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Desampling Capability of Polynomial Interpolators

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The paper deals with qualifying the capability of interpolating circuits to reconstruct an original analogue signal from a sampled-data signal. A method of comparison of the interpolators with an ideal low-pass filter was selected. Derived amplitude and phase frequency responses are plotted and compared with responses of the ideal low-pass reconstructing filter.

1. INTRODUCTION

For reconstruction of an original continuous form of a sampled signal, passive low-pass filters or active data hold circuits of the zeroth or first order are usually used. There is a possibility to utilize suitable properties of polynomial interpolators [2], [3] for such signal reconstruction. These active reconstruction filters reconstruct the original analogue signal by tracing parabolas through discrete values of sampled signal according to a given law. The quality and accuracy of reconstruction depends on many factors – mainly on the parabola order and on working interval selection (when using parabolas of higher orders than the second). Interpolating circuits are relatively simple. They consist of operational amplifiers, a circuit of the type "sample-hold" and usual computing impedances (resistors and capacitors).

Let us compare some results of the analogue signal sampling and its mathematical description with properties of polynomial interpolating circuits to determine a most convenient type of interpolator for given reconstruction requirements.

2. SAMPLED-DATA SIGNAL AND ITS MATHEMATICAL REPRESENTATION. ORIGINAL SIGNAL RECONSTRUCTION

Let us consider a sampled-data signal with fixed sampling intervals. It is given by a series of discrete values, for example by a series of narrow impulses or numbers, which can be marked $f(T_1), f(T_2), \ldots, f(T_n)$ for discrete instants $T_1, T_2, \ldots, T_n$. For these instants $T_1 = T$, $T_2 = 2T, \ldots, T_n = nT$, where T is the duration of the fixed sampling interval.

For investigation of such a signal we will use a procedure which is given in [4]. The signal, marked $f^*(t)$ (see Fig. 1), is given by

$$f^*(t) = f(t)\, p(t)\,. \tag{1}$$

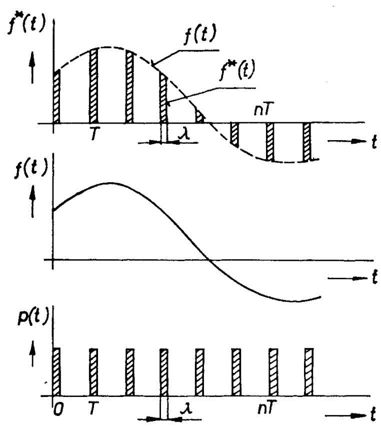


Fig. 1.

The sampling function $p(t)$ is supposed to be a series of impulses each of finite duration λ and the sampled function carrying the information to be $f(t)$. The sampling process is a process of modulation, i.e. multiplication of both these functions. It is possible to express the function $p(t)$ in a Fourier series

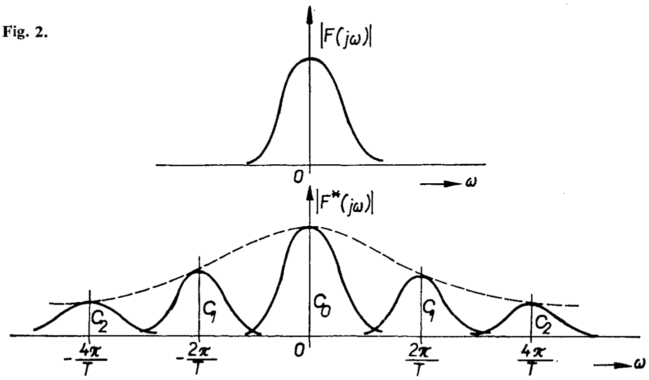
$$p(t) = \sum_{k=-\infty}^{+\infty} C_k e^{j2\pi k/T}\,. \tag{2}$$

The C_k's are Fourier coefficients. After putting (2) into (1) and using the shift theorem:

$$F^*(j\omega) = \sum_{k=-\infty}^{+\infty} C_k F\left[j\left(\omega - \frac{2\pi k}{T}\right)\right], \tag{3}$$

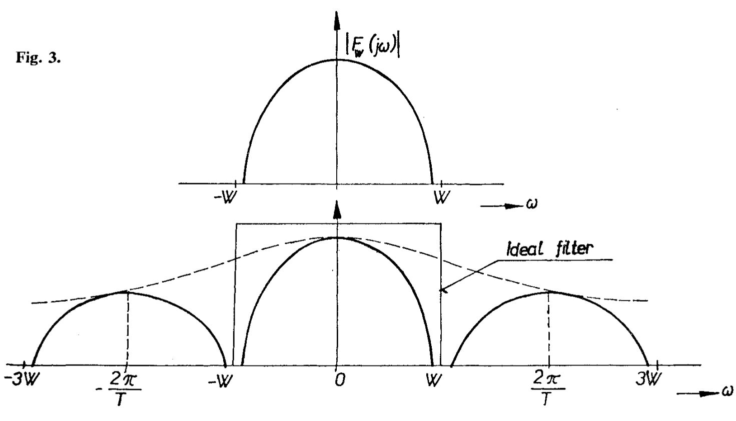
where $F(j\omega)$ is the transform of the function $f(t)$. The resulting expression is a sum of transforms of weighted spectra, each of which is equal the central spectrum except for the constant and the shift coefficient $j(\omega - 2\pi k/T)$. The expression (3) is illustrated in Fig. 2. A result of the sampling process is that to the basic frequency spectrum of the original signal are added new spurious spectra, which are shifted by a separation $2\pi k/T$ and whose weights are decreasing with increasing frequency.

