KYBERNETIKA — VOLUME 9 (1973), NUMBER 3

Algebraic Theory of Discrete Optimal Control for Single-Variable Systems II

Open-Loop Control

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This is a continuation of paper [4] on the algebraic theory of discrete optimal control for single-variable systems, Part I: Preliminaries. Having established mathematical machinery there, we are in the position to solve optimal control problems. This Part II is concerned with the open-loop control defined in the Introduction to Part I.

To recall, given a system s we seek for a control u which makes the system output y follow a given reference signal w in a prescribed fashion. The system configuration is shown in Fig. 1. We reiterate that this control is of feedforward type, i.e., no attempt is made to neutralize the effect of disturbances.

Two basic criteria are considered here, namely, the time optimal control and the least squares control [2], [3], [7], [8]. For rigorous definitions see the respective sections.

The theorems, examples, equations, etc. are numbered separately in each part of the tripaper. The usual system of references is used within this paper whereas cross-references are followed by a slash and the respective part number. The notation introduced in Part I is consistently adhered to throughout.

INTRODUCTION

The open-loop control problems are simplest and basic problems of control. The underlying idea is that of commanding the given system to produce a required output. Hence this type of control should be used whenever we are just to control, not to counteract disturbances, as will be discussed later. True, the actual system is almost always contaminated by disturbances but they can be eliminated in some other way, e.g. using an internal feedback. Another reason for developing the open-loop control theory is to compare the properties of the open-loop and the closed-loop controls. The comparison is instructive and will be done in Part III.

OPEN-LOOP OUTPUT TIME OPTIMAL CONTROL

In this problem the discrete system output is to exactly follow a given reference signal in the shortest time possible. The formal statement follows.

Given the configuration shown in Fig. 1, where

$$s = \frac{\zeta^d b}{a}, \quad w = \frac{q}{p},$$

the polynomials being arbitrary elements of $\mathfrak{F}[\zeta]$ but $d > 0$, $(a, \zeta b) = 1$, $(b, \zeta) = 1$, $(p, \zeta q) = 1$. Find a stable control

$$u = \frac{v}{u}$$

so as to zero the error e in a minimum time $k_{\min}$.

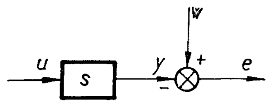

Fig. 1. Open-loop control system.

The solution of the problem is given in

Theorem 1. *The open-loop output time optimal control problem has a solution if and only if $p^- \mid a$. The solution is unique and is given by*

$$v = a_0 \hat{y},$$
$$u = p_0 b^+$$

where $\hat{x}$ and $\hat{y}$ is such solution of the Diophantine equation

$$px + \zeta^d b^- y = q$$

that

$$\partial \hat{x} = \min .$$

Moreover, $e = \hat{x}$ and $k_{\min} = 1 + \partial \hat{x}$.

Proof. Fig. 1 yields

$$e = w - su .$$

The e is to be zeroed in a finite time and hence it must be a polynomial, say x. Then

$$x = \frac{q}{p} - \frac{\zeta^d b}{a} \frac{v}{u} = \frac{q}{p_0 w} - \frac{\zeta^d b}{a_0 w} \frac{v}{u}$$

on taking (14/I) into account.

Having in mind that the degree of x is to be minimal so that e may vanish in a minimum time $k_{\min}$, the u and v should alter the common denominator to $p_0 w$ and u should reduce $\zeta^d b$ as much as possible and still remain stable. Hence

$$v = a_0 y \ ,$$
$$u = p_0 b^+ \ ,$$

and the above expression is converted into the equation

$$px + \zeta^d b^- y = q \ .$$

It follows that $\boldsymbol{e} = \hat{x}$, where $\partial \hat{x} = \min$, and $k_{\min} = 1 + \partial \hat{x}$.

The $\boldsymbol{u}$ is stable if and only if p_0 is stable, i.e., if $p^- \mid a$. This condition implies $(p, \zeta^d b^-) = 1$ and hence the Diophantine equation has a solution. □

Remark 1. In case $\boldsymbol{s}$ originated from a continuous system by the process of sampling, the error need not be zero in between sampling points. However, it is stable.

