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State Space Approach to Discrete Linear Control

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The state space approach to the synthesis of a class of discrete linear control systems is given. Both time-optimal and quadratic-cost problems are considered and a comparison to classical methods is made via the technique of pole assignment.

INTRODUCTION

During the past decade the state space approach to the optimal control theory has been given much attention. This paper makes a contribution to the synthesis of discrete linear control. We consider the time-optimal and the quadratic-cost problems, either in two modifications: the state and the output one.

By the *state time-optimal problem* we mean the problem of driving the state of the system to zero in minimum time, while in the *output time-optimal problem* we are to zero the (discrete) output of the system in minimum time.

The quadratic-cost problem is that of finding a control so as to minimize a quadratic cost functional. In the *state quadratic-cost problem* the cost functional involves the state and/or the input of the system. On the other hand, the *output quadratic-cost problem* involves the system output only. The precise definitions will be given at the respective sections.

The basic method of attack is the modern state space technique. However, the connection to the z-transform methods is discussed briefly for each problem.

Throughout the paper a prime denotes the transpose of a matrix, square brackets represent matrices or vectors made up of the inside symbols, and the standard notation $P \geqq 0\,(P > 0)$ for symmetric matrices means that P is nonnegative (positive) definite. Further we write $\mathscr{A}^{\perp}$ for the orthogonal complement of $\mathscr{A}$ and $\oplus$ denotes the direct summation of spaces.

SYSTEM DESCRIPTION

Consider the discrete, linear, time-invariant, single-input, single-output system $\mathcal{S}$ governed by

$$x_{k+1} = Ax_k + bu_k\,, \quad x_0 \text{ given,} \tag{1}$$

$$y_k = cx_k + du_k \tag{2}$$

where $x_k \in \mathscr{E}_n$, $u_k \in \mathscr{E}_1$ and $y_k \in \mathscr{E}_1$ are respectively the state, the input, and the output of the system at time k [21]. The matrices A, b, c and d are of dimensions $n \times n$, $n \times 1$, $1 \times n$ and 1×1 respectively. It is further assumed that det $A \neq 0$.

Here n is the *order* of the system. Set

$$\begin{aligned} h_0 &= d\,, \\ h_i &= cA^{i-1}b\,, \quad i = 1, 2, \ldots \end{aligned} \tag{3}$$

Then the number m defined by

$$m = \min_i \{i : h_i \neq 0\}$$

is called the *relative order* of the system [3]. In the z-transform parlance, m is the difference between the order of the denominator and that of the numerator of the system transfer function. Equivalently, m represents the delay of the discrete output response.

THE INVERSE SYSTEM

It will be seen later that the inverse system plays a fundamental role in the "output" optimal control problems. According to [3], [16], [17], $\hat{\mathcal{S}}$ is a left (right) inverse system of $\mathcal{S}$ if for any x_0 there exists $\hat{x}_0$ such that the cascade $\mathcal{S}\hat{\mathcal{S}}$ ($\hat{\mathcal{S}}\mathcal{S}$) acts as a delay of L time units, $L \geqq 0$.

The minimum L is called the inherent delay of the system and is equal to m. To get a state space representation of the inverse system, we find

$$\begin{aligned} y_k &= cA^k x_0 + cA^{k-1}bu_0 + \ldots + cbu_{k-1} + du_k = \\ &= cA^k x_0 + h_k u_0 + \ldots + h_1 u_{k-1} + h_0 u_k \end{aligned}$$

by (1), (2) and (3).

Let m be the relative order of the system. Then

$$y_{k+m} = cA^m x_k + h_m u_k\,. \tag{4}$$

On rearranging and substituting into (1) we obtain the inverse system representation

$$\hat{x}_{k+1} = \hat{A}_m \hat{x}_k + \hat{b}\hat{u}_k , \tag{5}$$

$$\hat{y}_k = \hat{c}\hat{x}_k + \hat{d}\hat{u}_k , \tag{6}$$

where

$$\begin{aligned} \hat{A}_m &= A - bh_m^{-1}cA^m , \\ \hat{b} &= bh_m^{-1} , \\ \hat{c} &= -h_m^{-1}cA^m , \\ \hat{d} &= h_m^{-1} \end{aligned} \tag{7}$$

and

$$\begin{aligned} \hat{y}_k &= u_k , \\ \hat{u}_k &= y_{k+m} . \end{aligned}$$

Observe that $\hat{\mathscr{S}}$ is again a linear system. The inverse system matrix, $\hat{A}_m$, $m \geqq 0$, has always m zero eigenvalues associated with the chain of generalized eigenvectors

$$A^{-1}b, A^{-2}b, \ldots, A^{-m}b$$

since

$$\begin{aligned} \hat{A}_m A^{-i}b = AA^{-i}b - bh_m^{-1}(cA^{m-i}b) &= A^{-(i-1)}b , \quad i = m, m-1, \ldots, 2 , \\ &= 0 , \qquad\qquad i = 1 \end{aligned}$$

by the definition of m and h_m.