Fig. 2.



For the original signal reconstruction we need to cut out these added spurious spectra by means of a low-pass filter. It is obvious, that even if we use a filter with a sharp cut-off, the resulting reconstruction will not be accurate if the original signal were not frequency limited. A spectrum of a sampled signal originating from a frequency limited signal is shown in Fig. 3 together with the illustration of an ideal reconstruction filter transfer function. One-half the sampling frequency is $W = \pi/T$.

Fig. 3.



If we supopse that the duration of sampling impulses is negligible with respect to the duration of the sampling interval T, we can imagine, the function $p(t)$ to be a series of Dirac impulses $\delta(t)$

(4) $$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) = \delta_T(t)\,.$$

$\delta(t - nT)$ represents an impulse of unite area at a time instant nT. The sampling process is then an impulse modulation.

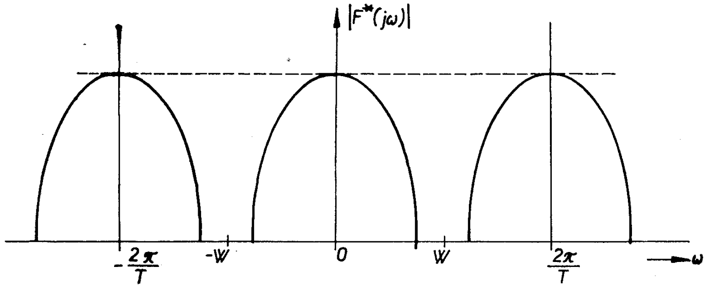


Fig. 4.

For the sampled function we can now write

(5) $$f^*(t) = f(t)\,.\,\delta_T(t) = f(nT)\,.\,\delta_T(t)$$

because the only utilizable values of the original function are given by the sampling instants. The series of impulses can now be considered as a series of impulses with areas proportional to the signal function. In order to investigate properties in the frequency domain let us determine an expression for the Fourier coefficients C_k

(6) $$C_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t)\, \mathrm{e}^{-j(2\pi/T)k}\, \mathrm{d}t\,.$$

The area of the impulse at the origin is unity. Therefore the integral also equals unity, and all Fourier coefficients are given by

(7) $$C_k = \frac{1}{T}\,,$$

independent of k. With respect to (3) the Fourier transform of the impulse modulated function $f^*(t)$ is given by

(8) $$F^*(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F[j(\omega - n\omega_0)]$$

where $\omega_0 = 2\pi/T$.

From this resulting expression a conclusion on the form of a sampled signal spectrum can be drawn. In the case of impulse sampling, spectra are added by this process to the basic signal spectrum, all of which have the same amplitude. They are repeated at intervals of $\omega_0 = 2\pi/T$. A form of the spectrum of an impulse sampled signal, which is frequency limited, is illustrated in Fig. 4. For the reconstruction of the original signal we need the same filter as was used in previous investigation.

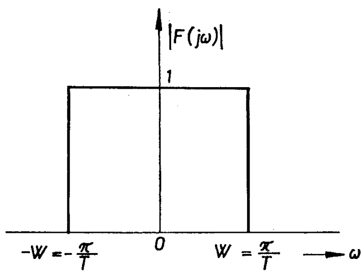


Fig. 5.

It is obvious, that the amplitude frequency response of this filter (sometimes called a "cardinal data hold") has to be unity along the frequency range from zero to one-half the sampling frequency. Above this frequency it vanishes. It is illustrated in Fig. 5. It implies that

(9) $$F(j\omega) = 1\,, \quad -W \leqq \omega \leqq W\,,$$
$$F(j\omega) = 0\,, \quad -W \geqq \omega \geqq W\,,$$

where $W = \pi/T$.

By means of the inverse transformation we obtain an impulse response of this ideal reconstruction filter:

(10) $$f(t) = \frac{W}{\pi} \frac{\sin Wt}{Wt}\,.$$

This impulse response is plotted in Fig. 6. It is obvious that such a filter is practically unrealizable due to finite output response at $t = 0$, when the input pulse is activated. Therefore, even for signals with limited frequency spectrum, it is practically impossible to get ideal reconstruction.

This ideal filter is able to be used as a prototype for comparison with implemented filters. In following sections we will compare it with amplitude frequency responses of polynomial interpolators.