Example 1. Consider

$$\boldsymbol{s} = \frac{\zeta(1 - 2{\cdot}5\zeta + \zeta^2)}{1 - 5\zeta + 4\zeta^2}, \quad \boldsymbol{w} = \frac{1}{1 - \zeta}$$

over $\mathfrak{R}$ and find the open-loop output time optimal control.

We have

$$a_0 = 1 - 4\zeta, \quad p_0 = 1,$$
$$b^+ = 1 - 0{\cdot}5\zeta, \quad b^- = 1 - 2\zeta.$$

Thus we are to solve the equation

$$(1 - \zeta)\, x + \zeta(1 - 2\zeta)\, y = 1 \ .$$

According to (9/I) we obtain

$$\begin{array}{cccc} 1 - \zeta & \zeta - 2\zeta^2 & 1 - \zeta & -1 \\ & 0 & 1 + 2\zeta & -1 + \zeta \\ 1 & 0 & 1 & -1 + \zeta \\ 0 & 1 & 1 + 2\zeta & -\zeta + 2\zeta^2 \end{array}$$

and hence

$$x = 1 + 2\zeta + (-\zeta + 2\zeta^2)\, t \ ,$$
$$y = \ -1 \ - (-1 + \zeta)\, t \ .$$

The condition $\partial \hat{x} = \min$ yields

$$\hat{x} = 1 + 2\zeta \ ,$$
$$\hat{y} = -1 \ ,$$

and, by Theorem 1,

$$u = \frac{-1 + 4\zeta}{1 - 0{\cdot}5\zeta}, \quad e = 1 + 2\zeta, \quad k_{\min} = 2.$$

Example 2. Obtain the open-loop output time optimal control for the finite automaton

$$s = \frac{\zeta^2}{1 + 2\zeta}, \quad w = \frac{1}{1 + \zeta}$$

defined over $\mathfrak{Z}_3$, the field of residue classes modulo 3. The Diophantine equation to be solved becomes

$$(1 + \zeta) x + \zeta^2 y = 1.$$

The solution is obtained by the use of the table

$$\begin{array}{llll} 1 + \zeta & \zeta^2 & 1 + \zeta & 1 \\ & 0 & 2 + \zeta & 1 + \zeta \\ 1 & 0 & 1 & 1 + \zeta \\ 0 & 1 & 2 + \zeta & \zeta^2 \end{array}$$

as

$$x = 1 + 2\zeta + \zeta^2 t,$$
$$y = 1 - (1 + \zeta) t.$$

Remember that all computations are carried out in the modulo 3 arithmetics.

The solution

$$\hat{x} = 1 + 2\zeta,$$
$$\hat{y} = 1$$

satisfies the condition $\partial\hat{x} = \min$. Hence

$$u = \frac{1 + 2\zeta}{1 + \zeta}, \quad e = 1 + 2\zeta, \quad k_{\min} = 2.$$

Observe that the control is not stable since p^- does not divide a. Nonetheless the solution may be acceptable for the engineer.

OPEN-LOOP STATE TIME OPTIMAL CONTROL

In contrast to the preceding problem we want not only polynomial e but polynomial u, too. This is equivalent to reaching equilibrium state in a finite time.

More formally, given the configuration shown in Fig. 1, where

$$s = \frac{\zeta^d b}{a}, \quad w = \frac{q}{p},$$

 the polynomials being arbitrary elements of $\mathfrak{F}[\zeta]$ but $d > 0$, $(a, \zeta b) = 1$, $(b, \zeta) = 1$, $(p, \zeta q) = 1$; generate such a finite control

$$\boldsymbol{u} = v$$

that the error $\boldsymbol{e}$ vanish in a minimum time $k_{\min}$.

We claim

Theorem 2. *The open-loop state time optimal control problem has a solution if and only if $p \mid a$. The solution is unique and is given by*

$$v = a_0 \hat{y}$$

where $\hat{x}$ and $\hat{y}$ is such solution of the Diophantine equation

$$px + \zeta^d by = q$$

that $\partial \hat{x} = \min$.

Moreover, $\boldsymbol{e} = \hat{x}$ and $k_{\min} = 1 + \partial \hat{x}$.