The other eigenvalues of $\hat{A}_m$ coincide with the zeros of the transfer function of $\mathscr{S}$. Hence system (1)–(2) is a minimum-phase (discrete) system if and only if the inverse system (5)–(6) is stable.

We note that (5)–(6) is both left and right inverse system for $\mathscr{S}$. The problem, however, is much more complex for multi-input multi-output systems [17].

POLE ASSIGNMENT

One of the most recent techniques of the optimum system synthesis is that of pole assignment. It is based on the following theorem [3], [20]: The system (1)–(2) can be assigned any pole configuration (with complex poles occurring in complex conjugate pairs, of course) via a suitable linear state feedback $u_k = fx_k$ if, and only if, system (1)–(2) is controllable.

Given that any configuration can be achieved, the construction of the feedback gain f is as follows.

Let T be a nonsingular transformation which brings system (1) to the controllable canonical form

$$v_{k+1} = A_0 v_k + b_0 u_k ,$$

where

$$v_k = T^{-1} x_k ,$$

$$A_0 = T^{-1} A T = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ \cdots & \cdots & \cdots & \cdots \\ & & & 1 \\ -\alpha_n & \cdots & & -\alpha_1 \end{bmatrix},$$

$$b_0 = T^{-1} b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Here $\chi(z) = z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n$ is the characteristic polynomial of A.

Let the desired pole configuration corresponds to a characteristic polynomial $\psi(z) = z^n + \beta_1 z^{n-1} + \ldots + \beta_n$.

Then

$$f_0 = f T = [\alpha_n - \beta_n, \ldots, \alpha_1 - \beta_1] \tag{8}$$

since

$$A_0 + b_0 f_0 = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ \cdots & \cdots & \cdots & \cdots \\ & & & 1 \\ -\alpha_n & \cdots & & -\alpha_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [\alpha_n - \beta_n, \ldots, \alpha_1 - \beta_1] =$$

$$= \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ \cdots & \cdots & \cdots & \cdots \\ & & & 1 \\ -\beta_n & \cdots & & -\beta_1 \end{bmatrix}.$$

This method forms a useful link between the state space and the z-transform approaches and will be made use of later on.

REVIEW OF STATE TIME-OPTIMAL PROBLEM

In the z-transform approach this problem is referred to as the deadbeat response problem [4], [18]. It has been posed in the state space and solved by Kalman [6], [8] and others as follows.

It is desired to find a control which brings any initial state $x_0 \in \mathscr{E}_n$ of system (1) to zero in a minimum time N.

Let $\mathscr{X}_j$ be the set of states that can be transferred to zero in no more than j time units. Then

$$\mathscr{X}_j = \operatorname{span}\{A^{-1}b, A^{-2}b, \ldots, A^{-j}b\} \tag{9}$$

and, evidently,

$$\mathscr{X}_j \subset \mathscr{X}_{j+1}, \quad j = 0, 1, \ldots,$$
$$\mathscr{X}_0 = \{0\}.$$

Let system (1) be controllable, i.e. rank $[A^{-1}b, A^{-2}b, \ldots, A^{-n}b] = n$, and define

$$\nu = \min_j \{j : \mathscr{X}_j = \mathscr{E}_n\}.$$

Then ν is called the controllability index of the system. By definition, it is the minimum transfer time N sought for. Using (9) and the Cayley-Hamilton theorem we conclude that

$$N = \nu = n. \tag{10}$$

The optimal control u_k^* is given by

$$u_k^* = f x_k, \tag{11}$$

where

$$f[A^{-1}b, A^{-2}b, \ldots, A^{-n}b] = [-1, 0, \ldots, 0]. \tag{12}$$

In fact, for any $x_k = \xi_1 A^{-1}b + \xi_2 A^{-2}b + \ldots + \xi_j A^{-j}b \in \mathscr{X}_j$ we have

$$x_{k+1} = (A + bf)\, x_k =$$
$$= \xi_1 b + \xi_2 A^{-1}b + \ldots + \xi_j A^{-j+1}b - \xi_1 b \in \mathscr{X}_{j-1}.$$

Further, since $0 = x_n = (A + bf)^n x_0$ for any $x_0 \in \mathscr{E}_n$, the closed-loop system matrix $A + bf$ is nilpotent with index n. Hence all its eigenvalues are zero. Moreover the associated generalized-eigenvector chain is

$$A^{-1}b, A^{-2}b, \ldots, A^{-n}b$$

because

$$\begin{aligned}(A + bf)\, A^{-i}b &= A^{-(i-1)}b, \quad & i &= n, n-1, \ldots, 2,\\ &= 0, & i &= 1\end{aligned}$$

by (12).

An alternative construction of f involves the pole assignment. The desired characteristic polynomial is

$$\psi(z) = z^n$$

and, therefore,

$$f_0 = fT = [\alpha_n, \ldots, \alpha_1]$$

and

$$A_0 + b_0 f_0 = T^{-1}(A + bf)\,T = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ \cdots & \cdots & \cdots & \cdots \\ & & & 1 \\ & & & 0 \end{bmatrix}$$

by virtue of (8).

