap's conditions of rationality cannot be satisfied (see [3]).

The aim of the present paper is to show that Carnap's conditions can always be satisfied for generalized rational credence functions; the generalization consists in allowing the values of a rational credence function to be elements of an ordered field, not necessarilly the field of real numbers.

It is shown in the Appendix that all Carnap's conditions can be satisfied within the framework of the theory of semisets when we do not require the rational credence function to be a set (i.e. if we admit semiset credence functions).*

* The results of this paper were presented at the seminar on applications of Mathematical Logic at the Faculty of Mathematics and Physics of the Charles University, Prague, partly in January 1971 and fully in October 1972. We thank to Dr. I. M. Havel for a valuable discussion and to Professor Dana Scott for his information about his work [7].

## 1. PRELIMINARIES

In this section we shall recall Carnap's notions and several notions of logic. Our (re)-formulation points out formal rigour and ignores nearly completely any informal derivation of the notions introduced. (For the latter see [1].)

By Carnap, subjective probability is "the probability assigned to a proposition or event $H$ by a subject $X$, say a person or a group of persons, in other words, the degree of belief of $X$ in $H$". Carnap supposes that the degree of belief is a real number. (Nevertheless, it seems he presents no arguments in favour of this assumption.)

Thus, we have the following

**1.1. Definition.** Let $\mathbf{T}$ be a non-contradictory theory in the first-order predicate calculus. $Form_{\mathbf{T}}$ denotes the set of closed formulas of $\mathbf{T}$. For $\varphi \in Form_{\mathbf{T}}$ we define $[\varphi]_{\mathbf{T}} = \{\psi; \mathbf{T} \vdash \varphi \equiv \psi\}$. $\boldsymbol{Lind}_{\mathbf{T}}$ denotes the *Lindenbaum-Tarski algebra of* $\mathbf{T}$, i.e. the Boolean algebra of classes of equivalent formulas of $\mathbf{T}$ with operations defined as usually. $\boldsymbol{Re}$ denotes the field of real numbers. $N$ is the set of all positive natural numbers. $Cr$ is a *credence function for* $\mathbf{T}$ if $Cr$ is a mapping of $Form_{\mathbf{T}}$ into $\boldsymbol{Re}$.

We take the first Carnap's requirement of rationality for the definition of a rational credence function:

**1.2. Definition.** $Cr$ is a *rational credence function* for a theory $\mathbf{T}$ if $Cr$ is a credence function for $\mathbf{T}$ and the following conditions hold for any $\varphi, \psi \in Form_{\mathbf{T}}$:

1) $0 \leqq Cr(\varphi) \leqq 1$,
2) $\mathbf{T} \vdash \neg(\varphi \wedge \psi) \Rightarrow Cr(\varphi \vee \psi) = Cr(\varphi) + Cr(\psi)$,
3) $\mathbf{T} \vdash \varphi \Rightarrow Cr(\varphi) = 1$,
4) $Cr(\varphi) = 0 \Rightarrow \mathbf{T} \vdash \neg\varphi$.

**1.3. Remark.** It follows from the conditions 1) to 3) that $\mathbf{T} \vdash (\varphi \equiv \psi) \Rightarrow Cr(\varphi) = Cr(\psi)$. Consequently, given $Cr$, we can define a function $\hat{C}r$ on $\boldsymbol{Lind}_{\mathbf{T}}$ putting $\hat{C}r([\varphi]) = Cr(\varphi)$. We can now reformulate 1.2 as follows: $Cr$ is a rational credence unction iff $Cr$ induces a strictly positive normalized measure on $\boldsymbol{Lind}_{\mathbf{T}}$.

**1.4. Definition.** Given a credence function $Cr$, Carnap defines the *conditional credence function* in the usual way:

$$Cr'(\varphi \mid E) = \frac{Cr(\varphi \wedge E)}{Cr(E)}$$

provided that $Cr(E) > 0$.

The second Carnap's requirement of rationality does not refer to a single credence function but to a system of credence functions.

**1.5.** Let $\mathfrak{S} = \langle \mathbf{T}_0, \{E_i\}_{i\in\omega}, \{Cr_i\}_{i\in\omega}\rangle$, where $E_i \in Form_{\mathbf{T}_0}$ and $Cr_i$ are mappings of $Form_{\mathbf{T}_0}$ into ***Re***. ($\omega = N \cup \{0\}$) Denote $\mathbf{T}_i = (\mathbf{T}_0, \bigwedge_{j<i} E_j)$. $\mathfrak{S}$ is an *evolving credence system* if $E_i$ is a non-contradictory formula of the theory $\mathbf{T}_i$ for any $i$. We shall call $Cr_0$ the *initial credence function* of $\mathfrak{S}$.

The second Carnap's requirement of rationality reads: If an evolving credence system is rational then

1) $Cr_i$ is a rational credence function for $\mathbf{T}_i$,

2) $Cr_{i+1}(\varphi) = Cr'_i(\varphi \mid E_i)$ for any $i \in \omega$ (equivalently, $Cr_{i+1}(\varphi) = Cr'_0(\varphi \mid \bigwedge_{j\leq i} E_j)$) for any $i \in \omega$).

*Remark.* It is evident from the definition that the initial credence function of an evolving credence system $\mathfrak{S}$ (together with the sequence $\{E_i\}_{i\in\omega}$) determines uniquely all the credence functions of $\mathfrak{S}$. Hence, the next requirement of rationality makes limitations just on initial credence functions.

**1.6. Definition.** A permutation $p$ of the alphabet of $\mathbf{T}$ is said to be a *substitution permutation* if the following holds:

1) $\{s; s \neq p(s)\}$ is a finite set,
2) if $s$ is a constant, then $p(s)$ is a constant,
3) if $s$ is an $n$-ary functor, then $p(s)$ is an $n$-ary functor,
4) if $s$ is an $n$-ary predicate, then $p(s)$ is an $n$-ary predicate.