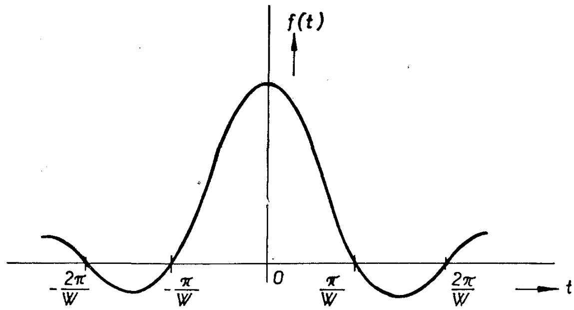


Fig. 6.

3. IMPULSE RESPONSES, TRANSFER FUNCTIONS AND FREQUENCY RESPONSES OF POLYNOMIAL INTERPOLATORS

By means of deriving a Laplace transform of the impulse response we will obtain a transfer function of a particular type of interpolator and after some arrangements we will get its amplitude and phase frequency responses. The used impulse responses

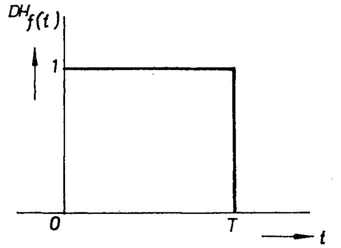


Fig. 7.

of interpolators are based on published properties of particular types (see references [2] and [3]). For the sake of simplicity we will consider interpolator output signals to be noninverted. The impulse response of the zeroth order interpolator (which is similar to a data-hold circuit) is obvious and is shown in Fig. 7. Let us mention it by way of introduction. An expression describing it is

$$^{DH}f(t) = u(t) - u(t - T) \tag{11}$$

where $u(t)$ is a step function. Let us put for the sake of simplicity $T = 1$. The Laplace transform of function (11) is then given by

(12) $$^{DH}F(s) = \frac{1}{s} - \frac{1}{s} e^{-s} = \frac{1 - e^{-s}}{s}.$$

Expression (12) is the transfer function of the zeroth order interpolator (data-hold).

First order (linear) interpolator

The impulse response of the linear interpolator is shown in Fig. 8. If we apply a unit impulse to interpolator input at $t = 0$, immediately a rising linear signal appears at the output. At the end of the first interval it will reach unit level and then at the end of the next interval it will sink to zero. This means that it consists of two parts. In this

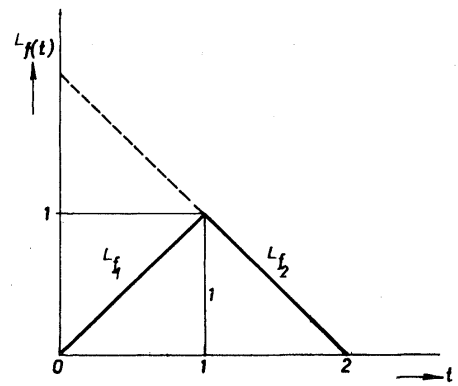


Fig. 8.

and next cases let us assume again that $T = 1$ to get rid of complicated and not on first sight clear expressions. This approach means that we suppose the sampling frequency to be $f_s = 1$. In frequency domain therefore all frequencies must be normalized with respect to the sampling frequency. It implies that

(13) $$^Lf(t) \begin{cases} ^Lf_1(t) = t & \text{for } 0 \leqq t \leqq 1, \\ ^Lf_2(t) = -t + 2 & \text{for } 1 \leqq t \leqq 2. \end{cases}$$

The Laplace transform of this function is

(14) $$\mathscr{L}[^Lf(t)] = {}^LF(s) = \int_0^1 t e^{-st} \, dt + \int_1^2 (-t + 2) e^{-st} \, dt = \frac{1}{s^2} (e^{-s} - 1)^2 .$$

We can reach the same result directly by considering the function $f(t)$ to be composed of several shifted ramp functions. Expression (14) is the transfer function of the linear interpolator.

We will substitute $s = j\omega$ and use the Euler expression for further arrangements. From (12) we will get an expression for the zeroth order interpolator (data hold):

$$(15) \qquad {}^{DH}F(j\omega) = \frac{1 - e^{-j}}{j\omega} = \frac{1}{\omega} \sin \omega + j \frac{1}{\omega} (-1 + \cos \omega)$$

and from (14) for the linear interpolator:

$${}^{L}F(j\omega) = \frac{1}{\omega^2} (-1 + 2 \cos \omega - \cos 2\omega) + j \frac{1}{\omega^2} (-2 \sin \omega + \sin 2\omega) .$$

When deriving impulse responses of interpolators of higher orders than 1, i.e. beginning with the quadratic interpolator, we have to decide what interval to use for the output parabolic signal – (See [2] and [3]). For example in the case of quadratic interpolation we can select one of two intervals, because a second order parabola passing through three points is sought, and we can use either the first interval between the first and second points or the second one between the second and third points. The number of intervals available for selection equals the order of the interpolating parabola.

