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Synthesis of Discrete Optimum Control Systems via Finite Impulse Response

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This paper contains a method for which the spectral density of a disturbance need not be necessarily known.

In this paper the method of the Z transformation will be used.

Let a linear discrete system be given by its transfer function $S(z)$ and its impulse response $H(z)$. We assume that the system output is contaminated by the stationary noise $U(z)$ with zero mean.

To eliminate this noise we shall find:

(i) a simple filter $K(z)$ that will minimize a quadratic performance index for the noise $U(z)$;

(ii) a filter $P(z)$ that will minimize a quadratic performance index for any stable input.

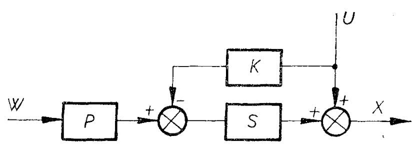


Fig. 1.

Either we can measure the disturbance U directly, see Fig. 1, or indirectly as it is shown in Fig. 2. In Fig. 1 there is shown the open-loop control system that could be formally recomputed as the closed-loop system.

Let the performance index be of the form

$$I = \frac{1}{2\pi j} \int_{\Gamma} [(1 - \bar{P}\bar{S})(1 - PS)\,\bar{W}W + (1 - \bar{K}\bar{S})(1 - KS)\,N]\,\frac{dz}{z} \tag{1}$$

where j is the imaginary unit, Γ is the unit circle, $\bar{S}(z) = S(z^{-1})$, W is the reference input, N is the spectral density of the noise U.

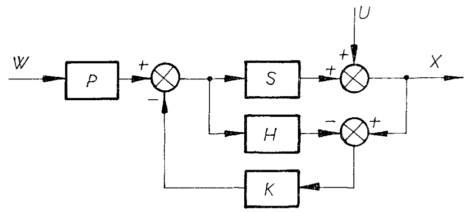


Fig. 2.

It is generally known that the minimization of the performance index I (see [1], [2]) gives the result of the form

(2) $$P = \frac{z}{S^+ W^+} [z^{-1}(S^-)^{-1} \bar{S} W^+]_+ ,$$

(3) $$K = \frac{z}{S^+ N^+} [z^{-1}(S^-)^{-1} \bar{S} N^+]_+$$

where W^+, S^+ and N^+ is obtained by the spectral factorization of $\bar{W}W$, $\bar{S}S$ and N respectively. ($W^+, (W^+)^{-1}, S^+, (S^+)^{-1}$ and $N^+, (N^+)^{-1}$) have no poles inside Γ), $[.]_+$ denotes the extraction of the poles lying outside Γ.

We can see that the filter P as well as K has S^+ in its denominator. This common part of the filters can be used for the compensation of the system. We shall obtain the new system

$$\Psi = (S^+)^{-1} S .$$

This new system Ψ has some interesting features:

(i) if S is a minimal phase system, then $\Psi = z^{-k}$, where k is the time delay,

(ii) if S is a nonminimal phase system, then $\Psi = z^{-k}(M^-/M^*)$, where M^- is a polynomial whose zeros lie outside Γ and M^* the polynomial reciprocal to M^- and its zeros lie inside Γ.

(iii) $\bar{\Psi}\Psi = 1$ because $\bar{S}S . (S^+\bar{S}^+)^{-1} = 1$.

The new control system is shown in Fig. 3.

We know that $\Psi = z^{-k}(M^-/M^*)$ and $\bar{\Psi}\Psi = 1$, hence $\Psi^+ = \Psi^- = 1$.

The filter G will be computed by (2) as the filter P in the following form

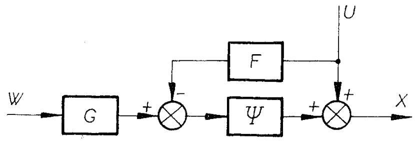
(4) $$G = \frac{z}{W^+} [z^{-1} \bar{\Psi} W^+]_+$$

and the filter F similarly in the form

$$F = \frac{z}{N^+} \left[z^{-1} \Psi N^+ \right]. \tag{5}$$

To compute the filter F we must know the spectral density of the noise U.