If $\varphi$ is a formula of $\mathbf{T}$, then $p(\varphi)$ denotes the formula which is obtained from $\varphi$ by replacing all constants, functors and predicates by their $p$-images.

A permutation is called $\mathbf{T}$-*conservative* if

$$(\forall\varphi \in Form_{\mathbf{T}})\,(\mathbf{T} \vdash \varphi \Leftrightarrow T \vdash p(\varphi))\,.$$

A credence function $Cr$ for $\mathbf{T}$ is called $\mathbf{T}$-*symmetric* if

$$(\forall\varphi \in Form_{\mathbf{T}})\,(Cr(\varphi) = Cr(p(\varphi))$$

holds for any $\mathbf{T}$-conservative permutation $p$.

**1.7.** The third Carnap's requirement of rationality requires the initial credence function of a rational evolving credence system to be $\mathbf{T}_0$-symmetric.

We shall not introduce any other requirements of rationality. Hence, we make the following

**1.8. Definition.** A credence function $Cr_0$ for $\mathbf{T}_0$ is a *rational initial credence function* for $\mathbf{T}_0$ if $Cr_0$ satisfies conditions 1) to 4) of 1.2 and if $Cr_0$ is $\mathbf{T}_0$-symmetric.

**1.9.** If $p$ is a $\mathbf{T}$-conservative permutation, then

$$(\forall \varphi, \psi \in Form_{\mathbf{T}})\,(\mathbf{T} \vdash \varphi \equiv \psi \Leftrightarrow \mathbf{T} \vdash p(\varphi) \equiv p(\psi))\,.$$

Hence, we can define $\hat{p}[\varphi] = [p(\varphi)]$, i.e. $\mathbf{T}$-conservative permutations induce automorphisms of $\boldsymbol{Lind}_{\mathbf{T}}$. Put $\boldsymbol{Consv}_{\mathbf{T}} = \{\hat{p};\, p$ is a $\mathbf{T}$-conservative permutation$\}$; $\boldsymbol{Consv}_{\mathbf{T}}$ is a locally finite group of automorphisms of $\boldsymbol{Lind}_{\mathbf{T}}$.

We have the following evident assertion: $Cr_0$ is a rational initial credence function for a theory $\mathbf{T}_0$ iff $Cr_0$ induces a strictly positive normalized measure on $\boldsymbol{Lind}_{\mathbf{T}_0}$ which is symmetric with respect to $\boldsymbol{Consv}_{\mathbf{T}_0}$. Consequently, we can study measures on Boolean algebras instead of functions defined on closed formulas of some theory.

## 2. INITIAL CREDENCE FUNCTIONS

The theory $\mathbf{T}$ of linear ordering with countably many constants $a_0, a_1, \ldots$, and with axioms $a_i \neq a_j$ for $i \neq j$ is an example of a theory for which no rational initial credence function exists. Let $\varphi_k$ be the formula $(\forall x)\,(a_k \leqq x)$ ("$a_k$ is the least element") and let $Cr$ be a rational initial credence function for $\mathbf{T}$. Then clearly $Cr(\varphi_k) > 0$ for any $k$, $Cr(\varphi_i) = Cr(\varphi_j)$ for any $i, j$ and $\mathbf{T} \vdash \neg(\varphi_i \wedge \varphi_j)$ for $i \neq j$. As a consequence, $Cr(\bigwedge_{i=0}^{k-1} \varphi_i) = k\,.\,Cr(\varphi_0)$, which is a contradiction: we obtain $Cr(\bigwedge_{i=0}^{k-1} \varphi_i) > 1$ for $k > 1/Cr(\varphi_0)$.

Thus, when studying rational initial credence functions in a general way it is inevitable to weaken the requirements on $Cr_0$. There are the following possibilities:

(i) It is possible to give up the requirement of strict positiveness. This possibility was investigated by Carnap and Kemeny.

(ii) Another possibility is to surrender symmetry. For this case the problem of existence of rational credence functions is solved by Kelley's result [4]. From this result it follows that a strictly positive finitely additive real measure exists on any countable Boolean algebra.

(iii) The third possibility is to remove the assumption that $Cr_0$ is a real function. (This solution was suggested by Kemeny [3].)

Carnap in [1] gives no arguments for the assumption that the values of credence functions are real numbers. Thus, we shall study generalized credence functions defined on any (not neccessarilly Archimedean) ordered field.

**2.1. Definition.** A *generalized credence function* for a theory $\mathbf{T}$ is a mapping of $Form_{\mathbf{T}}$ into an ordered field. A generalized measure $\mu$ on a Boolean algebra $\boldsymbol{B}$ is a mapping of $\boldsymbol{B}$ into an ordered field $\boldsymbol{F}$, for which

$$(\forall u \in \boldsymbol{B})\,(\mu(u) \geqq O_{\boldsymbol{F}})\,,$$

$$(\forall u, v \in \boldsymbol{B})\,(u \wedge_{\boldsymbol{B}} v = O_{\boldsymbol{B}} \rightarrow \mu(u \vee_{\boldsymbol{B}} v) = \mu(u) +_{\boldsymbol{F}} \mu(v))\,.$$

Quite similarly, *generalized normalized measure*, *strictly positive measure* etc. are defined. A generalized measure with values in an ordered field $\boldsymbol{F}$ is called an $\boldsymbol{F}$*-measure*. We say "measure" instead of "$\boldsymbol{Re}$-measure".

**2.2. Remark.** The whole Carnap's discussion about the rationality of credence functions can be performed without complications for functions with values in any ordered field.

**2.3. Theorem.** Let $\boldsymbol{B}$ be a non-degenerate Boolean algebra, let $\boldsymbol{P}$ be a locally finite group of automorphisms of $\boldsymbol{B}$. Then there exists an ordered field $\boldsymbol{F}$ and a strictly positive normalized $\boldsymbol{F}$-measure $\mu^*$ on $\boldsymbol{B}$ such that

$$(\forall p \in \boldsymbol{P})\,(\forall u \in \boldsymbol{B})\,(\mu^*(u) = \mu^*(p(u)))$$

(i.e. $\mu^*$ is *symmetric with respect to* $\boldsymbol{P}$).