OUTPUT TIME-OPTIMAL PROBLEM

This is the classical problem solved by the z-transform approach in [4], [18]. It has been first formulated in terms of state space and partially solved by Kučera [9]. The unified and complete solution is given below.

It is desired to zero the (discrete) output in a minimum time M, starting at any $x_0 \in \mathscr{E}_n$. Note that if the underlying system is continuous the output need not be zero in between sampling points.

Let m be the relative order of system (1)–(2).

To solve the problem, we set

$$0 = y_{k+m} = cA^m x_k + h_m u_k\,,$$

see (4). Hence

$$u_k^* = -h_m^{-1} c A^m x_k \tag{13}$$

and the closed-loop system matrix is

$$A - bh_m^{-1} cA^m\,,$$

the inverse system matrix $\hat{A}_m$. It also follows that $M = m$.

However, this *optimal* system will not be stable if the inverse system is not, i.e. if (1)–(2) is not a minimum-phase system.

The matrix $\hat{A}_m$ has m zero eigenvalues associated with the chain of generalized eigenvectors $A^{-1}b, A^{-2}b, \ldots, A^{-m}b$. Further let λ_i^+, $i = 1, 2, \ldots, s$ be the remaining stable ($|\lambda_i^+| < 1$) eigenvalues of $\hat{A}_m$ and write a_i^+ for the associated eigenvectors. Similarly let λ_i^- be the unstable eigenvalues of $\hat{A}_m$ and a_i^- be the associated eigenvectors. For the sake of simplicity we shall assume that all λ_i^+ and λ_i^- be distinct and not equal to zero. This assumption is by no means essential, however.

Now the state space can be decomposed as follows:

$$\mathscr{E}_n = \mathscr{A}_m \oplus \mathscr{A}_+ \oplus \mathscr{A}_-$$

where $\mathscr{A}_m$, $\mathscr{A}_+$ and $\mathscr{A}_-$ are the eigensubspaces of $\hat{A}_m$ associated with the zero, stable, and unstable eigenvalues respectively.

Observe that

$$\dim \mathscr{A}_m = m\,,$$

$$\dim \mathscr{A}_+ = s\,,$$

$$\dim \mathscr{A}_- = n - m - s\,.$$

It is easy to see that the optimal system will be stable as well if and only if the initial state is transferred to $\mathscr{A}_m \oplus \mathscr{A}_+$ and will remain there forever.

In a like manner, write $\mathscr{Y}_j$ for the set of states that can be transferred to $\mathscr{A}_m \oplus \mathscr{A}_+$ in no more than j time units. Then

$$\mathscr{Y}_j \subset \mathscr{Y}_{j+1}\,, \quad j = 0, 1, \ldots,$$

$$\mathscr{Y}_0 = \mathscr{A}_m \oplus \mathscr{A}_+$$

and

(14) $$\mathscr{Y}_j = \operatorname{span}\{A^{-1}b, \ldots, A^{-m}b, \ldots, A^{-m-j}b, A^{-j}a_1^+, \ldots, A^{-j}a_s^+\}\,.$$

Let $\mathscr{Y}_j = \mathscr{E}_n$ for some j and set

$$\mu = \min_j \{j : \mathscr{Y}_j = \mathscr{E}_n\}\,.$$

Then the *optimal as well as stable* control is given by

(15) $$u_k^* = f x_k\,,$$

where

$$f[A^{-1}b, \ldots, A^{-m}b, \ldots, A^{-m-\mu}b, A^{-\mu}a_1^+, \ldots, A^{-\mu}a_s^+] =$$
$$= [-1, 0, \ldots, 0]\,.$$

In fact, for any $x_k = \xi_1 A^{-1}b + \ldots + \xi_m A^{-m}b + \ldots + \xi_{m+j}A^{-m-j}b + \eta_1 A^{-j}a_1^+ + \ldots + \eta_s A^{-j}a_s^+ \in \mathscr{Y}_j$ we have

$$x_{k+1} = (A + bf)\,x_k =$$
$$= \xi_1 b + \ldots + \xi_{m+j}A^{-m-j+1}b + \eta_1 A^{-j+1}a_1^+ + \ldots +$$
$$+ \eta_s A^{-j+1}a_s^+ - \xi_1 b \in \mathscr{Y}_{j-1}\,.$$

As a result, $x_\mu \in \mathscr{A}_m \oplus \mathscr{A}_+$. It remains to prove that $x_{\mu+i} \in \mathscr{A}_m \oplus \mathscr{A}_+$ for $i = 0, 1, \ldots$

First let $x_\mu = \xi_1 A^{-1} b + \ldots + \xi_m A^{-m} b \in \mathscr{A}_m$. Then

$$x_{\mu+1} = (A + bf)\, x_\mu =$$

$$= \xi_1 b + \xi_2 A^{-1} b + \ldots + \xi_m A^{-m+1} b - \xi_1 b \in \mathscr{A}_{m-1} \subset \mathscr{A}_m$$

and hence

$$x_{\mu+m} = 0 \in \mathscr{A}_m .$$

Before proceeding we recall that

$$\hat{A}_m a_i^+ = A a_i^+ - b h_m^{-1} c A^m a_i^+ = \lambda_i^+ a_i^+ , \quad i = 1, 2, \ldots, s .$$