Proof. In a like manner,

$$(1) \qquad \boldsymbol{e} = x = \frac{q}{p} - \frac{\zeta^d b}{a} v = \frac{a_0 q - p_0 \zeta^d b v}{a_0 w p_0}.$$

By letting

$$v = a_0 y$$

we reduce expression (1) as much as possible and convert it into the equation

$$w p_0 x + p_0 \zeta^d b y = q .$$

Since p_0 does not divide q, this equation has a solution if and only if $p_0 = 1$ or $p \mid a$ by (14/I). Then

$$px + \zeta^d by = q$$

and $\boldsymbol{e} = \hat{x}$, $k_{\min} = 1 + \partial \hat{x}$, where $\partial \hat{x} = \min$.

This Diophantine equation has a solution since $p \mid a$ implies $(p, \zeta^d b) = 1$. □

Remark 2. It is worth noting that the equilibrium state need not be achieved in $k_{\min}$ time units, in general. This situation occurs when $\partial v > \partial \hat{x}$, i.e., when the control time exceeds the follow-up time. See Example 4.

Example 3. Consider again

$$\boldsymbol{s} = \frac{\zeta(1 - 2{\cdot}5\zeta + \zeta^2)}{1 - 5\zeta + 4\zeta^2}, \quad \boldsymbol{w} = \frac{1}{1 - \zeta}$$

over $\mathfrak{R}$ and find the open-loop state time optimal control.

The respective Diophantine equation reads

$$(1 - \zeta)\,x + \zeta(1 - 2{\cdot}5\zeta + \zeta^2)\,y = 1$$

and the desired solution is

$$\hat{x} = 1 + 3\zeta - 2\zeta^2\,,$$

$$\hat{y} = -2\,.$$

Hence

$$u = -2 + 8\zeta\,, \quad e = 1 + 3\zeta - 2\zeta^2\,, \quad k_{\min} = 3\,.$$

Example 4. Consider a continuous system over $\mathfrak{R}$ represented by its Laplace transform $(s + 2)/s^2$ and sampled at $k = 0, 1, \ldots$ We get

$$s = \frac{2\zeta}{(1 - \zeta)^2}$$

and let

$$w = \frac{1}{1 - \zeta}\,.$$

Then the open-loop state time optimal control problem leads to

$$(1 - \zeta)\,x + 2\zeta y = 1\,, \quad \partial\hat{x} = \min\,,$$

the solution being

$$\hat{x} = 1\,,$$

$$\hat{y} = 0{\cdot}5\,.$$

Hence

$$u = 0{\cdot}5 - 0{\cdot}5\zeta\,, \quad e = 1\,, \quad k_{\min} = 1\,.$$

Note that it may well happen that $k_{\min} < n$, the dimension of the given system. The equilibrium state, however, is reached at $k = n$.

OPEN-LOOP LEAST SQUARES CONTROL

This sort of control problems was introduced as those involving minimization of a quadratic functional.

More formally, given the configuration shown in Fig. 1, where

$$s = \frac{\zeta^d b}{a}\,, \quad w = \frac{q}{p}\,,$$

the polynomials being arbitrary elements of $\mathfrak{F}[\zeta]$ but $d > 0$, $(a, \zeta b) = 1$, $(b, \zeta) = 1$, $(p, \zeta q) = 1$; find a stable control

$$u = \frac{v}{u}$$

so as to minimize the cost functional

$$\varphi = \sum_{k=0}^{\infty} \varepsilon_k^2 ,$$

where

$$e = \varepsilon_0 + \varepsilon_1\zeta + \varepsilon_2\zeta^2 + \dots .$$

Theorem 3. *The open-loop output least squares control problem has a solution if and only if* $p^- \mid a$ *and* $\tilde{b}^-/((\hat{x}, \hat{y}), \tilde{b}^-)$ *is stable. The solution is unique and is given as*

$$v = a_0\hat{y}$$

$$u = p_0 b^*$$

where $\hat{x}$ *and* $\hat{y}$ *is such solution of the Diophantine equation*

$$px + \zeta^d b^- y = \tilde{b}^- q$$

that $\partial\hat{x} = \min$.