Second order (quadratic) interpolator with the first working interval

Its impulse response is in Fig. 9. Unused segments of parabolas are plotted in dashed lines.

$$(17) \qquad {}^{Q_1}f(t) \begin{cases} {}^{Q_1}f_1(t) = \frac{1}{2}t^2 - \frac{1}{3}t & \text{for} \quad 0 \leqq t \leqq 1 , \\ {}^{Q_1}f_2(t) = -t^2 + 4t - 3 & \text{for} \quad 1 \leqq t \leqq 2 , \\ {}^{Q_1}f_3(t) = \frac{1}{2}t^2 - \frac{7}{2}t + 6 & \text{for} \quad 2 \leqq t \leqq 3 . \end{cases}$$

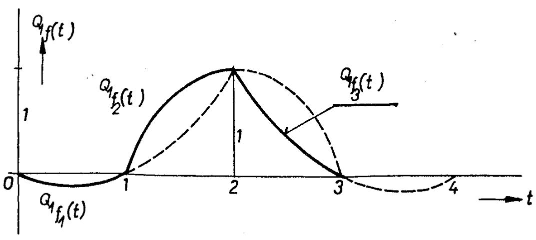


Fig. 9.

Outside the given intervals all functions vanish.

$$(18)\qquad {}^{Q_1}F(s) = \frac{1}{s^2}\left(-\tfrac{1}{2} + \tfrac{3}{2}e^{-s} - \tfrac{3}{2}e^{-2s} + \tfrac{1}{2}e^{-3s}\right) + \frac{1}{s^3}\left(1 - 3e^{-s} + 3e^{-2s} - e^{-3s}\right),$$

$$(19)\qquad {}^{Q_1}F(j\omega) = \frac{1}{\omega^2}\left(\tfrac{1}{2} - \tfrac{3}{2}\cos\omega + \tfrac{3}{2}\cos 2\omega - \tfrac{1}{2}\cos 3\omega\right) + \frac{1}{\omega^3}\left(-3\sin\omega + 3\sin 2\omega - \sin 3\omega\right) + j\left[\frac{1}{\omega^2}\left(\tfrac{3}{2}\sin\omega - \tfrac{3}{2}\sin 2\omega + \tfrac{1}{2}\sin 3\omega\right) + \frac{1}{\omega^3}\left(1 - 3\cos\omega + 3\cos 2\omega - \cos 3\omega\right)\right].$$

Second order (quadratic) interpolator with the second working interval

The impulse response is shown in Fig. 10. The output signal reaches unity in one sampling interval. The whole response is in fact the response from Fig. 9 (quadratic

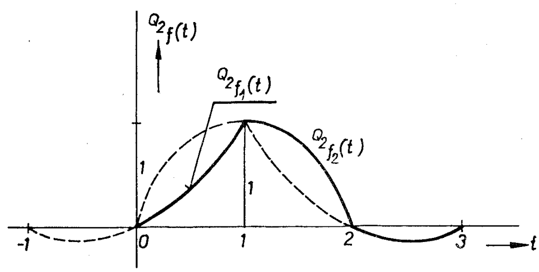


Fig. 10.

interpolator with the first working interval), which is shifted and rotated about a vertical axis.

$$(20)\qquad \begin{aligned} {}^{Q_2}f_1(t) &= \tfrac{1}{2}t^2 + \tfrac{1}{2}t && \text{for } 0 \leqq t \leqq 1\,, \\ {}^{Q_2}f_2(t) &= -t^2 + 2t && \text{for } 1 \leqq t \leqq 2\,, \\ {}^{Q_2}f_3(t) &= \tfrac{1}{2}t^2 - \tfrac{5}{2}t + 3 && \text{for } 2 \leqq t \leqq 3\,. \end{aligned}$$

 Outside the given intervals all functions vanish.

$$(21)\qquad {}^{Q_2}F(s) = \frac{1}{s^2}\left(\tfrac{1}{2} - \tfrac{3}{2}e^{-s} + \tfrac{3}{2}e^{-2s} - \tfrac{1}{2}e^{-3s}\right) + \frac{1}{s^3}\left(1 - 3e^{-2s} - e^{-3s}\right),$$

$$(22)\qquad F(j\omega) = \frac{1}{\omega^2}\left(-\tfrac{1}{2} + \tfrac{3}{2}\cos\omega - \tfrac{3}{2}\cos 2\omega + \tfrac{1}{2}\cos 3\omega\right) + \frac{1}{\omega^3}\left(-3\sin\omega + 3\sin 2\omega - \sin 3\omega\right) + j\left[\frac{1}{\omega^2}\left(-\tfrac{3}{2}\sin\omega + \tfrac{3}{2}\sin 2\omega - \tfrac{1}{2}\sin 3\omega\right) + \frac{1}{\omega^3}\left(1 - 3\cos\omega + 3\cos 2\omega - \cos 3\omega\right)\right].$$

Third order (cubic) interpolator with the first working interval

The impulse response is shown in Fig. 11. The whole response is composed of parts of third order parabolas, each of which passes through four points.