On multiplying the above equation by A^{-j-1} we get

$$(16) \qquad A^{-j} a_i^+ = h_m^{-1}(c A^m a_i^+)\, A^{-j-1} b + \lambda_i^+ A^{-j-1} a_i^+ ,$$

$$i = 1, 2, \ldots, s; \; j = 0, 1, \ldots$$

Now let $x_\mu = \eta_1 a_1^+ + \ldots + \eta_s a_s^+ \in \mathscr{A}_+$. Then (16) yields

$$a_i^+ = h_m^{-1}(c A^m a_i^+)\, A^{-1} b + \lambda_i^+ A^{-1} a_i^+ ,$$

$$A^{-1} a_i^+ = h_m^{-1}(c A^m a_i^+)\, A^{-2} b + \lambda_i^+ A^{-2} a_i^+ ,$$

etc. so that

$$a_i^+ = h_m^{-1}(c A^m a_i^+)\, A^{-1} b + \lambda_i^+ h_m^{-1}(c A^m a_i^+)\, A^{-2} b + \ldots$$

$$\ldots + (\lambda_i^+)^{\mu-1} h_m^{-1}(c A^m a_i^+)\, A^{-\mu} b + (\lambda_i^+)^{\mu} A^{-\mu} a_i^+ , \quad i = 1, 2, \ldots, s .$$

It follows that

$$(17) \quad x_{\mu+1} = (A + bf)\, x_\mu =$$

$$= \eta_1 A a_1^+ + \ldots + \eta_s A a_s^+ - \eta_1 b h_m^{-1} c A^m a_1^+ - \ldots - \eta_s b h_m^{-1} c A^m a_s^+ =$$

$$= \hat{A}_m x_\mu \in \mathscr{A}_+ .$$

As a result, $x_\mu \in \mathscr{A}_m \oplus \mathscr{A}_+$ implies not only $x_{\mu+i} \in \mathscr{A}_m \oplus \mathscr{A}_+$ for $i = 0, 1, \ldots$ but even more:

$$x_{\mu+m+i} \in \mathscr{A}_+ , \quad i = 0, 1, \ldots$$

It also results from equation (17) that the state $x_j \in \mathscr{A}_+$ obeys the same equation as if it were controlled according to (13).

Thus

$$y_{\mu+m+i} = 0 , \quad i = 0, 1, \ldots$$

and the proof of (15) has been completed.

It follows from the above that the minimum transfer time M is given as

$$M = \mu + m .$$

To arrive at an expression for μ we recall that $\mu \geqq n - s - m$; otherwise it would not be enough vectors in (14) to span $\mathscr{Y}_\mu$. On the other hand, $\mu \leqq n - s - m$. If this were not true, we could write

$$A^{-m-\mu}b \in \operatorname{span}\{A^{-1}b, \ldots, A^{-m-\mu+1}b, A^{-\mu}a_1^+, \ldots, A^{-\mu}a_s^+\}$$

and since

$$A^{-\mu}a_i^+ \in \operatorname{span}\{A^{-\mu}b, A^{-\mu+1}a_i^+\}$$

by (16), we would get $\mathscr{Y}_{\mu-1} = \mathscr{Y}_\mu$, a contradiction. Hence

$$\mu = n - s - m .$$

Thus the minimum transfer time is

$$M = \mu + m = n - s . \tag{18}$$

To compare (10) and (18), $M \leqq N$ but at the expense of reaching equilibrium in finite time.

The optimal closed-loop system matrix $A + bf$ has eigenvalues $\lambda = 0$ and $\lambda_1^+, \ldots, \lambda_s^+$. The zero eigenvalue is associated with the generalized-eigenvector chain

$$A^{-1}b, \ldots, A^{-m}b, \ldots, A^{-m-\mu}b$$

and the eigenvalue λ_i^+ with the eigenvector a_i^+, $i = 1, 2, \ldots, s$.

In comparison to (13), it can be seen that stabilizing the system has cost μ time units; we had to remove $n - s - m = \mu$ unstable eigenvalues of $\hat{A}_m$ and introduce extra μ zero eigenvalues instead.

An alternate construction of f involves again the pole assignment. The desired characteristic polynomial is

$$\psi(z) = z^{n-s}(z - \lambda_1^+)\ldots(z - \lambda_s^+) = z^n + \beta_1 z^{n-1} + \ldots + \beta_s z^{n-s}$$

and, therefore,

$$f_0 = fT = [\alpha_n, \ldots, \alpha_{s+1}, \alpha_s - \beta_s, \ldots, \alpha_1 - \beta_1]$$

where $\chi(z) = z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n$ is the characteristic polynomial of the given system.