Moreover

$$e = \frac{\hat{x}}{\tilde{b}^-}$$

and

$$\varphi_{\min} = \left\langle \left(\frac{\hat{x}}{b^-}\right)^{\sim} \left(\frac{\hat{x}}{b^-}\right) \right\rangle .$$

Proof. Write

$$e = \varepsilon_0 + \varepsilon_1\zeta \quad + \varepsilon_2\zeta^2 \quad + \dots ,$$

$$\tilde{e} = \varepsilon_0 + \varepsilon_1\zeta^{-1} + \varepsilon_2\zeta^{-2} + \dots ,$$

then

$$\varphi = \langle \tilde{e}e \rangle$$

provided e is stable.

We will manipulate the expression for φ in such a way as to make the minimizing choice of u obvious. Rewrite

$$\varphi = \left\langle \left(\frac{\tilde{b}^-}{\zeta^d b^-} e\right)^{\sim} \left(\frac{\tilde{b}^-}{\zeta^d b^-} e\right) \right\rangle .$$

By inspection of Fig. 1,

$$e = \frac{q}{p} - \frac{\zeta^d b}{a}\frac{v}{u} .$$

Therefore

$$\frac{\tilde{b}^-}{\zeta^d b^-} e = \frac{\tilde{b}^- q}{\zeta^d b^- p} - \frac{b^* v}{au} \tag{2}$$

and consider the decomposition

$$\frac{\tilde{b}^- q}{\zeta^d b^- p} = \frac{x}{\zeta^d b^-} + \frac{y}{p}$$

of the first term in (2), so that x and y are coupled by the Diophantine equation

(3) $$px + \zeta^d b^- y = \tilde{b}^- q \,.$$

Collecting the terms gives us

$$\frac{\tilde{b}^-}{\zeta^d b^-} \boldsymbol{e} = \frac{x}{\zeta^d b^-} + \boldsymbol{a}$$

where

(4) $$\boldsymbol{a} = \frac{y}{p} - \frac{b^* v}{au} = \frac{a_0 yu - p_0 b^* v}{p_0 w a_0 u} \,.$$

Hence on substituting

(5) $$\varphi = \left\langle \left(\frac{x}{b^-}\right)^{\sim} \left(\frac{x}{b^-}\right) \right\rangle + \langle \tilde{\boldsymbol{a}} \boldsymbol{a} \rangle + 2 \left\langle \left(\frac{x}{\zeta^d b^-}\right)^{\sim} \boldsymbol{a} \right\rangle .$$

Further we refine (5) be setting

(6) $$x = \hat{x} + \zeta^d b^- t \,, \quad \partial \hat{x} < \partial \zeta^d b^- \,.$$

The key observation is that

$$\left(\frac{\hat{x}}{\zeta^d b^-}\right)^{\sim} = \frac{\tilde{\hat{x}}}{\tilde{b}^-} \zeta^{\partial \zeta^d b^- - \partial \hat{x}}$$

is divisible by ζ due to (6). Hence

$$\left\langle \left(\frac{\hat{x}}{\zeta^d b^-}\right)^{\sim} t \right\rangle = 0$$

and also

$$\left\langle \left(\frac{\hat{x}}{\zeta^d b^-}\right)^{\sim} \boldsymbol{a} \right\rangle = 0$$

as long division into positive powers of ζ shows.

Thus (5) becomes

(7) $$\varphi = \left\langle \left(\frac{\hat{x}}{b^-}\right)^{\sim} \left(\frac{\hat{x}}{b^-}\right) \right\rangle + \langle \tilde{\boldsymbol{a}} \boldsymbol{a} \rangle + 2 \langle \tilde{\boldsymbol{a}} t \rangle + \langle \tilde{t} t \rangle \,.$$

The first term in (7) cannot be further reduced. The best we can do to minimize φ is to set $\boldsymbol{a} = 0$, $t = 0$ and the theorem follows.

Indeed, $v = a_0\hat{y}$, $u = p_0 b^*$ by virtue of (4) and $\partial\hat{x} = \min$ in (6).
Moreover

$$e = \frac{q}{p} - \frac{\zeta^d b}{a}\frac{a_0\hat{y}}{p_0 b^*} = \frac{(\tilde{b}^- q - \zeta^d b^- \hat{y})\, b^+}{p b^*} = \frac{\hat{x}}{\tilde{b}^-}$$

by (3). Note that e is stable if and only if $\tilde{b}^-/(\hat{x}, \tilde{b}^-)$ is stable, and that u will be stable if and only if $\tilde{b}^-/(\hat{y}, \tilde{b}^-)$ is stable and p_0 is stable, that is, $p^- \mid a$. Hence both e and u will be stable if and only if $p^- \mid a$ and $\tilde{b}^-/((\hat{x}, \hat{y}), \tilde{b}^-)$ is stable.