Fig. 3.



It is very difficult to compute the spectral density in the form of a rational fraction function.

Now we describe a method for which the spectral density need not be necessarily known.

THE CALCULATION OF THE FILTER F

Assume $W = 0$. Then

$$X = U + \Psi Z = U \left(1 - z^{-k} \frac{M^-}{M^*} F\right). \tag{6}$$

The performance index will be given as

$$I = \frac{1}{2\pi j} \int_\Gamma \left(1 - z^k \frac{\overline{M}^-}{\overline{M}^*} \overline{F}\right) \left(1 - z^{-k} \frac{M^-}{M^*} F\right) N \frac{dz}{z}. \tag{7}$$

Let M have all its zeros inside Γ. Then $\Psi = z^{-k}$ and F is a prediction filter which predicts the disturbance U for k steps ahead.

If we calculated the optimal prediction filters by the mean-squared value of prediction error we could see that:

(i) a noise, the correlation function of which is $R(i) = a \, . \, e^{-bi}$ (where $a > 0$, $b > 0$, i is an integer) is optimally predicted by the filter $F = f_0$,

(ii) a noise, the correlation function of which is $R(i) = a \cos bi$, is optimally predicted by the filter $F = f_0 + f_1 z^{-1}$ with zero error,

(iii) from the numerical point of view a flat noise is practically predictable by the filter $F = f_0$,

(iv) an almost white noise is practically predictable by the filter $F = f_0 + f_1 z^{-1}$.

Now we summarize the above discussion in the

Lemma. *The optimal predictor is the filter of the form*

$$F = f_0 + f_1 z^{-1} .$$

Let us return to the nonminimal phase system Ψ and consider $F = f_0 + f_1 z^{-1}$. The minimum performance index is achieved if $\partial I / \partial f_i = 0$ for $i = 0, 1$.
Hence

(8) $$\frac{1}{2\pi j} \int_\Gamma \left(1 - z^{-k} \frac{M^-}{M^*} F\right) \frac{M^-}{M^*} z^k N z^{i-1} \, dz = 0$$

for $i = 0, 1$, where N is the spectral density of the disturbance U. We know that

$$\frac{M^- \overline{M}^-}{M^* \overline{M}^*} = 1 .$$

Then

(9) $$\frac{1}{2\pi j} \int_\Gamma \frac{1}{\overline{M}^*} (\overline{M}^- z^k - \overline{M}^* F) \, N z^{i-1} \, dz = 0 .$$

The polynomial $\overline{M}^*$ has all its zeros outside Γ and hence it has no influence upon the equation $\partial I / \partial f_i = 0$ and therefore is can be deleted.

Substituting $M^- = \sum_{l=0}^{r} m_l z^{-l}$ into (9) gives us

(10) $$\frac{1}{2\pi j} \int_\Gamma \left(\sum_{l=0}^{r} m_l z^{k+l} - f_0 \sum_{l=0}^{r} m_{r-l} z^l - f_1 \sum_{l=0}^{r} m_{r-l} z^{l-1}\right) N z^{i-1} \, dz = 0 .$$

We know that

(11) $$N = \sum_{l \to -\infty}^{+\infty} R(l) \, z^{-l}$$

where $R(l)$ is the value of the correlation function of the disturbance U at a point i.
Substitute (11) into (10) and integrate for $i = 0, 1$, then

(12) $$i = 0 ; \quad \sum_{l=0}^{r} m_l R(k + l) - f_0 \sum_{l=0}^{r} m_{r-l} R(l) - f_1 \sum_{l=0}^{r} R(l - 1) = 0 ,$$

(13) $$i = 1 ; \quad \sum m_l R(k + l + 1) - f_0 \sum_{l=0}^{r} m_{r-l} R(l + 1) - f_1 \sum_{l=0}^{r} m_{r-l} R(l) = 0 .$$

This equations yield f_0 and f_1.

It is very important that we need not know the spectral density N. We must know only the finite number of values of the correlation function.

Let us show that the optimal filter F reduces only to f_0 for any disturbance whose correlation function is of the form $R(i) = a^{|i|}$ ($a > 0$, i is an integer) and for any system Ψ.