Example: To illustrate the theory, consider system (1)—(2) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0{\cdot}3679 & -1{\cdot}5809 & 2{\cdot}2130 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$c = [0{\cdot}0792 \quad 0{\cdot}4094 \quad 0{\cdot}1306], \quad d = [0] .$$

 It can readily be shown that

$$G(z) = \frac{0{\cdot}1306z^2 + 0{\cdot}4094z + 0{\cdot}0792}{z^3 - 2{\cdot}2130z^2 + 1{\cdot}5809z - 0{\cdot}3679} = \\ = \frac{0{\cdot}1306(z + 0{\cdot}2071)(z + 2{\cdot}9276)}{(z - 1)(z - 0{\cdot}6065)^2}$$

is the transfer function of the system. The discrete system in question can also be viewed as the continuous system

$$S(p) = \frac{1}{p(p + 0{\cdot}5)^2}$$

sampled at $t = 0, 1, 2, \ldots$

Since

$$h_0 = [0],$$
$$h_1 = [0{\cdot}1306] \neq 0,$$

the relative order $m = 1$ and the inverse system matrix reads

$$\hat{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0{\cdot}6065 & -3{\cdot}1348 \end{bmatrix}.$$

The eigenvalues of $\hat{A}_1$ are

$$\lambda = 0,$$
$$\lambda^+ = -0{\cdot}2071,$$
$$\lambda^- = -2{\cdot}9276$$

and the associated eigenvectors

$$A^{-1}b = \begin{bmatrix} 2{\cdot}7183 \\ 0 \\ 0 \end{bmatrix}, \quad a^+ = \begin{bmatrix} 4{\cdot}8286 \\ -1 \\ 0{\cdot}2071 \end{bmatrix}, \quad a^- = \begin{bmatrix} 0{\cdot}3416 \\ -1 \\ 2{\cdot}9276 \end{bmatrix}.$$

Hence

$$\mathscr{A}_1 = \operatorname{span}\{A^{-1}b\},$$
$$\mathscr{A}_+ = \operatorname{span}\{a^+\}, \qquad s = 1,$$
$$\mathscr{A}_- = \operatorname{span}\{a^-\}.$$

The minimum transfer time M is equal to $n - s = 2$.

The optimal as well as stable control law f follows from (15). We obtain

$$[A^{-1}b,\ A^{-2}b,\ A^{-1}a^+] = \begin{bmatrix} 2{\cdot}7183 & 11{\cdot}6814 & 27{\cdot}3284 \\ 0 & 2{\cdot}7183 & 4{\cdot}8286 \\ 0 & 0 & -1 \end{bmatrix}$$

and hence

$$f = [-0{\cdot}3679 \;\; 1{\cdot}5809 \;\; -2{\cdot}4201] .$$

Since the system is already exhibited in the controllable canonical form and

$$\chi(z) = z^3 - 2{\cdot}2130z^2 + 1{\cdot}5809z - 0{\cdot}3679 ,$$
$$\psi(z) = z^2(z + 0{\cdot}2071) =$$
$$= z^3 + 0{\cdot}2071z^2 ,$$

the pole assignment technique can readily be applied to yield the same result.

REVIEW OF STATE QUADRATIC-COST PROBLEM

This problem is quite involved to solve by means of z-transform [18], even for single-input single-output systems. In the state space form it has been introduced by Kalman [7], [8]; see also [2].

We are to control system (1) in such a way that the following cost functional

$$J = \tfrac{1}{2} \sum_{k=0}^{\infty} x_k' Q x_k + r u_k^2 \tag{19}$$

be minimized for any $x_0 \in \mathscr{E}_n$.

Here Q and r are $n \times n$ and 1×1 symmetric matrices respectively. It will be assumed that $Q \geqq 0$ and $r > 0$.

We invoke dynamic programming [1], [19] to obtain the optimal control

$$u_k^* = -(r + b'Pb)^{-1} \, b'PAx_k . \tag{20}$$

Here P is an $n \times n$ symmetric nonnegative definite matrix solution of the algebraic equation

$$P - A'PA + A'Pb(r + b'Pb)^{-1} \, b'PA - Q = 0 . \tag{21}$$

The minimal value J^* of J is given by

$$J^* = \tfrac{1}{2} x_0' P x_0 .$$

Before proceeding any further we define the pair (A, b) to be *stabilizable* [20] if there exists an $1 \times n$ matrix f such that $A + bf$ is stable, and, dually, the pair (c, A) is said to be *detectable* if (A', c') is stabilizable.

Another characterization of stabilizability and detectability utilizes the concept of controllable and observable eigenvalues of A [5] : (A, b) is stabilizable if and only if the unstable eigenvalues of A are controllable through b, and (c, A) is detectable if and only if the unstable eigenvalues of A are observable in c.

Unfortunately, the solution P of (21), if it exists at all, is not generally unique.

There can be negative definite, indefinite, or even nonsymmetric solutions, which are of no value to us. Kučera has recently proved the following fundamental theorem [10], [12]:

Let $br^{-1}b' = b_1 b_1'$ and $Q = C_1' C_1$, where b_1 and C_1 are matrices of full rank such that rank b_1 = rank $br^{-1}b'$ and rank C_1 = rank Q. Then stabilizability of (A, b_1) and detectability of (C_1, A) is necessary and sufficient for equation (21) to have a unique solution $P \geqq 0$ yielding a stable closed-loop system.