Now $p^- \mid a$ guarantees that $(p, \zeta^d b^-) = 1$ and, in turn, (3) has a solution. □

Remark 3. Caution! $\varphi = \langle \tilde{e}e \rangle$ if and only if e is stable, as the example

$$e = \frac{2 - \zeta}{1 - 2\zeta} = 2 + 3\zeta + 6\zeta^2 + 12\zeta^3 + \dots$$

demonstrates. Indeed, $\langle \tilde{e}e \rangle = 1$ whereas $\varphi \to \infty$.

Remark 4. For a minimum-phase system ($b^- = 1$) we have $b = b^+ = b^*$. As a result the open-loop output time optimal and least squares controls turn out to be the same. However, they may equal in other cases as well, see Example 7.

Example 5. Let $\mathfrak{F} = \mathfrak{R}$ and

$$s = \frac{\zeta}{1 - \zeta}, \quad w = \frac{1 - 2\zeta}{1 - \zeta}.$$

To find the open-loop least squares control, we realize that $b^- = b^* = b^+ = 1$ and solve

$$(1 - \zeta)\, x + \zeta y = 1 - 2\zeta.$$

The respective table reads

$1 - \zeta$	ζ	1
	-1	$1 - \zeta$
1	-1	$1 - \zeta$
0	1	ζ

whence

$$x = 1 - 2\zeta + \zeta t$$
$$y = 1 - 2\zeta - (1 - \zeta)\, t.$$

The condition $\partial\hat{x} = \min$ results in

$$\hat{x} = 1,$$
$$\hat{y} = -1$$

and thus, by Theorem 3,

$$u = -1\,, \quad e = 1 \text{ and } \varphi_{\min} = 1\,.$$

Example 6. Consider the running example

$$s = \frac{\zeta(1 - 2{\cdot}5\zeta + \zeta^2)}{1 - 5\zeta + 4\zeta^2}\,, \quad w = \frac{1}{1 - \zeta}$$

and $\mathfrak{F} = \mathfrak{R}$. Obtain the open-loop least squares control.

It is seen that

$$a_0 = 1 - 4\zeta \;, \quad p_0 = 1$$
$$b^+ = 1 - 0{\cdot}5\zeta\,, \quad b^- = 1 - 2\zeta\,.$$

The Diophantine equation to be solved reads

$$(1 - \zeta)\,x + \zeta(1 - 2\zeta)\,y = \zeta - 2\,.$$

Consult Example 1 to get

$$x = (1 + 2\zeta)(\zeta - 2) + (2\zeta^2 - \zeta)\,t\,,$$
$$y = 2 - \zeta \qquad\qquad - (\zeta - 1)\,t\,.$$

The condition $\partial \hat{x} = \min$ produces

$$\hat{x} = -2 - 2\zeta\,,$$
$$\hat{y} = 1$$

and by Theorem 3

$$u = \frac{1 - 4\zeta}{(1 - 0{\cdot}5\zeta)(\zeta - 2)}\,, \quad e = \frac{2 + 2\zeta}{2 - \zeta}\,, \quad \varphi_{\min} = 4\,.$$

For effective computation of $\varphi_{\min}$ see Example 9.

Example 7. Given $\mathfrak{F} = \mathfrak{R}$,

$$s = \frac{\zeta(1 + \zeta)^2}{3 - \zeta}\,, \quad w = \frac{2 + 2\zeta^2 + \zeta^3}{2 - \zeta}\,,$$

compute the open-loop least squares control.

Evidently the Diophantine equation

$$(2 - \zeta)\,x + \zeta(1 + \zeta)^2\,y = (1 + \zeta)^2\,(2 + 2\zeta^2 + \zeta^3)$$

is to be solved. We obtain

$$\hat{x} = (1 + \zeta)^2\,,$$
$$\hat{y} = (1 + \zeta)^2$$

 and hence

$$u = \frac{3 - \zeta}{2 - \zeta}, \quad e = 1, \quad \varphi_{\min} = 1$$

by Theorem 3.