Substituting $R(i) = a^{|i|}$ into (12) and (13) gives us

$$\sum_{l=0}^{r} m_l a^{k+l} - f_0 \sum_{l=0}^{r} m_{r-l} a^l - f_1 \sum_{l=0}^{r} m_{r-l} a^{|l-1|} = 0 ,$$

$$\sum_{l=0}^{r} m_l a^{k+l+1} - f_0 \sum_{l=0}^{r} m_{r-l} a^{l+1} - f_1 \sum_{l=1}^{r} m_{r-l} a^l = 0 .$$

Premultiplying the first equation by a and subtracting it from the second one gives

$$f_1 \sum_{l=0}^{r} m_{r-l}(a^{|l-1|} - a^l) = 0 .$$

Hence evidently $f_1 = 0$.

Example. Let a system S be given by the approximation of its impulse response as follows

$$H(z) = z^{-1}(1 + 2{\cdot}7z^{-1} + 1{\cdot}41z^{-2} + 0{\cdot}02z^{-3}) ,$$

and let U is the Z transform of the realization of a stationary random process. The values of the correlation function $R(i)$ are:

$$R(0) = 1 ; \quad R(1) = 0{\cdot}7 ; \quad R(2) = 0 ; \quad R(3) = -0{\cdot}7 ;$$

and consider the step input.

Compute the optimal control for the quadratic performance index (1).

To obtain the new system Ψ we have to compute the spectral factorization of $\bar{H}H = H^+H^-$.

The method described in [3] is used in computer programme and we find

$$H^+ = 2 + 2{\cdot}4z^{-1} + 0{\cdot}72z^{-2} + 0{\cdot}01z^{-3} .$$

It is generally known that if

$$H = z^{-1}k \prod_{i=r+1}^{r} (1 - \alpha_i z^{-1}) . \prod_{j=1}^{r} (1 - \beta_j z^{-1})$$

where n is the order of the polynomial H $|\alpha_i| < 1$, $\beta_j > 1$, k is a constant, then

$$H^+ = k \prod_{i=r+1}^{n} (1 - \alpha_i z^{-1}) . \prod_{j=1}^{r} (z^{-1} - \beta_j)$$

and

(14) $$\Psi = \frac{H}{H^+} = z^{-1} \frac{M^-}{M^*}$$

where

$$M^- = \prod_{j=1}^{r} (1 - \beta_j z^{-1}) ,$$

$$M^* = \prod_{j=1}^{r} (z^{-1} - \beta_j) .$$

Because in our case $H \neq H^+$, there exists at least one zero of the polynomial H which lie outside Γ.

Assume that only one such zero exists, then

$$M^- = 1 + \gamma z^{-1} \quad \text{and} \quad M^* = z^{-1} + \gamma .$$

The equation (14) gives us

$$z^{-1}M^-H^+ = HM^* ,$$

(i)
$$z^{-1}(1 + \gamma z^{-1})(2 + 2\cdot4z^{-1} + 0\cdot72z^{-2} + 0\cdot01z^{-3}) =$$
$$= z^{-1}(1 + 2\cdot7z^{-1} + 1\cdot41z^{-2} + 0\cdot02z^{-3})(z^{-1} - \gamma) .$$

On equating the absolute terms in equation (i) we can see that $\gamma = 2$.

If the equation (i) is satisfied for $M^- = 1 + 2z^{-1}$, then $M^- = 1 + 2z^{-1}$, but if it is not, then we assume $M = 1 + \gamma_1 z^{-1} + \gamma_2 z^{-2}$, etc.

In our case $M^- = 1 + 2z^{-1}$ and

$$\Psi = z^{-1} \frac{1 + 2z}{z^{-1} + 2} .$$

In order to eliminate a disturbance U we need to know $r + k + 2 = 1 + 1 + 2 = 4$ values of the correlation function, that is, $R(0)$, $R(1)$, $R(2)$, $R(3)$.