We find it convenient to introduce the *costate* p_k by

$$p_k = \frac{\partial J^*}{\partial x_k} = x_k' P .$$

Then we have the following two-point boundary value problem [15]

$$x_{k+1} = A x_k - br^{-1}b' p_{k+1}' , \tag{22}$$
$$p_k = x_k' Q + p_{k+1} A$$

to be solved instead of equation (21).

On rearranging,

$$\begin{bmatrix} x_{k+1} \\ p_{k+1}' \end{bmatrix} = H \begin{bmatrix} x_k \\ p_k' \end{bmatrix}$$

where H is the $2n \times 2n$ matrix below:

$$H = \begin{bmatrix} A + br^{-1}b'A'^{-1}Q , & -br^{-1}b'A'^{-1} \\ -A'^{-1}Q , & A'^{-1} \end{bmatrix}$$

Let

$$H \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} , \quad \lambda \neq 0 .$$

Then it is easy to show that

$$[-y', x'] H^{-1} = \lambda [-y', x']$$

and hence

$$[-y', x'] H = \lambda^{-1} [-y', x'] .$$

To put it in words, if H has an eigenvalue λ it has also the eigenvalue λ^{-1}.

It has been established in [14], [13] that every solution P of (21) takes the form

$$P = Y X^{-1} ,$$

where

$$X = [x_1, x_2, \ldots, x_n] ,$$
$$Y = [y_1, y_2, \ldots, y_n]$$

and $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$, $i = 1, 2, \ldots, n$ are the eigenvectors or generalized eigenvectors of H associated with such an n-tuple of eigenvalues λ_i that X^{-1} exists. Moreover, with any generalized eigenvector all lower-ranking ones must also be used.

It is important to note that $A - b(r + b'Pb)^{-1} b'PA$, the closed-loop system matrix yielded by P, has λ_i, $i = 1, 2, \ldots, n$ as its eigenvalues and x_i as the associated eigenvectors. It implies that P generates a stable closed-loop system if, and only if, it corresponds to the choice of stable eigenvalues λ_i of H, $i = 1, 2, \ldots, n$.

In case (C_1, A) is not detectable, equation (21) will have at least two nonnegative solutions and the *optimal* solution will no more coincide with the *stable* one [11].

OUTPUT QUADRATIC-COST PROBLEM

This problem has been completely solved by the z-transform approach in [4], [18]. From the state-space point of view it can be regarded as a special case of the previous problem provided $r \geqq 0$ is allowed. Nevertheless, we are providing a deeper insight, which, to the author's knowledge, originates here.

It is desired to control system (1)–(2) so as to minimize the cost

$$J = \tfrac{1}{2} \sum_{k=0}^{\infty} y_k^2 \tag{23}$$

for any $x_0 \in \mathscr{E}_n$.

Unlike (19), functional (23) involves no control ($r = 0$) and hence the above results are not directly applicable.

Let $m \geqq 0$ be the relative order (discrete output delay) of system (1)–(2). Then (23) is minimized if and only if the cost

$$J_m = \tfrac{1}{2} \sum_{k=0}^{\infty} y_{k+m}^2 \tag{24}$$

is minimized. Making use of (4),

$$J_m = \tfrac{1}{2} \sum_{k=0}^{\infty} x_k' Q x_k + 2x_k' s u_k + r u_k^2 \,, \tag{25}$$

where

$$\begin{aligned} Q &= A'^m c' c A^m \,, \\ s &= A'^m c' h_m \,, \\ r &= h_m^2 > 0 \,. \end{aligned} \tag{26}$$

The substitution

$$\tilde{u}_k = u_k + r^{-1} s' x_k \tag{27}$$

eliminates the cross-term in (25). The state term in (25) also vanishes due to (26),

$$\tilde{Q} = Q - sr^{-1}s' = 0,$$

and (25) simplifies to

$$J_m = \tfrac{1}{2}\sum_{k=0}^{\infty} h_m^2 u_k^2 .$$

Equation (1) is thus modified to

$$x_{k+1} = \tilde{A}x_k + b\tilde{u}_k$$

where

$$\tilde{A} = A - br^{-1}s' \equiv \hat{A}_m ,$$

the inverse system matrix.