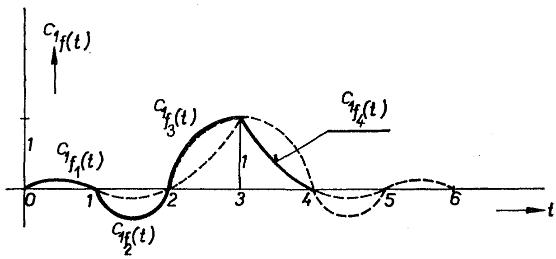


Fig. 11.

$$(23)\qquad {}^{C_1}f(t)\begin{cases} {}^{C_1}f_1(t) = \tfrac{1}{6}t^3 - \tfrac{1}{2}t^2 + \tfrac{1}{3}t & \text{for } 0 \leqq t \leqq 1\,, \\ {}^{C_1}f_2(t) = -\tfrac{1}{2}t^3 + \tfrac{7}{2}t^2 - 7t + 4 & \text{for } 1 \leqq t \leqq 2\,, \\ {}^{C_1}f_3(t) = \tfrac{1}{2}t^3 - \tfrac{11}{2}t^2 + 19t - 20 & \text{for } 2 \leqq t \leqq 3\,, \\ {}^{C_1}f_4(t) = -\tfrac{1}{6}t^3 + \tfrac{5}{2}t^2 - \tfrac{37}{3}t + 20 & \text{for } 3 \leqq t \leqq 4\,. \end{cases}$$

Outside the given intervals all functions vanish.

$$(24)\qquad {}^{C_1}F(s) = \frac{1}{s^2}\left(\tfrac{1}{3} - \tfrac{4}{3}e^{-s} + 2e^{-2s} - \tfrac{4}{3}e^{-3s} + \tfrac{1}{3}e^{-4s}\right) +$$

$$+ \frac{1}{s^3}(-1 + 4e^{-s} - 6e^{-2s} + 4e^{-3s} - e^{-4s}) +$$

$$+ \frac{1}{s^4}(1 - 4e^{-s} + 6e^{-2s} - 4e^{-3s} + e^{-4s}),$$

(25) $$^{C_1}F(j\omega) = \frac{1}{\omega^2}(-\tfrac{1}{3} + \tfrac{4}{3}\cos\omega - 2\cos 2\omega + \tfrac{4}{3}\cos 3\omega - \tfrac{1}{3}\cos 4\omega) +$$

$$+ \frac{1}{\omega^3}(4\sin\omega - 6\sin 2\omega + 4\sin 3\omega - \sin 4\omega) +$$

$$+ \frac{1}{\omega^4}(1 - 4\cos\omega + 6\cos 2\omega - 4\cos 3\omega + \cos 4\omega) +$$

$$+ j\left[\frac{1}{\omega^2}(-\tfrac{4}{3}\sin\omega + 2\sin 2\omega - \tfrac{4}{3}\sin 3\omega + \tfrac{1}{3}\sin 4\omega) +\right.$$

$$+ \frac{1}{\omega^3}(-1 + 4\cos\omega - 6\cos 2\omega + 4\cos 3\omega - \cos 4\omega) +$$

$$\left.+ \frac{1}{\omega^4}(4\sin\omega - 6\sin 2\omega + 4\sin 3\omega - \sin 4\omega)\right].$$

Third order (cubic) interpolator with the second working interval

See the impulse response in Fig. 12.

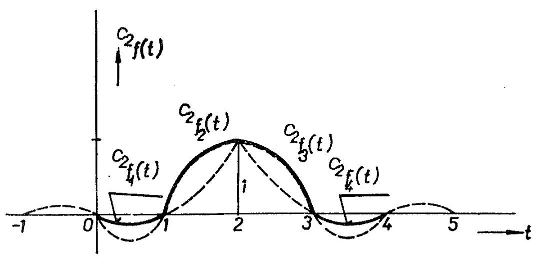


Fig. 12.

(26) $$^{C_2}f(t)\begin{cases} ^{C_2}f_1(t) = \frac{1}{6}t^3 - \frac{1}{6}t & \text{for } 0 \leqq t \leqq 1, \\ ^{C_2}f_2(t) = -\frac{1}{2}t^3 + 2t^2 - \frac{3}{2}t & \text{for } 1 \leqq t \leqq 2, \\ ^{C_2}f_3(t) = \frac{1}{2}t^3 - 4t^2 + \frac{19}{2}t - 6 & \text{for } 2 \leqq t \leqq 3, \\ ^{C_2}f_4(t) = -\frac{1}{6}t^3 + 2t^2 - \frac{47}{6}t + 10 & \text{for } 3 \leqq t \leqq 4. \end{cases}$$

 Outside the given intervals all functions vanish.