There are two points about this example. Firstly it illustrates that it is $\tilde{b}^- / ((\hat{x}, \hat{y}), \tilde{b}^-)$, not $\tilde{b}^-$ itself, that determines the system stability and secondly the obtained control is identical to the output time optimal control even though $b^- \neq 1$.

THE EFFECT OF DISTURBANCES

To determine the effect of disturbances upon the open-loop optimal control, consider a nonzero initial state $\mathbf{x}_0$ of the system s as a typical disturbance (Fig. 2).

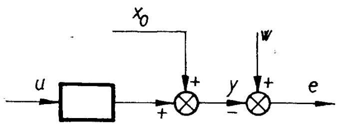

Fig. 2. The effect of disturbances.

Then

$$y = \frac{\zeta^d b}{a} u + \frac{g}{a} \tag{8}$$

where $g \in \mathfrak{F}[\zeta]$ is an arbitrary polynomial of degree one less than dimension of s, characterizing $\mathbf{x}_0$, the output produced by $\mathbf{x}_0$.

Computing the error e results in

$$e = w - \frac{\zeta^d b}{a} u - \frac{g}{a}.$$

Therefore the open-loop control strategy is not the right one to handle disturbances because the a remains unaffected and may introduce instability. However, it is the best approach in the absence of uncertainty or when uncertainty is eliminated in some other way.

SOME COMPUTATIONAL ASPECTS

Synthesizing optimal controls we needed certain polynomials like m^+, m^-, m^* etc., which may be nontrivial to obtain. This section presents effective methods for their computation and, in turn, demonstrates the power and elegance of the algebraic approach.

If the system is defined over the field $\mathfrak{Z}_p$, no stable polynomials exist save the units of $\mathfrak{Z}_p[\zeta]$ and $m^- = m$, $m^+ = 1$, $m^* = \tilde{m}$ for any $m \in \mathfrak{Z}_p[\zeta]$. Hence no stability tests and factorizations are involved. If the ground field is $\mathfrak{Q}$, $\mathfrak{R}$ or $\mathfrak{C}$, however, some control problems call for the computations described below.

A crucial role plays the computation of m^*. The following ingenious algorithm is proposed in [6], for another refer to [9]. Given $m \in \mathfrak{F}[\zeta]$, perform the recurrent division

$$\tilde{m}(\tilde{m})^\sim = f_k q_k + r_k\,, \quad \partial r_k < \partial f_k\,, \quad k = 0, 1, \ldots$$

by f_k, where

$$f_0 = \zeta^{\partial \tilde{m}(\tilde{m})^\sim/2}\,,$$
$$f_k = \tilde{q}_{k-1}\,, \quad k = 1, 2, \ldots$$

Then $q_k \to m^*$ up to a normalizing constant as $k \to \infty$.

Having computed m^*, we can take

$$m^+ = \frac{(m, m^*)}{(\tilde{m}^*, m^*)}\,,$$

$$m^- = \zeta^{\partial m - \partial m^*}(m, \tilde{m}^*)\,.$$

Example 8. Let

$$m = 2\zeta - 3\zeta^2 - 2\zeta^3 \in \mathfrak{R}[\zeta]$$

and apply the above iterative technique to obtain m^*.

Evidently

$$\tilde{m} = -2 - 3\zeta + 2\zeta^2\,,$$
$$(\tilde{m})^\sim = 2 - 3\zeta - 2\zeta^2$$

and hence

$$\tilde{m}(\tilde{m})^\sim = -4 + 17\zeta^2 - 4\zeta^4\,.$$

Initializing with $f_0 = \zeta^2$, we get

$$q_0 = 17.000\,000\,.\,(1 - 0{\cdot}235\,294\zeta^2)\,,$$
$$q_1 = 0{\cdot}944\,637\,.\,(1 - 0{\cdot}249\,084\zeta^2)\,,$$
$$q_2 = 16{\cdot}941\,606\,.\,(1 - 0{\cdot}249\,943\zeta^2)\,,$$
$$q_3 = 0{\cdot}944\,434\,.\,(1 - 0{\cdot}249\,996\zeta^2)\,,$$
$$q_4 = 16{\cdot}941\,379\,.\,(1 - 0{\cdot}250\,000\zeta^2)\,,$$
$$q_5 = 0{\cdot}944\,433\,.\,(1 - 0{\cdot}250\,000\zeta^2)\,,$$

etc. and, therefore,

$$m^* = 4\,.\,(1 - 0{\cdot}25\zeta^2) = 4 - \zeta^2$$

 modulo ± 1. It further follows that

$$m^{+} = 2 + \zeta\,, \quad m^{-} = \zeta(1 - 2\zeta)$$

modulo a unit of $\mathfrak{R}[\zeta]$.