This is exactly the state quadratic-cost problem with

$$Q = 0, \quad r = h_m^2, \quad A = \hat{A}_m, \tag{28}$$

that is, equation (21) reads

$$P - \hat{A}_m' P \hat{A}_m + \hat{A}_m' P b(h_m^2 + b'Pb)^{-1} b'P\hat{A}_m = 0 . \tag{29}$$

The only difficulty involved is that $\hat{A}_m^{-1}$, $m > 0$, does not exist. The nature of the problem, however, implies that a solution does exist. To circumvent the difficulty, write

$$\mathscr{E}_n = (\mathscr{A}_+ \oplus \mathscr{A}_-) \oplus (\mathscr{A}_+ \oplus \mathscr{A}_-)^{\perp}$$

and define the *generalized inverse* $\hat{A}_m^{\#}$ of $\hat{A}_m$ as follows:

$$\hat{A}_m \hat{A}_m^{\#} x = x, \quad x \in \mathscr{A}_+ \oplus \mathscr{A}_-$$
$$\hat{A}_m^{\#} y = 0, \quad y \in (\mathscr{A}_+ \oplus \mathscr{A}_-)^{\perp} .$$

Roughly speaking, $\hat{A}_m^{\#}$ acts as $\hat{A}_m^{-1}$ on $\mathscr{A}_+ \oplus \mathscr{A}_-$ and as zero on $(\mathscr{A}_+ \oplus \mathscr{A}_-)^{\perp}$. It follows that $(\hat{A}_m\hat{A}_m^{\#})' = \hat{A}_m\hat{A}_m^{\#}$.

Now we are able to solve equation (22) for p_{k+1}:

$$p_{k+1} = x_k' Q \hat{A}_m^{\#} + p_k \hat{A}_m^{\#} .$$

In view of (28) the matrix H becomes

$$H = \begin{bmatrix} \hat{A}_m, & -bh_m^{-2}b'\hat{A}_m^{\#\prime} \\ 0, & \hat{A}_m^{\#\prime} \end{bmatrix} .$$

The eigenvalues of H are those of $\hat{A}_m$ and $\hat{A}_m^{\#\prime}$. Let for $\lambda_i \neq 0$

$$\hat{A}_m a_i = \lambda_i a_i ,$$
$$r_i \hat{A}_m = \lambda_i r_i .$$

Then

$$\hat{A}_m^{\#} a_i = \lambda_i^{-1} a_i$$
$$r_i \hat{A}_m^{\#} = \lambda_i^{-1} r_i$$

by definition of $\hat{A}_m^{\#}$.

It follows by direct verification for $\lambda_i \neq 0$ that

$$H\begin{bmatrix} a_i \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} a_i \\ 0 \end{bmatrix},$$
$$H\begin{bmatrix} R_i' r_i' \\ r_i' \end{bmatrix} = \lambda_i^{-1} \begin{bmatrix} R_i' r_i' \\ r_i' \end{bmatrix}$$

where $R_i' = (\lambda_i \hat{A}_m - I)^{-1}\, b h_m^{-2} b'$.

In the case of zero eigenvalues,

$$H\begin{bmatrix} A^{-i} b \\ 0 \end{bmatrix} = \begin{bmatrix} A^{-(i-1)} b \\ 0 \end{bmatrix}, \quad i = m, m-1, \ldots, 2 ,$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad i = 1$$

and also

$$H\begin{bmatrix} 0 \\ z_i' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad i = 1, 2, \ldots, m ,$$

where $z_i \hat{A}_m^{\#} = 0$. Note that the eigenvectors $\begin{bmatrix} 0 \\ z_i' \end{bmatrix}$ can form no solution of (29) since X^{-1} would not exist.

There is always the solution $P = 0$ to equation (29). The corresponding control $\tilde{u}_k^* = 0$ is certainly *optimal* since $J_m^* = 0$. By virtue of (27) and (26) we obtain

$$u_k^* = -h_m^{-1} c A^m$$

while the closed-loop system matrix becomes

$$A - b h_m^{-1} c A^m = \hat{A}_m .$$

As a result the optimal control system is stable if and only if the inverse system is so, or equivalently, if the given system enjoys the minimum-phase property. Moreover, the optimal strategy for the output quadratic-cost problem coincides with that of the output time-optimal problem.

System (1)–(2) is assumed to be stabilizable. It follows that the pair $(\hat{A}_m, b_1)$ is stabilizable, too. Since $Q = 0$, however, the pair $(C_1, \hat{A}_m)$ is never observable. In case $\hat{A}_m$ is stable, $(C_1, \hat{A}_m)$ is detectable and $P = 0$ is the only nonnegative solution of (29). If $\hat{A}_m$ is not stable, i.e. system (1)–(2) is not of minimum phase, the pair $(C_1, \hat{A}_m)$ is not detectable. Hence there exist at least two nonnegative solutions to (29): $P = 0$, the *optimal but not stable* one, and another solution which is *optimal as well as stable.*

To extract the latter solution we have to choose the stable eigenvalues of H, that is to set

$$P = YX^{-1} \tag{30}$$

where

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} A^{-1}b, \ldots, A^{-m}b, & a_1^+, \ldots, a_s^+, & R_1' r_1^{-\prime}, \ldots, R_{n-m-s}' \, r_{n-m-s}^{-\prime} \\ 0, \quad \ldots, 0, & 0, \ \ldots, 0, & r_1^{-\prime}, \ldots, \qquad r_{n-m-s}^{-\prime} \end{bmatrix}.$$

This is the state space equivalent to spectral factorization.