The proof is a modification of the proof in [5]. First we prove the following lemma:

**2.4. Lemma.** Let $\boldsymbol{B}$ and $\boldsymbol{P}$ satisfy assumptions of the foregoing theorem; let $u_1, \ldots$ $\ldots, u_n \in \boldsymbol{B}$, $u_1 > O_{\boldsymbol{B}}, \ldots, u_n > O_{\boldsymbol{B}}$, $p_1, \ldots, p_k \in \boldsymbol{P}$. Then there exists a real measure $\mu$ on $\boldsymbol{B}$ satisfying the following conditions:

$$(\forall j)\,(\mu(u_j) > 0)\,, \quad \mu(1_{\boldsymbol{B}}) = 1$$
$$(\forall i, j)\,(\mu(p_i(u_j)) = \mu(u_j))\,.$$

Proof. Put $U = \{p_i(u_j);\ i = 1, \ldots, k,\ j = 1, \ldots, n\}$. Without loss of generality we can suppose $\bigvee U = 1_{\boldsymbol{B}}$ (otherwise we add $u_{n+1} = 1_{\boldsymbol{B}} - \bigvee U$). Similarly we can suppose that $p_1, \ldots, p_k$ constitute a group (since $\boldsymbol{P}$ is locally finite). We define $U_0 = \{ \bigwedge_{i,j} \varepsilon_{ij}\, p_i(u_j);\ \varepsilon_{ij} \in \{1, -1\},\ \bigwedge_{i,j} \varepsilon_{ij}\, p_i(u_j) \neq O_{\boldsymbol{B}}\}$. For any $w \in U_0$ let $\boldsymbol{B} \mid w$ be the partial algebra determined by $w$. Let $j_w$ be an ultrafilter on $\boldsymbol{B} \mid w$. Let $U_0$ contain $q$ elements. We define $\mu_w$ on $\boldsymbol{B} \mid w$ as follows:

$$\mu_w(w') = \begin{cases} \dfrac{1}{q} & \text{if} \quad w' \in j_w\,, \\ 0 & \text{if} \quad w' \notin j_w\,. \end{cases}$$

The measure $\mu(u) = \sum_{w \in U_0} \mu_w(u \wedge w)$ satisfies the requirements of the lemma.

Proof of Theorem 2.3.

Put $D = \{\mu;\ \mu \text{ is a normalized measure on } \boldsymbol{B}\}$. For any $u_1, \ldots, u_n \in \boldsymbol{B}$, $u_1 >$ $> O_{\boldsymbol{B}}, \ldots, u_n > O_{\boldsymbol{B}}$, $p_1, \ldots, p_k \in \boldsymbol{P}$ we define $D^{u_1,\ldots,u_n}_{p_1,\ldots,p_k}$ as a set of all $\mu \in D$ satisfying conditions of the lemma 2.4. By this lemma $D^{u_1,\ldots,u_n}_{p_1,\ldots,p_k} \neq \emptyset$. The set $\sigma_0 = \{D^{u_1,\ldots,u_n}_{p_1,\ldots,p_k};$

$n, k \in N,\ u_1, \ldots, u_n \in \boldsymbol{B},\ p_1, \ldots, p_k \in \boldsymbol{P}\}$ generates a filter on $D$ (because $D^{u_1,\ldots,u_n}_{p_1,\ldots,p_k} \cap \cap D^{v_1,\ldots,v_m}_{q_1,\ldots,q_l} \supseteq D^{u_1,\ldots,u_n,v_1,\ldots,v_m}_{p_1,\ldots,p_k,q_1,\ldots,q_l}$). Let $j$ be an ultrafilter containing the filter generated by $\sigma_0$. Let $\boldsymbol{F}$ be the ultrapower $\boldsymbol{Re}^D/j$.

(On the set of all functions $f : D \to \boldsymbol{Re}$ we define an equivalence relation

$$f \sim_j g \equiv \{\mu; f(\mu) = g(\mu)\} \in j\,.$$

Let $\tilde{f} = \{g; g \sim_j f\}$. Then $\boldsymbol{F} = \{\tilde{f}; f : D \to \boldsymbol{Re}\}$ i.e. elements of $\boldsymbol{F}$ are equivalence classes of the equivalence relation $\sim_j$. We define algebraic operations on $\boldsymbol{F}$ as follows:

$$\tilde{h} = \tilde{f} + \tilde{g} \equiv \{\mu; h(\mu) = f(\mu) + g(\mu)\} \in j\,,$$
$$\tilde{h} = \tilde{f}\ .\ \tilde{g} \equiv \{\mu; h(\mu) = f(\mu)\ .\ g(\mu)\} \in j\,,$$
$$\tilde{f} \leqq \tilde{g} \equiv \{\mu; f(\mu) \leqq g(\mu)\} \in j\,.$$

Then $\boldsymbol{F}$ is an ordered field (see e. g. [6]).)

We define a measure $\tilde{\mu}^*$: If $u \in \boldsymbol{B}$, then $\tilde{\mu}^*(u)$ is the elemenet of $\boldsymbol{F}$ determined by the function $\mu^*(u)$ which has the value $\mu(u)$ for each $\mu \in D$, i.e. $(\forall \mu \in D)(\mu^*(u)(\mu) = = \mu(u))$. Evidently $\tilde{\mu}^*$ is a normalized $\boldsymbol{F}$-measure. Let $u > 0_{\boldsymbol{B}}$. Then $\mu^*(u)(\mu) > 0$ for all $\mu \in D^u_i$, where $i$ is the identical permutation. $D^u_i \in j$, consequently $\tilde{\mu}^*$ is a strictly positive measure. Finally we have the following:

$$(\forall p \in \boldsymbol{P})(\mu^*(p(u))(\mu) = \mu^*(u)(\mu) \quad \text{for} \quad \mu \in D^u_p \in j)\,,$$

i.e. $\tilde{\mu}^*$ is symmetric with respect to $\boldsymbol{P}$.