In Part I a stable polynomial m was defined via the log division of $1/m$. This is very impractical to compute, however. A well-known check for stability is provided by the following algorithm [5], [7].

The polynomial

$$m = \mu_0 + \mu_1\zeta + \ldots + \mu_n\zeta^n$$

of degree n is stable if and only if

$$\left|\frac{\langle \tilde{m}^{(k)} \rangle}{\langle m^{(k)} \rangle}\right| < 1\,, \quad k = 0, 1, \ldots, n-1 \tag{9}$$

where, recursively

$$m^{(k+1)} = m^{(k)} - \frac{\langle \tilde{m}^{(k)} \rangle}{\langle m^{(k)} \rangle}\,\tilde{m}^{(k)}\,, \quad k = 0, 1, \ldots, n-1\,,$$

$$m^{(0)} = m\,.$$

If

$$m^{(k)} = \mu_0^{(k)} + \mu_1^{(k)}\zeta + \ldots + \mu_{n-k}^{(k)}\zeta^{n-k}\,,$$

the above recursive steps can be arranged in a table as follows:

$$\begin{array}{cccccc}
\mu_0^{(0)} & \mu_1^{(0)} & \ldots & \mu_{n-1}^{(0)} & \mu_n^{(0)} & \\
\mu_n^{(0)} & \mu_{n-1}^{(0)} & \ldots & \mu_1^{(0)} & \mu_0^{(0)} & \dfrac{\mu_n^{(0)}}{\mu_0^{(0)}} \\
\mu_0^{(1)} & \mu_1^{(1)} & \ldots & \mu_{n-1}^{(1)} & 0 & \\
\mu_{n-1}^{(1)} & \mu_{n-2}^{(1)} & \ldots & \mu_0^{(1)} & 0 & \dfrac{\mu_{n-1}^{(1)}}{\mu_0^{(1)}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\mu_0^{(n-1)} & \mu_1^{(n-1)} & \ldots\, 0 & & 0 & \\
\mu_1^{(n-1)} & \mu_0^{(n-1)} & \ldots\, 0 & & 0 & \dfrac{\mu_1^{(n-1)}}{\mu_0^{(n-1)}} \\
\mu_0^{(n)} & 0 & \ldots\, 0 & & 0\,. &
\end{array} \tag{10}$$

The multipliers $\mu_{n-k}^{(k)}/\mu_0^{(k)}$, $k = 0, 1, \ldots, n-1$ are those appearing in the absolute value in (9).

To evaluate the performance of a least squares control we have to compute expressions of the type

$$\varphi = \langle \tilde{e}e \rangle = \left\langle \left(\frac{l}{m}\right)^{\sim} \left(\frac{l}{m}\right) \right\rangle$$

where

$$l = \lambda_0 + \lambda_1\zeta + \ldots + \lambda_n\zeta^n ,$$
$$m = \mu_0 + \mu_1\zeta + \ldots + \mu_n\zeta^n$$

and m is a stable polynomial. Notice that both l and m are assumed to be of same formal degree n.

According to [1], [5] this can effectively be done as follows:

$$\varphi = \frac{1}{\langle m^{(0)} \rangle} \sum_{k=0}^{n} \frac{\langle \tilde{l}^{(k)} \rangle^2}{\langle m^{(k)} \rangle} \tag{11}$$

where, recursively,

$$m^{(k+1)} = m^{(k)} - \frac{\langle \tilde{m}^{(k)} \rangle}{\langle m^{(k)} \rangle} \tilde{m}^{(k)} , \quad k = 0, 1, \ldots, n-1 ,$$
$$m^{(0)} = m$$

and

$$l^{(k+1)} = l^{(k)} - \frac{\langle \tilde{l}^{(k)} \rangle}{\langle m^{(k)} \rangle} \tilde{m}^{(k)} , \quad k = 0, 1, \ldots, n-1 ,$$
$$l^{(0)} = l .$$