Solution (30) yields, by equation (20), the control

$$\tilde{u}_k^* = -(h_m^2 + b'Pb)^{-1}\, b'P\hat{A}_m x_k\,,$$

i.e.,

$$u_k^* = f x_k$$

where

$$f = -(h_m^2 + b'Pb)^{-1}\, b'P\hat{A}_m - h_m^{-1} c A^m\,. \tag{31}$$

The first term in (31) has the desired stabilizing effect.

The optimal as well as stable closed-loop system therefore obeys the equation

$$x_{k+1} = [\hat{A}_m - b(h_m^2 + b'Pb)^{-1}\, b'P\hat{A}_m]\, x_k$$

and has eigenvalues $\lambda = 0$, λ_i^+, $i = 1, 2, \ldots, s$ and $(\lambda_i^-)^{-1}$, $i = 1, 2, \ldots, n - m - s$. The chain $A^{-1}b, \ldots, A^{-m}b$ is associated with $\lambda = 0$, the eigenvectors a_i^+ with λ_i^+ and $R_i' r_i^{-\prime}$ with $(\lambda_i^-)^{-1}$.

It is of interest to note that the eigenvalues and eigenvectors of $\hat{A}_m$ only, not of H, are required. Also the generalized inverse $\hat{A}_m^{\#}$ does not enter any computations and was introduced for formal reasons only.

The cost J_m^* of the optimal control takes the value

$$J_m^* = \tfrac{1}{2} x_0' P x_0$$

consistently with the previous section.

Another possible synthesis technique is that of pole placement. The desired characteristic polynomial is

$$\psi(z) = z^m(z - \lambda_1^+) \dots (z - \lambda_s^+)(z - (\lambda_1^-)^{-1}) \dots (z - (\lambda_{n-m-s}^-)^{-1}) =$$
$$= z^n + \beta_1 z^{n-1} + \dots + \beta_{n-m} z^{n-m} .$$

Hence the optimal as well as stable control gain satisfies

$$f_0 = fT = [\alpha_n, \dots, \alpha_{n-m+1}, \alpha_{n-m} - \beta_{n-m}, \dots, \alpha_1 - \beta_1]$$

where $\chi(z) = z^n + \alpha_1 z^{n-1} + \dots + \alpha_n$ is the characteristic polynomial of the given system.

Example. As an illustrative example consider again the system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0{\cdot}3679 & -1{\cdot}5809 & 2{\cdot}2130 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$c = [0{\cdot}0792 \quad 0{\cdot}4094 \quad 0{\cdot}1306], \quad d = [0] .$$

To compute f we extra need the vectors

$$r^{-\prime} = \begin{bmatrix} 0 \\ 0{\cdot}2071 \\ 1 \end{bmatrix}, \quad R' r^{-\prime} = \begin{bmatrix} 168{\cdot}6425 \\ -57{\cdot}6045 \\ 19{\cdot}6764 \end{bmatrix}.$$

In the light of the previous computations we can write

$$X = \begin{bmatrix} 2{\cdot}7183 & 4{\cdot}8286 & 168{\cdot}6425 \\ 0 & -1 & -57{\cdot}6045 \\ 0 & 0{\cdot}2071 & 19{\cdot}6764 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0{\cdot}2071 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$P = YX^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0{\cdot}0055 & 0{\cdot}0267 \\ 0 & 0{\cdot}0267 & 0{\cdot}1290 \end{bmatrix}.$$

Therefore (31) results in

$$f = [-0{\cdot}3679 \quad 1{\cdot}5101 \quad -2{\cdot}7617] .$$

The same formula is obtained by invoking the pole assignment technique for

$$\psi(z) = z(z + 0{\cdot}2071)(z + 0{\cdot}3416) = z^3 + 0{\cdot}5487z^2 + 0{\cdot}0708z .$$

CONCLUSIONS

In this paper a unified state space approach to the synthesis of the time-optimal and quadratic-cost controls has been established. In either case the "state" problem has been reviewed and the "output" problem posed and completely solved. Illustrative examples have been included to help the reader.

The highlight of the paper is the state space theory of the "output" problems. This theory is limited to single-input single-output systems. This is mostly due to the inverse system problem, which becomes very involved for multivariable systems and has not been completely solved yet.

It should be emphasized that the entire state of the system is needed to generate the optimal control, even in the "output" problems.

Throughout the paper the following important idea accompanies the theory: optimality does not necessarily imply stability. As a matter of fact, certain measures had to be taken to ensure stability.

Last but not least, the pole assignment method is briefly discussed for each problem.

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VÝTAH

Syntéza diskrétního lineárního řízení metodou stavového prostoru

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V článku je formulován diskrétní časově optimální problém (konečný počet kroků regulace) i problém syntézy diskrétního řízení podle kvadratického kritéria ve stavovém prostoru. Jsou shrnuty známé poznatky a poprvé jsou ve stavovém prostoru systematicky zkoumány úlohy zahrnující výstup soustavy. Je dáno jednotné a kompletní řešení pro jednoparametrové obvody. Metoda přiřazení pólů umožňuje jednak účinnou syntézu optimálního obvodu, a jednak srovnání s klasickým řešením užívajícím z-transformace.

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