This completes the proof.

**2.5. Corollary.** For any non-contradictory theory $\mathbf{T}$ there exists a generalized rational initial credence function $Cr$.

## 3. PREFERENCE QUASIORDERINGS

Let us consider the possibility of eliminating credence functions completely. We shall investigate in what sense a rational credence function may be replaced by a quasiordering.

**3.1. Definition.** A linear quasiordering $\lesssim$ of a Boolean algebra $\boldsymbol{B}$ is *realizable by a generalized measure* $\mu$ if

$$(\forall u, v \in \boldsymbol{B})(u \lesssim v \Leftrightarrow \mu(u) \leqq \mu(v))\,.$$

A linear quasiordering $\lesssim$ of a Boolean algebra $\boldsymbol{B}$ is a *preference quasiordering* if there is a normalized strictly positive generalized measure $\mu$ such that $\precsim$ is realizable by $\mu$.

Scott ([7]) proved the following:

**3.2. Theorem.** Let $\boldsymbol{B}$ be a finite Boolean algebra and let $\succsim$ be a binary relation on $\boldsymbol{B}$. $\succsim$ is realizable by a normalized measure on $\boldsymbol{B}$ if and only if the following conditions hold:

1) $1_{\boldsymbol{B}} \succ 0_{\boldsymbol{B}}$,
2) $(\forall x \in \boldsymbol{B})(x \succsim 0_{\boldsymbol{B}})$,
3) $(\forall x, y \in \boldsymbol{B})(x \succsim y \vee y \succsim x)$,
4) $(\forall x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n \in \boldsymbol{B})(((\forall i = 1, \ldots, n)(x_i \succsim y_i) \mathbin{\&} x_0 + x_1 + \ldots \ldots + x_n = y_0 + y_1 + \ldots + y_n) \rightarrow (x_0 \precsim y_0))$ for any $n \in N$.

(The sum in 4) means the sum of characteristic functions of the sets corresponding to elements $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n$ in the Stone space of the algebra $\boldsymbol{B}$.)

**3.3. Corollary.** Any preference quasiordering $\precsim$ on a finite Boolean algebra $\boldsymbol{B}$ is realizable by a normalized real measure.

Proof. If $\mu^*$ is the generalized measure realizing $\precsim$, then

$$(\forall u, v \in \boldsymbol{B})(u \precsim v \Leftrightarrow \mu^*(u) \leqq \mu^*(v)) .$$

It follows from the axioms of an ordered field that $\precsim$ satisfies the conditions of the foregoing theorem, and therefore is realizable by a normalized real measure.

**3.4. Notation.** Let $\boldsymbol{B}$ be a Boolean algebra, $x \in \boldsymbol{B}$. Then $\chi_x^{\boldsymbol{B}}$ is the characteristic function of the set corresponding to the element $x$ in the Stone space of the algebra $\boldsymbol{B}$. Put

$$\mathbf{Sc}\,(\boldsymbol{B}, \succsim) \Leftrightarrow (\forall n \in N)(\forall x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n \in \boldsymbol{B})$$
$$(((\forall i = 1, \ldots, n)(x_i \succsim y_i) \mathbin{\&} \sum_{i=0}^{n} \chi_{x_i}^{\boldsymbol{B}} = \sum_{i=0}^{n} \chi_{y_i}^{\boldsymbol{B}}) \rightarrow x_0 \precsim y_0) .$$

**3.5. Lemma.** Let $\boldsymbol{B}$ be a Boolean algebra and let $\succsim$ be a binary relation on $\boldsymbol{B}$. Then $\mathbf{Sc}(\boldsymbol{B}, \succsim) \Leftrightarrow (\forall \boldsymbol{A})(\boldsymbol{A}$ is a finite subalgebra of $\boldsymbol{B} \rightarrow \mathbf{Sc}(\boldsymbol{A}, \succsim))$.

Proof. Suppose $\mathbf{Sc}(\boldsymbol{B}, \succsim)$. Let $\boldsymbol{A}$ be a finite subalgebra of $\boldsymbol{B}$ and let $x_0, x_1, \ldots \ldots, x_n, y_0, y_1, \ldots, y_n \in \boldsymbol{A}$ be such that $x_i \succsim y_i$ for any $i = 1, \ldots, n$ and

$$(!) \qquad \chi_{x_0}^{\boldsymbol{A}} + \chi_{x_1}^{\boldsymbol{A}} + \ldots + \chi_{x_n}^{\boldsymbol{A}} = \chi_{y_0}^{\boldsymbol{A}} + \chi_{y_1}^{\boldsymbol{A}} + \ldots + \chi_{y_n}^{\boldsymbol{A}} .$$

Denote by $u_1, \ldots, u_m$ atoms of the algebra $\boldsymbol{A}$. Then there is a one-to-one correspondence between functions $\chi_x^{\boldsymbol{A}}$ and vectors $(e_1^x, \ldots, e_m^x)$, where $e_i^x = 1$ if $u_i \leqq x$, and

 $e_i^x = 0$ otherwise, and we have the following equality

$$\text{(!!)} \qquad \sum_{i=0}^{n} (e_1^{x_i}, \ldots, e_m^{x_i}) = \sum_{i=0}^{n} (e_1^{y_i}, \ldots, e_m^{y_i}) .$$

Clearly, $\chi_x^{\boldsymbol{B}} = e_1^x \chi_{u_1}^{\boldsymbol{B}} + \ldots + e_m^x \chi_{u_m}^{\boldsymbol{B}}$ for all $x \in A$ and it follows from (!) and (!!) that

$$\sum_{i=0}^{n} \chi_{x_i}^{\boldsymbol{B}} = \sum_{i=0}^{n} \chi_{y_i}^{\boldsymbol{B}} .$$

By our assumptions we have necessarily $x_0 \precsim y_0$. We have proved $\mathbf{Sc}(A, \succsim)$.

