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An Outerplanar Test of Linguistic Projectivity

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In the present paper, a mathematical model (called here an L-tree) of the dependency structure of the sentence is considered. From the linguistic point of view the most important L-trees are the projective ones. For any L-tree L we define a graph G such that G uniquely determines L (Theorem 1) and that L is projective if and only if G is outerplanar (Theorem 2). The outerplanar test of projectivity of L-trees given by Theorem 2 is relative to the planar test of projectivity of L-trees given in [5].

In [5] we defined an L-tree as an quadruple $L = (V_0, E_0, r, \leqq_L)$ such that (V_0, E_0) is a tree, r is one of the vertices of V_0 and $\leqq_L$ is a complete ordering of V_0. We said that an L-tree L is projective if for every vertices u, v and w such that uw is an edge of E_0 and that either $u <_L v <_L w$ or $w <_L v <_L u$ it holds, that if u lies on the path from r to w, then u also lies on the path from r to v (notice that in the present paper we use a rather different graphical terminology and notation than in [5].

The concept of L-trees is an apparatus useful for modelling the sentence structure in dependency syntax; the most important L-trees are the projective ones. For position of the concept of projectivity in algebraic linguistics, see Marcus [3], Chapter VI (our concept of L-trees corresponds to Marcus' concept of simple strings, but Marcus studied projectivity more generally, not only for simple strings). For another mathematical discussion of projectivity of L-trees, see, for example, [4], Chapter IV. For linguistic questions of projectivity or non-projectivity of sentence structures, see, for example, Novák [7] and Uhlířová [8].

In the present paper, for any L-tree L we shall construct a certain graph G and prove that L is projective if and only if G is outerplanar. Outerplanar graphs represent a simple class of planar graphs. A graph G is outerplanar if it can be embedded in the plane such that all the vertices of G lie on the exterior region. Chartrand and Harary [2] proved that a graph is outerplanar if and only if it contains no subgraph homeomorphic from the complete graph K_4 or the complete bipartite graph $K_{2,3}$. A graph H is homeomorphic from a graph H_0 if H is isomorphic either to H_0 or to a graph which can be obtained from H_0 by a suitable insertion of vertices of degree 2 into the edges of H_0 (the concept „homeomorphic from" is different from the concept „homeomorphic with"; see [1] and [2]).

Now, we shall define the main concept of the present paper:

Definition. Let $L = (V_0, E_0, r, \leqq_L)$ be an L-tree such that $V_0 = \{v_1, \ldots, v_n\}$, $n \geqq 1$, and $v_1 <_L \ldots <_L v_n$. We say that a graph $G = (V, E)$ is a graphical expansion of L if there is a set $W = \{w_0, \ldots, w_{n+1}\}$ disjoint with V_0 and such that $V = V_0 \cup W$ and

$$E = E_0 \cup \{rw_{n+1}\} \cup \{w_0v_1, v_1w_1, \ldots, w_{n-1}v_n, v_nw_n, w_nw_{n+1}\} .$$

Obviously, any two graphical expansion of an L-tree L are isomorphic. A close connection between L-trees and their graphical expansions is given in the following theorem:

Theorem 1. *Let G be a graphical expansion of an L-tree L. Then G is a graphical expansion of the only L-tree.*

Proof. We can assume that L and G are the same as in the definition. For every $u \in V$ it holds that $u \in V_0$ if and only if u has degree at least 3 in G. Similarly, for every $uv \in E$ it holds that $uv \in E_0$ if and only if both u and v are in V_0. There is exactly one vertex of degree 1 in G; it is w_0. Further, we have $w_0v_1 \in E$. For any i, $1 \leqq i < n$, there is exactly one vertex $w \in W$ and exactly one vertex $v \in V_0$ such that $v \neq v_i$ and v_iw, $wv \in E$; obviously $w = w_i$ and $v = v_{i+1}$. There are exactly two vertices w', $w'' \in W - \{w_0, \ldots, w_{n-1}\}$; obviously, $w'w'' \in E$. If v_nw', $v_nw'' \in E$, then $r = v_n$. Otherwise, there is j, $1 \leqq j < n$, such that either v_jw', $v_nw'' \in E$, or v_jw'', $v_nw' \in E$; then $r = v_j$. This means that G uniquely determines L. Hence the theorem.