$$(27)\qquad {}^{C_2}F(s) = \frac{1}{s^2}\left(-\tfrac{1}{6} + \tfrac{2}{3}e^{-s} - e^{-2s} + \tfrac{2}{3}e^{-3s} - \tfrac{1}{6}e^{-4s}\right) + \frac{1}{s^4}\left(1 - 4e^{-s} + 6e^{-2s} - 4e^{-3s} + e^{-4s}\right),$$

$$(28)\qquad {}^{C_2}F(j\omega) = \frac{1}{\omega^2}\left(\tfrac{1}{6} - \tfrac{2}{3}\cos\omega + \cos 2\omega - \tfrac{2}{3}\cos 3\omega + \tfrac{1}{6}\cos 4\omega\right) + \frac{1}{\omega^4}\left(1 - 4\cos\omega + 6\cos 2\omega - 4\cos 3\omega + \cos 4\omega\right) + j\left[\frac{1}{\omega^2}\left(\tfrac{2}{3}\sin\omega - \sin 2\omega + \tfrac{2}{3}\sin 3\omega - \tfrac{1}{6}\sin 4\omega\right) + \frac{1}{\omega^4}\left(4\sin\omega - 6\sin 2\omega + 4\sin 3\omega - \sin 4\omega\right)\right].$$

Third order (cubic) interpolator with the third working interval

The impulse response is shown in Fig. 13. Let us mention that it is similar to the response given in Fig. 11 (the cubic interpolator with the first working interval),

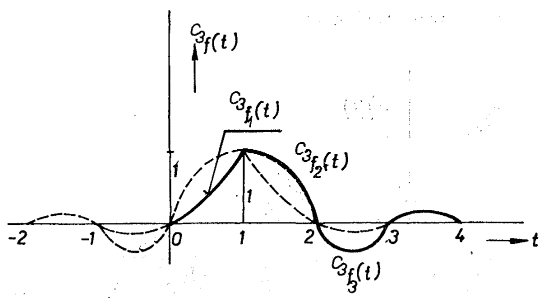


Fig. 13.

which is shifted in time and rotated about a vertical axis.

$$(29)\qquad {}^{C_3}f(t)\begin{cases} {}^{C_3}f_1(t) = \tfrac{1}{6}t^3 + \tfrac{1}{2}t^2 + \tfrac{1}{3}t & \text{for } 0 \leqq t \leqq 1, \\ {}^{C_3}f_2(t) = -\tfrac{1}{2}t^3 + \tfrac{1}{2}t^2 + t & \text{for } 1 \leqq t \leqq 2, \\ {}^{C_3}f_3(t) = \tfrac{1}{2}t^3 - \tfrac{5}{2}t^2 + 3t & \text{for } 2 \leqq t \leqq 3, \\ {}^{C_3}f_4(t) = -\tfrac{1}{6}t^3 + \tfrac{2}{3}t^2 - \tfrac{13}{3}t + 4 & \text{for } 3 \leqq t \leqq 4. \end{cases}$$

Outside the given intervals all functions vanish.

(30)
$$
{}^{C_3}F(s) = \frac{1}{s^2}\left(\tfrac{1}{3} - \tfrac{4}{3}e^{-s} + 2e^{-2s} - \tfrac{4}{3}e^{-3s} + \tfrac{1}{3}e^{-4s}\right) + \frac{1}{s^3}\left(1 - 4e^{-s} + 6e^{-2s} - 4e^{-3s} + e^{-4s}\right) + \frac{1}{s^4}\left(1 - 4e^{-s} + 6e^{-2s} - 4e^{-3s} + e^{-4s}\right),
$$

(31)
$$
\begin{aligned}
{}^{C_3}F(j\omega) = {} & \frac{1}{\omega^2}\left(-\tfrac{1}{3} + \tfrac{4}{3}\cos\omega - 2\cos 2\omega + \tfrac{4}{3}\cos 3\omega - \tfrac{1}{3}\cos 4\omega\right) + \\
& + \frac{1}{\omega^3}\left(-4\sin\omega + 6\sin 2\omega - 4\sin 3\omega + \sin 4\omega\right) + \\
& + \frac{1}{\omega^4}\left(1 - 4\cos\omega + 6\cos 2\omega - 4\cos 3\omega + \cos 4\omega\right) + \\
& + j\left[\frac{1}{\omega^2}\left(-\tfrac{4}{3}\sin\omega + 2\sin 2\omega - \tfrac{4}{3}\sin 3\omega + \tfrac{1}{3}\sin 4\omega\right) + \right. \\
& + \frac{1}{\omega^3}\left(1 - 4\cos\omega + 6\cos 2\omega - 4\cos 3\omega + \cos 4\omega\right) + \\
& \left. + \frac{1}{\omega^4}\left(4\sin\omega - 6\sin 2\omega + 4\sin 3\omega - \sin 4\omega\right)\right].
\end{aligned}
$$

Let us now limit ourselves in deriving transfer function and frequency responses to the cubic interpolator.

All expressions (15), (16), (19), (22), (25), (28) and (31) are functions of a complex variable and both their real and imaginary parts are sums of sine and cosine functions of frequency harmonics up to the $(m + 1)$-th order, if m is the order of interpolation. This form is suitable for computing particular amplitude frequency responses $R(\omega) = |F(j\omega)|$ and phase frequency responses $\psi(\omega) = \operatorname{arctg}[F(j\omega)]$. The responses of both kinds are plotted in Fig. 14. The amplitude frequency response of an ideal reconstruction filter is also plotted for comparison there.