Conversely, suppose that $\mathbf{Sc}(A, \succsim)$ holds for any finite subalgebra $A$ of $\boldsymbol{B}$. Let $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n \in \boldsymbol{B}$, $x_i \succsim y_i$ $(i = 1, \ldots, n)$ and let

$$\text{(!!!)} \qquad \sum_{i=0}^{n} \chi_{x_i}^{\boldsymbol{B}} = \sum_{i=0}^{n} \chi_{y_i}^{\boldsymbol{B}} .$$

The elements $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n$ generate a finite subalgebra $\bar{A}$ of $\boldsymbol{B}$. If $u_1, u_2, \ldots, u_m$ are atoms of $\bar{A}$, we can express (!!!) (using the notation from the first part of the proof) in the following equivalent form:

$$\sum_{i=0}^{n} e_1^{x_i} \chi_{u_1}^{\boldsymbol{B}} + \ldots + \sum_{i=0}^{n} e_m^{x_i} \chi_{u_m}^{\boldsymbol{B}} = \sum_{i=0}^{n} e_1^{y_i} \chi_{u_1}^{\boldsymbol{B}} + \ldots + \sum_{i=0}^{n} e_m^{y_i} \chi_{u_m}^{\boldsymbol{B}} .$$

From this we obtain

$$\sum_{i=0}^{n} (e_1^{x_i}, \ldots, e_m^{x_i}) = \sum_{i=0}^{n} (e_1^{y_i}, \ldots, e_m^{y_i})$$

and consequently

$$\sum_{i=0}^{n} \chi_{x_i}^{\bar{A}} = \sum_{i=0}^{n} \chi_{y_i}^{\bar{A}} .$$

Then, by $\mathbf{Sc}(\bar{A}, \succsim)$, we have $x_0 \precsim y_0$. This completes the proof of the lemma.

**3.6. Lemma.** Let $\boldsymbol{B}$ be a Boolean algebra and let $\precsim$ be a linear quasiordering of $\boldsymbol{B}$. Suppose that for any finite subalgebra $A$ of $\boldsymbol{B}$ there is a strictly positive normalized measure realizing the quasiordering $\precsim \cap (A \times A)$ (i.e. realizing the restriction $\precsim$ on $A$). Then there is a strictly positive normalized generalized measure on $\boldsymbol{B}$ realizing $\precsim$.

Proof. Similarly as in the proof of Theorem 2.3, put

$$D = \{\mu;\ \mu \text{ is a normalized measure on } \boldsymbol{B}\} .$$

Let $A$ be a finite subalgebra of $\boldsymbol{B}$. We define $D^A = \{\mu;\ \mu \in D \;\&\; \mu \restriction A$ to be a strictly positive normalized measure on $A$ realizing $\precsim$ on $A\}$. $D^A$ is non-empty: By the as-

sumption, there is a strictly positive normalized measure $\mu'$ on $A$ realizing $\precsim$ on $A$. 
Then, if $u_1, \ldots, u_m$ are atoms of $A$, we can define measures $\mu_1, \ldots, \mu_m$ on partial algebras $B \mid u_1, \ldots, B \mid u_m$ putting

$$\mu_i(v) = \begin{cases} \mu'(u_i) & \text{if } v \in j_i, \\ 0 & \text{otherwise}, \end{cases}$$

where $j_i$ is an ultrafilter on the partial algebra $B \mid u_i$ $(i = 1, \ldots, m)$. We define the required measure by

$$\mu(v) = \sum_{i=1}^{m} \mu_i(v \wedge u_i).$$

The set $\sigma = \{D^A; A \text{ is a finite subalgebra of } B\}$ generates a filter on $D$. $(D^{A_1} \cap D^{A_2} \supseteq D^A$, where $A$ is the algebra generated by the set $A_1 \cup A_2$.) Let $j$ be an ultrafilter on $D$ containing $\sigma$. The ultrapower $F = Re^D/j$ is an ordered field. We now define a measure $\tilde{\mu}^*$ on $B$ with values in $F$: If $u \in B$, then $\tilde{\mu}^*(u)$ is the element of $F$ which is determined by the function $\mu^*(u)$ having the value $\mu(u)$ in any $\mu \in D$, i.e.

$$(\forall u \in B)(\forall \mu \in D)(\mu^*(u)(\mu) = \mu(u)).$$

The measure $\tilde{\mu}^*$ has the desired properties.

**3.7. Theorem.** Let $B$ be a Boolean algebra and let $\succsim$ be a binary relation on $B$. There exists a strictly positive normalized generalized measure realizing $\succsim$ iff $\succsim$ satisfies the following conditions:

1) $(\forall x \in B)(x \neq O_B \rightarrow x \succ O_B)$,
2) $(\forall x, y \in B)(x \succsim y \vee y \succsim x)$,
3) $\mathrm{Sc}(B, \succsim)$.

Proof. If there exists a strictly positive normalized generalized measure on $B$ realizing $\succsim$, then it follows immediately that conditions 1) to 3) hold.

Suppose conversely that $\succsim$ satisfies contitions 1) to 3). Then (by lemma 3.5) the restriction of this relation on any finite subalgebra of $B$ satisfies assumptions of 3.2, and consequently a strictly positive normalized real measure exists on any finite subalgebra of $B$. Hence (by lemma 3.6) a strictly positive normalized generalized measure realizing $\succsim$ exists on $B$.