An outerplanar test of projectivity of L-trees is given in the following theorem:

Theorem 2. *Let L be an L-tree and G be a graphical expansion of L. A necessary and sufficient condition for L to be projective is that G be outerplanar.*

Proof. We assume that L and G are the same as in the definition.

Necessity: Let L be projective. If $1 \leqq i \leqq n$, then by d_i we denote the distance between r and v_i in (V_0, E_0). For every vertex v in V we denote the points P_v and Q_v in the cartesian plane as follows:

$$P_{v_i} = (i - 1, -d_i), \quad \text{for} \quad 1 \leqq i \leqq n ;$$
$$P_{w_0} = (-1/2, -d_1) ;$$
$$P_{w_j} = (j - (1/2), -\max(d_j, d_{j+1})), \quad \text{for} \quad 1 \leqq j \leqq n - 1 ;$$
$$P_{w_n} = (n - (1/2), -d_n) ;$$

$$P_{w_{n+1}} = (n, 1);$$
$$\text{if } P_v = (x, y), \quad \text{then} \quad Q_v = (x, -n), \quad \text{for every} \quad v \in V.$$

If P and P' are points then by PP' we denote the straight-line segment which connects P and P'. Denote $S_0 = \{P_u P_v \mid uv \in E_0\}$, $S = \{P_u P_v \mid uv \in E\}$, $T_0 = \{P_u Q_u \mid u \in \in V_0\}$ and $T = \{P_u Q_u \mid u \in V\}$. As L is projective then no two straight-line segments in $S_0 \cup T_0$ cross; cf. [3], pp. 237–240. The set S gives an embedding of G in the plane. It is easy to see that no two straight-line segments in $S \cup T$ cross. This means that G is outerplanar.

Sufficiency: Let L be not projective. Then, there are u, v and w in V_0 such that (i) uw is in E_0, (ii) u lies on the path from r to w, (iii) u does not lie on the path from r to v, and (iv) either $u <_L v <_L w$ or $w <_L v <_L u$. It is obvious that $u \neq r \neq w$. Without loss of generality we assume that $u <_L v <_L w$.

Let either $r <_L u$ or $w <_L r$. Then there is an edge st in E_0 such that either $t <_L u <_L s <_L w$ or $u <_L s <_L w <_L t$. Without loss of generality we assume that $u <_L s <_L w <_L t$. There are i, j such that $1 < i < j - 1 < n$ and $s = v_i$, $t = v_j$. It is evident that G contains a subgraph which includes the vertices u, w_{i-1}, s, w_i, w, w_{j-1}, t and which is homeomorphic from $K_{2,3}$.

Let $u <_L r <_L w$. There is k such that $1 \leqq k \leqq n$ and $r = v_k$. It is evident that G contains a subgraph which includes the vertices u, w_{k-1}, r, w_k, w, w_n, w_{n+1} and which is homeomorphic from $K_{2,3}$. Thus G is not outerplanar which completes the proof.

The test of projectivity of L-trees given by Theorem 2 is relative to the planar test of projectivity of L-trees given in [5] (cf. also [6]).

Notice that there is an L-tree with a non-planar graphical expansion; for example an L-tree $(V_0, E_0, r, \leqq_L)$ with $V = \{v_1, \ldots, v_6\}$, $E_0 = \{v_3v_6, v_6v_1, v_1v_4, v_4v_2, v_2v_5\}$, $r = v_1$, $v_1 <_L \ldots <_L v_6$.

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