4. DISCUSSION OF RESULTS

The judgement on the desampling capability of particular interpolators will be based on the plotted amplitude and phase frequency responses. It is obvious that the quality of reconstruction increases with the order of interpolation. The higher the order of interpolation, the better the approximation of the ideal amplitude response obtained. This fact fits the results of the computation of sine function reconstruction accuracy which were published in [3]. These results are shown in Fig. 15. The reconstruction error is $\varepsilon = |\sin \Omega t - f_r(\Omega t)|$, where $\sin \Omega t$ $(\Omega < W)$ is the original sine wave and $f_r(\Omega t)$ is the reconstructed signal. The error is plotted there as a function of the number of samples N per period $T = 2\pi/\Omega$ of the original sine wave. The meaning the of different line-types is there the same as in Fig. 14.

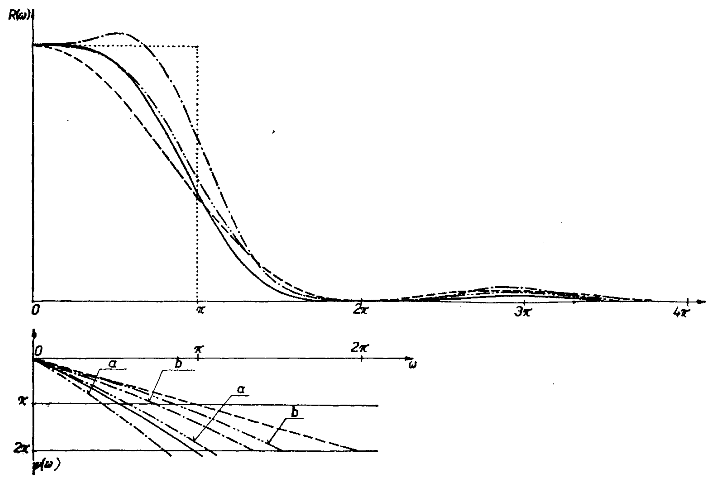


Fig. 14. Plot of the amplitude and phase frequency responses of the polynomial interpolators (— — — — linear interpolation; — ·· — ·· — quadratic ((a) first or (b) second working interval); — · — · — cubic ((a) first or (b) third working interval); ——— cubic (second working interval); · · · · · · ideal low-pass filter).

If we compare the amplitude frequency responses of interpolators with their relevant impulse responses and the same thing with the ideal filter responses we can judge roughly the quality of reconstruction from the very form of the impulse response. We can see that with increasing order of interpolation we get a better approximation to the ideal filter impulse response and therefore, of course, a better approximation to the relevant amplitude frequency response.

The reconstruction capabilities of both types of quadratic interpolators are the same (independently of the phase shift). The same property is in cubic interpolators with the first and third working interval. These pairs of interpolators have impulse responses which enclose the same area, in spite of their shift and rotation. The fact is reflected in different phase frequency responses only, or in other words, in different phase shifts of their output signals.

The same statement will always be valid for pairs of interpolators with working intervals symmetrically situated with respect to the center of the traced group of discrete values. Interpolators whose working intervals are situated in the center of this group are worth more interest. The impulse responses of these types are strictly symmetrical. Let us discuss from this point of view the cubic interpolator with the second working interval.

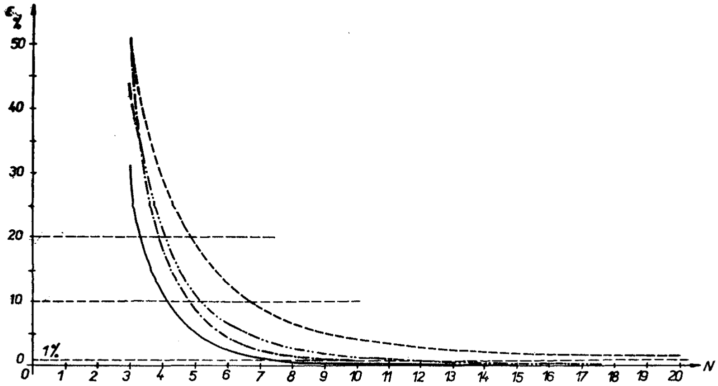


Fig. 15. Plot of the error of sine wave restoration by means of the first, second and third order polynomial interpolators versus the number of samples per period (— — — — linear interpolation; —··—··— quadratic (first or second working interval); —·—·— cubic (first or third working interval); ——— cubic (second working interval)).

Of all types discussed this type give the best approximation (ignoring the phase shift again) to the impulse response of the ideal low-pass filter. As expected, this type also has a relatively flat amplitude frequency response for lowest frequencies. Furthermore, it gives the best suppression of frequencies in the neighbourhood of the central frequencies of added spectra, i.e. of frequencies close to $\omega_k = 2\pi k$ (where $k = 1, 2, \ldots$). Therefore its reconstruction capability is the best of all presented types.