**3.8. Remark.** It is easy to see that the relation $\succsim$ from the foregoing theorem is realizable by a strictly positive normalized generalized measure which is symmetric with respect to a group $P$ of automorphisms of the algebra $B$ if and only if

4) $(\forall p \in P)(\forall x \in B)(x \succsim p(x) \;\&\; p(x) \succsim x)$

holds in addition to the conditions 1) to 3) given above.

What does the foregoing tell us about rational credence functions?

**3.9. Definition.** A linear quasiordering $\precsim$ of closed formulas of a theory $\mathbf{T}$ is a *(rational) preference quasiordering for* $\mathbf{T}$ if there is a (rational) generalized credence function $Cr$ such that

$$(\forall\varphi, \psi \in Form_{\mathbf{T}})\,(\varphi \precsim \psi \Leftrightarrow Cr(\varphi) \leqq Cr(\psi))\,.$$

We say that $\precsim$ *extends* $\mathbf{T}$*-implication* if

$$(\forall\varphi, \psi \in Form_{\mathbf{T}})\,(\mathbf{T} \vdash \varphi \rightarrow \psi \Rightarrow \varphi \precsim \psi)\,.$$

**3.10. Remark.** A rational preference quasiordering of $Form_{\mathbf{T}}$ extends $\mathbf{T}$-implication.

**3.11. Notation.** Let $E^n = \{\langle\varepsilon_1, \ldots, \varepsilon_n\rangle;\ \varepsilon_i$ is the symbol of negation or the void symbol$\}$. For any $\varphi_1, \ldots, \varphi_n,\ \varphi \in Form_{\mathbf{T}},\ \Phi = \langle\varphi_1, \ldots, \varphi_n\rangle,\ \varepsilon = \langle\varepsilon_1, \ldots, \varepsilon_n\rangle \in E^n$ we put

$$C_\varepsilon^\Phi(\varphi) = \begin{cases} 1 & \text{if} \quad \mathbf{T} \vdash \bigwedge_{i=1}^{n} \varepsilon_i\varphi_i \rightarrow \varphi\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

**3.12. Theorem.** A linear quasiordering $\precsim$ of closed formulas of a theory $\mathbf{T}$ is a rational preference quasiordering iff the following conditions hold:

1) $\precsim$ extends $\mathbf{T}$-implication.

2) If $\mathbf{T} \vdash \neg\psi$ then the relation $\psi \prec \varphi$ holds for any non-contradictory formula $\varphi \in Form_{\mathbf{T}}$.

3) Let $\Phi = \langle\varphi_0, \varphi_1, \ldots, \varphi_n\rangle,\ \Psi = \langle\psi_0, \psi_1, \ldots, \psi_n\rangle$. If $(\forall i = 1, \ldots, n)\,(\varphi_i \succsim \psi_i)$ and if $(\forall\varepsilon \in E^{2n+2})\,(\sum_{i=0}^{n} C_\varepsilon^{\Phi,\Psi}(\varphi_i) = \sum_{i=0}^{n} C_\varepsilon^{\Phi,\Psi}(\psi_i)$, then $\varphi_0 \precsim \psi_0$.

Proof. As soon as 1) is satisfied we can define a quasiordering on $\boldsymbol{Lind}_{\mathbf{T}}$ as follows: $[\varphi] \precsim_* [\psi] \Leftrightarrow \varphi \precsim \psi$. Since the zero element of the algebra $\boldsymbol{Lind}_{\mathbf{T}}$ is the class of all contradictory formulas, the algebra $\boldsymbol{Lind}_{\mathbf{T}}$ and the quasiordering $\precsim_*$ satisfy conditions 1) and 2) of the Theorem 3.7. Elements of $\{[\bigwedge_{i=0}^{n} \varepsilon_i\varphi_i \wedge \bigwedge_{j=0}^{n} \eta_j\psi_j];\ \langle\varepsilon_0, \ldots, \varepsilon_n, \eta_0, \ldots, \eta_n\rangle \in E^{2n+2}\}$ different from $O_{\boldsymbol{Lind}_{\mathbf{T}}}$ are just all the atoms of the finite subalgebra $A$ of $\boldsymbol{Lind}_{\mathbf{T}}$ generated by the elements $[\varphi_0], \ldots, [\varphi_n], [\psi_0], \ldots, [\psi_n]$. Hence, the equality $\sum_{i=0}^{n} C_\varepsilon^{\Phi,\Psi}(\varphi_i) = \sum_{i=0}^{n} C_\varepsilon^{\Phi,\Psi}(\psi_i)$ means that the number of atoms of the algebra $A$ included in the elements $[\varphi_0], \ldots, [\varphi_n]$ is equal to the number of atoms of $A$ included in $[\psi_0], \ldots, [\psi_n]$ or that

$$\sum_{i=0}^{n} \chi_{[\varphi_i]}^{\boldsymbol{Lind}_{\mathbf{T}}} = \sum_{i=0}^{n} \chi_{[\psi_i]}^{\boldsymbol{Lind}_{\mathbf{T}}}\,.$$

Hence, by Lemma 3.5, $\mathbf{Sc}(\boldsymbol{Lind}_{\mathbf{T}}, \gtrsim_*)$ holds and Theorem 3.12 follows immediately from Theorem 3.7. 

**3.13. Remark.** A rational preference quasiordering $\lesssim$ for a theory $\mathbf{T}$ is realizable by a rational generalized initial credence function (say, $\lesssim$ is a rational initial preference quasiordering) iff for any $\mathbf{T}$-conservative permutation $p$ the following holds:

$$(\forall \varphi \in Form_{\mathbf{T}})\,(\varphi \lesssim p(\varphi)\ \&\ p(\varphi) \lesssim \varphi)\,.$$

**3.14. Remark.** Let $\langle \mathbf{T}_0, \{E_i\}_{i\in\omega}, \{Cr_i\}_{i\in\omega}\rangle$ be a rational evolving credence system (see 1.5). If $\lesssim_i$ is a preference quasiordering realizable by a rational credence function $Cr_i$, then the quasiordering $\lesssim_{i+1}$ defined as $(\forall \varphi, \psi \in Form_{\mathbf{T}})\,(\varphi \lesssim_{i+1} \psi \Leftrightarrow \Leftrightarrow \varphi \wedge E_i \lesssim_i \psi \wedge E_i)$ is realizable by the function $Cr_{i+1}$.