Substituting this into (12) and (13) gives us

$$0\cdot7 = f_0\, 2\cdot7 + f_1\, 2\cdot4 ,$$
$$-1\cdot4 = f_0\, 1\cdot4 + f_1\, 2\cdot7 .$$

Hence $f_0 = 1\cdot335, f_1 = -122.$

Now we shall compute the closed-loop control system.

By Fig. 3 we can obtain the output in the form

(ii)
$$X = (1 - \Psi F)\, U + \Psi G W .$$

The closed-loop system, which will be assumed in the form gives the output

(iii)
$$X = \frac{R\Psi}{1 + R\Psi} W + \frac{1 - \Phi\Psi}{1 + R\Psi} U .$$

Let us compare (ii) and (iii). We can see that

$$G = \frac{R}{1 + R\Psi}$$

and

$$1 - \Psi F = \frac{1 - \Phi\Psi}{1 + R\Psi} .$$

Hence

$$R = \frac{G}{1 - \Psi G}$$

and

$$\Phi = \frac{1}{1 - \Psi G}(F - G).$$

In our example the results are

$$G = z(1 - z^{-1})\left[z^{-1}\frac{z(1+2z)}{2+z}\,\frac{z}{z-1}\right] = 1 \quad \text{by} \quad (4),$$

$$F = 1{\cdot}335 - 1{\cdot}22z^{-1}$$

and

$$R = \frac{2 + z^{-1}}{2(1 - z^{-2})},$$

$$\Phi = \frac{2 + z^{-1}}{2(1 - z^{-1})}(0{\cdot}335 - 1{\cdot}22z^{-1})$$

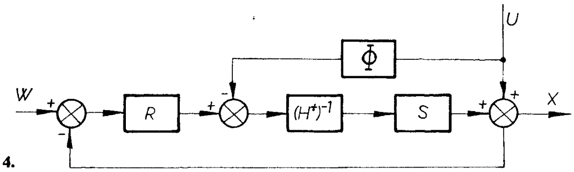


Fig. 4.

The closed-loop system in Fig. 4 can be rearranged in the form shown in Fig. 5 where

$$L = \frac{1}{1 - \Psi G} H^{+\,-1}.$$

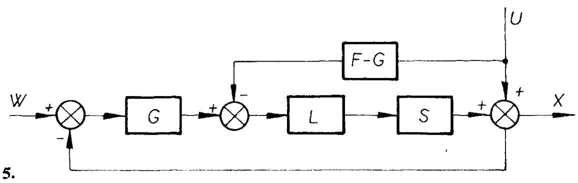


Fig. 5.

In our case

$$L = \frac{2 + z^{-1}}{2(1 - z^{-2})}\,\frac{1}{2 + 2{\cdot}4z^{-1} + 0{\cdot}72z^{-2} + 0{\cdot}01z^{-3}} =$$

$$= \frac{1}{2(1 - z^{-2})(1 + 0{\cdot}7z^{-1} + 0{\cdot}01z^{-2})}.$$

Computation of this optimum control system is based on the knowledge of the zeros of the impulse response. In [4] a theorem is proved about the relation between the system transfer function zeros lying outside Γ and the respective finite impulse response.

This theorem justifies the synthesis from the measured impulse response.

The above method of the synthesis is convenient because we need not know the zeros of the impulse response and it is known that the impulse response is the polynomial of order 20 or more.

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VÝTAH

Syntéza optimálních diskrétních regulačních obvodů daných impulsní charakteristikou konečné délky

ZDENĚK VOSTRÝ

Při syntéze optimálních diskrétních regulačních obvodů z impulsních charakteristik podle kvadratických kritérií se setkáváme se dvěma problémy: impulsní charakteristika je polynomem značného stupně (může být větší než 20), což ztěžuje výpočty hlavně z numerického hlediska; je velmi obtížné získat spektrální hustoty poruch ve tvaru racionálně lomených funkcí.

Oba tyto problémy jsou řešeny v tomto článku. První je řešen pomocí kompenzace soustavy speciálním filtrem a druhý je řešen tak, že stačí znát jen konečný počet hodnot autokorelační funkce poruch. Na závěr článku je uveden příklad.

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