This fact is verified by the computation of the sine reconstruction accuracy (see the diagram in Fig. 15). Let us, for example, demand sine reconstruction accuracy better then 1%. Then, using the lineaı interpolator, we need at least 23 samples per period of the reconstruction sine wave; for both types of quadratic interpolators 12 samples; for cubic types with the first or third working intervals 10 samples; but for the cubic interpolator with the second (i.e. central) working interval only 8 samples. For this computation, all possible shifts of sampling instants with respect to the period of the reconstructed sine wave were taken into account.

In this paper we have limited ourselves to the reconstruction capability derivation of interpolators of order not higher then 3. From the qualitative point of view with respect to impulse responses of interpolators with the central working interval, we can estimate that the approximation of the shifted impulse response of the ideal low-pass filter will be much better in the case of higher order symetrical interpolators (for example for the fifth order interpolator with the third working interval).

If we summarize facts on central working interval interpolators from previous paragraphs, supported by results from Fig. 15, we can say: An ideal low-pass filter is, of course, unrealizable. Interpolating circuits introduce the possibility of achieving by relatively simple means a sufficient approximation to the ideal low-pass filter impulse response. This approximation is, however, shifted in time. In this way, we can implement a reconstruction filter (desampler) with an amplitude frequency response which approximates that of the ideal filter more and more closely as the order of interpolation increases.

There exists, however, some limitation in increasing the order of interpolation. Noise decreases the accuracy of higher order interpolation, when differences of higher orders are used. We must take into account not only the noise originating in quantization of the signal in the sampling process, but also a random noise originating in the interpolator itself.

Up to the last paragraph we have investigated the reconstruction signal from the point of view of its amplitude accuracy. Let us investigate now its phase shift with respect to the sampled data input signal.

The phase shifts, as shown in Fig. 14 are high for all types of interpolators. Relatively smallest phase shifts we get in the case of linear, quadratic with the second and cubic with the third working interval. These three phase responses create a bunch and for the lowest frequencies they practically fuse. Another bunch of phase responses is made by the quadratic interpolator with the first and the cubic with the second working interval. Relatively the highest is the phase shift in the case of the cubic interpolator with the first working interval. It is worth to notice that at symmetrical types of interpolation (linear and cubic with the second working interval) the phase frequency responses are linear.

The all frequency responses reach the phase shift $\psi(\omega) = \pi$ at relatively low frequencies. This fact is important especially in the case when the interpolator is used in a larger feedback system. Phase shifts originating in such a system, together with the phase shift introduced by the interpolator, can give rise to undesired oscillation. The danger is increased when using higher order interpolators with working intervals at the beginning of the traced group of points.

Therefore, the interpolators (and especially of higher order) are more suitable to be used in open-loop systems. There their capability to reconstruct the analog signal from a relatively small number of discrete values can be fully utilized.

Finally let us mention some practical difficulties in implementation of interpolators.

5. REMARKS ON IMPLEMENTATION OF INTERPOLATORS

Let us consider the problem of the highest practically realizable order of interpolation. All three types of cubic interpolators were implemented without any serious difficulty by means of an all-purpose analogue computer. It is convenient to use an electronic contact, and a current amplifier in addition to the usual operational amplifier for radical shortening of the setting time of the memory capacitor in the hold circuit. Then one can shorten the sampling impulse and many difficulties relevant to the nonnegligible time duration of this impulse with respect to the sampling interval are overcome. In the case of higher order interpolators, there is no limitation on the output of the sample-hold circuit due to higher order differencies, as was stated in [2] and [3]. The sampling frequency must, of course, be selected appropriately. The step input signal is suitable for use only as a test signal for final adjustment of the whole circuit (computing impedances and sample-hold circuit).

An advantage of interpolating circuits is that they can be very easily switched to different types of interpolation and, of course, to different sampling frequencies as well. Their implementation by temporary technology of integrated operational amplifiers is not a difficult task. Suitable sample-hold circuits are offered in miniature version. From this point of view the interpolators are more suitable than passive RLC filters, using expensive and sizable elements.

6. CONCLUSION

It can be said that use of interpolators as active low-pass filters for sampled signal reconstruction is convenient, especially there where the primary demand is to reconstruct the original analog signal from the lowest possible number of samples. The whole system in which such an interpolator is used must not be sensitive to phase shifts of the reconstructed signal.

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VÝTAH

Rekonstrukční schopnost interpolačních obvodů

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Článek se zabývá hodnocením schopnosti interpolačních obvodů rekonstruovat původní analogový signál ze signálu vzorkovaného, tj. diskrétně definovaného. Byla zvolena metoda srovnání interpolátorů s ideálním dolnofrekvenčním filtrem. Z impulsních odezev jsou vypočteny frekvenční amplitudové a fázové charakteristiky jednotlivých typů interpolátorů, jsou vyneseny a porovnány s charakteristikou ideálního dolnofrekvenčního filtru. Nepřihlížíme-li k fázovému posuvu výstupního signálu, stoupá restaurační schopnost interpolátorů se stoupajícím stupněm interpolačního polynomu. Výhodné se jeví zejména typy se symetrickým umístěním pracovního intervalu v prokládané skupině bodů.

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