**3.15.** Thus, we have a "pure" definition of rational initial preference quasiorderings (a definition not mentioning generalized credence functions, cf. 3.12, 3.13) and we know that for any non-contradictory theory $\mathbf{T}$ there exists a rational initial preference quasiordering of $Form_{\mathbf{T}}$ (see 2.5). But it is easy to see that different rational credence functions can realize the same preference quasiordering. Hence, it is impossible to replace a credence function by a preference quasiordering in the case when the values of the function are important. (E.g., for calculation of subjective values of individiual acts in the course of rational decision (see [1]) a preference quasiordering is insufficient.)

## 4. APPENDIX: THE THEORY OF SEMISETS AND RATIONAL INITIAL CREDENCE FUNCTIONS

The purpose of this Appendix is to show that our result on generalized rational initial credence functions has a nice corollary in the theory of semisets. We think that the corollary could be of some philosophical interest and that, consequently, the theory of semisets might be interesting as a frame theory for some parts of philosophical logic.

The theory of semisets was introduced in [8] (for quite other purposes); there are some survey papers, e.g. [9]. We shall not repeat here the axioms of the theory of semisets (hereafter denoted by **TSS**); instead, we shall try to evoke some intuition concerning semisets.

One may say that semisets satisfy the intuitive equation

$$\frac{\text{semisets}}{\text{sets}} = \frac{\text{complex numbers}}{\text{real numbers}}$$

in the sense that proper semisets (semisets that are not sets) are some new (imaginary) objects that cannot exist from the point of view of the set theory but that are introduced in such a way that when speaking on semisets we cannot prove any new statement concerning only sets. This can be also expressed by saying that **TSS** extends the set theory conservatively with respect to statements concerning only sets.

In the Zermelo-Fraenkel system of set theory (**ZF**) one consisders only one sort of objects (called sets) and the membership relation $\in$ between sets. One has some axioms of existence of sets, some operations with sets etc. For example, for each set $x$ one has the set of all subsets of $x$. (We use letters $x, y, \ldots$ to denote sets.) It is well-known (and provable in **ZF**) that there is no set having all sets as its elements (the set of all sets). The Gödel-Bernays system of set theory (**GB**) considers two sorts of objects: sets and classes (and membership). In this theory each set is a class but not each class is a set. A class is a set iff it is an element of some class. We shall denote classes by capital letters $X, Y, \ldots$ It is provable in **GB** that each class $X$ has the following property:

*for each set a, the intersection $X \cap a$ is a set.*

Call classes with this property *real classes*; then in **GB** we have real classes only. It follows easily that each real class which is included in a set (as a subclass) is a set. Hence, in **GB** one proves: each subclass of a set is a set. Define a *semiset* to be a subclass of a set. Hence, all sets are semisets and in **GB** all semisets are sets. One knows that **GB** extends **ZF** conservatively, i.e. each statement of **ZF** is provable in **ZF** iff it is provable in **GB**.

Imagine now a theory of three sorts of objects: sets, real classes and classes (both real and non-real = imaginary). The axioms assure that sets together with real classes behave as in **GB** (i.e., sets behave as in **ZF**) but there may be some imaginary classes (equivalently, there may be some proper semisets, i.e. semisets that are not sets). The aim is that the new theory extends both **ZF** and **GB** *conservatively*, i.e. we cannot prove anything new on sets (and on real classes), but that we have some interesting proper semisets. This is the motivation for **TSS** (or, better, for various systems of the theory of semisets), at least from one point of view. We shall now describe some reasonable assumptions on existence of proper semisets.

(a) First, observe that we have two sorts of collections of natural numbers: *sets* of natural numbers and (proper) *semisets* of natural numbers. Each non-empty set of natural numbers have a least element (since sets obey axioms of **ZF**); but there can be non-empty semisets of natural numbers having no least element. It can be metamathematically demonstrated that the axiom "there is a non-empty semiset of natural numbers with no least element" can be added to the axioms of **TSS** and that the resulting theory extends **ZF** conservatively.

(b) Secondly, we have two notions of cardinality. Either we consider one-to-one mappings that are sets or we admit one-to-one mappings being (proper) semisets. Two semisets $\sigma$, $\varrho$ are *absolutely equivalent* if there exists a semiset $\tau$ which is a one-to-one mapping of $\sigma$ onto $\varrho$. Denotation: $\sigma \approx \varrho$.

In particular, one may have sets $x$, $y$ that are absolutely equivalent but not equivalent, i.e. no one-to-one mapping which is a set maps $x$ onto $y$. We give two sorts of examples: 

(i) Call a natural number $n$ *absolute* if the $n$-element set $\{0, 1, \ldots, n-1\}$ is not absolutely equivalent to the $(n+1)$-element set $\{0, 1, \ldots, n-1, n\}$. Let $An$ be the semiset of absolute natural numbers. Of course, in set theory, all numbers are absolute (i.e. $An = \omega$); but in **TSS** we may have $An \neq \omega$, i.e., we may have an $n$ such that $\{0, 1, \ldots, n-1\}$ can be one-to-one mapped onto $\{0, 1, \ldots, n-1, n\}$ by a semiset-mapping.

(ii) We can have an uncountable set $x$ (i.e., $\omega$ can be one-to-one mapped into $x$ by a set-mapping but not onto) which is absolutely countable, i.e. there is a semiset one-to-one mapping $\tau$ of $\omega$ onto $x$. Or we may require $\tau$ to be a one-to-one mapping of $An$ onto $x$. This means that $x$ can be enumerated by absolute natural numbers; the enumeration is not a set.