These computations can be carried out simultaneously with the stability check. Write

$$m^{(k)} = \mu_0^{(k)} + \mu_1^{(k)}\zeta + \ldots + \mu_{n-k}^{(k)}\zeta^{n-k} ,$$
$$l^{(k)} = \lambda_0^{(k)} + \lambda_1^{(k)}\zeta + \ldots + \lambda_{n-k}^{(k)}\zeta^{n-k} .$$

Then table (10) can be combined with the table below

$$\begin{array}{llllll}
\lambda_0^{(0)} & \lambda_1^{(0)} & \ldots & \lambda_{n-1}^{(0)} & \lambda_n^{(0)} & \\
\mu_n^{(0)} & \mu_{n-1}^{(0)} & \ldots & \mu_1^{(0)} & \mu_0^{(0)} & \dfrac{\lambda_n^{(0)}}{\mu_0^{(0)}} \\
\lambda_0^{(1)} & \lambda_1^{(1)} & \ldots & \lambda_{n-1}^{(1)} & 0 & \\
\mu_{n-1}^{(1)} & \mu_{n-2}^{(1)} & \ldots & \mu_0^{(1)} & 0 & \dfrac{\lambda_{n-1}^{(1)}}{\mu_0^{(1)}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\lambda_0^{(n-1)} & \lambda_1^{(n-1)} & \ldots & 0 & 0 & \\
\mu_1^{(n-1)} & \mu_0^{(n-1)} & \ldots & 0 & 0 & \dfrac{\lambda_1^{(n-1)}}{\mu_0^{(n-1)}} \\
\lambda_0^{(n)} & 0 & \ldots & 0 & 0 & \\
\mu_0^{(n)} & 0 & \ldots & 0 & 0 &
\end{array} \tag{12}$$

 where the coefficients $\lambda_{n-k}^{(k)}$ and $\mu_0^{(k)}$, $k = 0, 1, \ldots, n$ are those employed in (11).

Example 9. Check the optimal control

$$u = \frac{1 - 4\zeta}{-2 + 2\zeta - 0{\cdot}5\zeta^2}$$

and the resulting error

$$e = \frac{2 + 2\zeta}{2 - \zeta}$$

of Example 6 for stability and compute the optimal cost $\varphi_{\min}$.

Table (10) for u becomes

$$\begin{array}{ccccc}
-2 & 2 & -\frac{1}{2} & & \\
-\frac{1}{2} & 2 & -2 & \frac{1}{4}, & \left|\frac{1}{4}\right| < 1 \\
-\frac{15}{8} & \frac{3}{2} & 0 & & \\
\frac{3}{2} & -\frac{15}{8} & 0 & -\frac{4}{5}, & \left|-\frac{4}{5}\right| < 1 \\
-\frac{27}{40} & 0 & 0 & &
\end{array}$$

and hence u is stable.

Table (10) for e reads

$$\begin{array}{cccc}
2 & -1 & & \\
-1 & 2 & -\frac{1}{2}, & \left|-\frac{1}{2}\right| < 1 \\
\frac{3}{2} & 0 & &
\end{array}$$

and hence e is also stable.

To get $\varphi_{\min}$, we further compute

$$\begin{array}{ccc}
2 & 2 & \\
-1 & 2 & 1 \\
3 & 0 & \\
\frac{3}{2} & 0 &
\end{array}$$

according to (12). The result is

$$\varphi_{\min} = \frac{1}{2}\left(\frac{2^2}{2} + \frac{3^2}{\frac{3}{2}}\right) = 4\,.$$

CONCLUSIONS

This paper is the second part of a tripaper on the algebraic theory of discrete optimal control for single-variable systems. Here we have discussed the open-loop control problems, namely, the time optimal problems and the least squares problem.

Certain computational techniques have been included to demonstrate the power and elegance of the algebraic approach.

The section dealing with the effect of disturbances indicated that applicability of this type of control is practically limited to stable or prestabilized systems. We intended this material as a springboard to discuss more complicated closed-loop problems in the remaining part of the tripaper.

(Received June 30, 1972.)

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