One has the following metamathematical result (Balcar - Hájek - Vopěnka):

**4.1. Metatheorem.** If one adds the following axioms

1) $An \neq \omega$,

2) $(\forall x \text{ infinite})\ (x \approx An)$

to the (usual) axioms of **TSS** then the resulting theory **TSS**$^+$ is a conservative extension of **ZF** and of **GB**.

The theory **TSS**$^+$ will be called the *rich theory of semisets*. Recall the notions of a relational structure and elementary equivalence of relational structures in their usual set-theoretical meanings (see e.g. [6]). We restrict ourselves to structures such that the arities of all relations and functors are absolute natural numbers and the number of all relations and functors is absolute. (In fact, we are interested only in structures for the language of ordered fields, i.e. with two binary predicates $=$, $<$, two binary functors $+$, . and two constants 0, 1.)

Vopěnka recently proved the following important theorem in **TSS**$^+$ (not yet published):

**4.2. Theorem (TSS$^+$).** If $\boldsymbol{m}_1$, $\boldsymbol{m}_2$ are two elementarily equivalent relational structures, then there is a semiset-isomorphism of $\boldsymbol{m}_1$, $\boldsymbol{m}_2$, i.e. a semiset one-to-one mapping of the field of $\boldsymbol{m}_1$ onto the field of $\boldsymbol{m}_2$ preserving all the structure.

Since one knows that each relational structure is elementarily equivalent to each of its ultrapowers, we obtain from 2.3 (better, from the proof of 2.3, which makes sense in **TSS**$^+$) to any Boolean algebra $\boldsymbol{b}$ (a set), and to any locally finite group $\boldsymbol{p}$ of automorphisms of $\boldsymbol{b}$ ($\boldsymbol{p}$ a set), an ordered field $\boldsymbol{f}$ (which is a set) elementarily equivalent to the field $\boldsymbol{Re}$ of real numbers such that there is a strictly positive $\boldsymbol{p}$-symmetric $\boldsymbol{f}$-measure $\boldsymbol{m}$ on $\boldsymbol{b}$ and the preceding theorem assures a semiset-isomorphism $\tau$

 between $f$ and $\boldsymbol{Re}$. Composing $m$ and $\tau$ we obtain a semiset-mapping $\mu$ of $\boldsymbol{b}$ into $\boldsymbol{Re}$ which is a strictly positive $\boldsymbol{p}$-symmetric measure on $\boldsymbol{b}$.

Thus, we have the following.

**4.3. Corollary** ($\mathbf{TSS}^+$). If $\boldsymbol{b}$ is a Boolean algebra and if $\boldsymbol{p}$ is a locally finite group of automorphisms of $\boldsymbol{b}$ then there is a semiset $\mu$ which is a normalized strictly positive $\boldsymbol{p}$-symmetric measure on $\boldsymbol{b}$, i.e.,

$$(\forall u \in \boldsymbol{b})\,(\mu(u) \in \boldsymbol{Re}\ \&\ O \leqq \mu(u) \leqq 1)\,,$$
$$\mu(1_{\boldsymbol{b}}) = 1\,,$$
$$(\forall u \in \boldsymbol{b})\,(\mu(u) = O \equiv u = O_{\boldsymbol{b}})\,,$$
$$(\forall u, v \in \boldsymbol{b})\,(u \wedge_{\boldsymbol{b}} v = O_{\boldsymbol{b}} \rightarrow \mu(u \vee_{\boldsymbol{b}} v) = \mu(u) + \mu(v))\,,$$
$$(\forall u \in \boldsymbol{b})\,(\forall i \in \boldsymbol{p})\,(\mu(u) = \mu(i(u)))\,.$$

*Remark.* We may prove from the preceding that if $u_0, \ldots, u_n$ are pairwise disjoint elements of $\boldsymbol{b}$ and if $n$ is an absolute natural number, then $\mu(u_0 \vee \ldots \vee u_n) = \sum_{i=0}^{n} \mu(u_i)$.

**4.4. Corollary** ($\mathbf{TSS}^+$). For any non-contradictory theory $\mathbf{t}$ ($\mathbf{t}$ a set) there exists a rational initial credence function $Cr$ which is a semiset.

(Received January 31, 1973.)

REFERENCES


[1] Carnap R.: The Aim of Inductive Logic. In: "Logic, Methodology and Philosophy of Science" (ed. by E. Nagel, P. Suppes, A. Tarski). Stanford (USA) 1962, p. 303.

[2] Carnap R.: The Continuum of Inductive Methods. Chicago 1952.

[3] Kemeny J. G.: Carnap's Theory of Probability and Induction. In: "The Philosophy of R. Carnap", series The Library of Living Philosophers.

[4] Kelley J. L.: Measures on Boolean Algebras. Pacific Journal of Mathematics *9* (1959), 1165.

[5] Luxemburg W. A. J.: Two Applications of the Method of Construction by Ultrapower to Analysis. Bull. Amer. Mathemat. Soc. *68* (1962), 416.

[6] Bell J. L., Slomson A. B.: Models and Ultraproducts. North-Holland Publ. Comp., 1969.

[7] Scott D.: Measurement Structures and Linear Inequalities. Journal of Mathemat. Psychology *1* (1964), 233.

[8] Vopěnka P., Hájek P.: The Theory of Semisets. North-Holland Publ. Comp., Amsterdam and Academia, Prague 1972.

[9] Hájek P.: On Semisets. In: "Logic Colloquium '69" (ed. by R. C. Gandy and C. M. E. Yates). North-Holland Publ. Comp., Amsterdam 1971.



*RNDr. Petr Hájek, CSc.; Matematický ústav ČSAV (Mathematical Institute — Czechoslovak Academy of Sciences), Žitná 25, 115 67 Praha 1. Czechoslovakia.*

*Dagmar Harmancová, prom. mat.; Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sicences), Vyšehradská 49, 128 48 Praha 2. Czechoslovakia.*