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On Generalized Linear Discrete Inversion Filters

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In this article, the methods and results of [1] are generalized to cover the case of disturbing signals. The connection of the inversion filter for this case and of the matched filter is shown.

1. INTRODUCTION

Let the finite sequence

$$\{b_j\} = b_0, b_1, \ldots, b_h \qquad (1)$$

$$(h \geqq 1,\ b_j \text{ real},\ b_0 \neq 0,\ b_h \neq 0)$$

be interpreted as the unit impulse response of a linear discrete filter $\mathscr{B}$ – the "coding" or "distorting" filter.

Let $N \geqq h$ be a natural number. Let the real sequence

$$\{a_j\} = a_0, a_1, \ldots, a_N \qquad (2)$$

be the unit impulse response of a linear discrete filter $\mathscr{A}$. For the filters $\mathscr{B}$, $\mathscr{A}$ in cascade, the output sequence

$$\{c_j\} = c_0, c_1, \ldots, c_{N+h} \qquad (3)$$

satisfies the relations

$$\begin{aligned} c_0 &= b_0 a_0\,, \\ c_1 &= b_1 a_0 + b_0 a_1\,, \\ &\ \vdots \\ c_T &= b_T a_0 + b_{T-1} a_1 + \ldots + b_0 a_T\,, \\ &\ \vdots \\ c_{N+h} &= \qquad\qquad\qquad\qquad b_h a_N \end{aligned} \qquad (4)$$

with $b_j = 0$ for $j > h$. In what follows we will always suppose $0 \leqq T \leqq N + h$.

The filter $\mathscr{A}$ will be called the inversion filter to the given $\mathscr{B}$ if for the $\{c_j\}$ given by (4) the condition

$$(5) \qquad c_0^2 + c_1^2 + \ldots + (1 - c_T)^2 + \ldots + c_{N+h}^2 = \min$$

is satisfied.

Denoting

$$(6) \qquad A(z) = a_0 + a_1 z^{-1} + \ldots + a_N z^{-N},$$

$$(7) \qquad B(z) = b_0 + b_1 z^{-1} + \ldots + b_h z^{-h},$$

the condition (5) can be expressed as

$$(8) \qquad \frac{1}{2\pi \mathrm{i}} \int_{C_1} |z^{-T} - A(z)\, B(z)|^2 \frac{\mathrm{d}z}{z} = \min .$$

C_1 is the unit circle.

Introducing the matrix notation, (4) is

$$(9) \qquad \boldsymbol{c} = \boldsymbol{B}\boldsymbol{a}$$

and

$$(10) \qquad \sum_{i=0}^{N+h} c_j^2 = \boldsymbol{c}'\boldsymbol{I}\boldsymbol{c} = \boldsymbol{a}'\boldsymbol{B}'\boldsymbol{B}\boldsymbol{a} = \boldsymbol{a}'\boldsymbol{M}\boldsymbol{a}$$

where $\boldsymbol{I}$ is the unit matrix and

$$(11) \qquad \boldsymbol{M} = \boldsymbol{B}'\boldsymbol{B} = \begin{pmatrix} \mu_0, & \mu_{-1}, & \ldots, & \mu_{-h-N} \\ \mu_1, & \mu_0, & \ldots, & \mu_{-h+1-N} \\ & & \vdots & \\ \mu_{N+h}, & & \ldots, & \mu_0 \end{pmatrix}$$

where

$$(12) \qquad \mu_{-j} = \sum_{k=0}^{h-j} b_{k+j} b_k = \sum_{k=0}^{h-j} b_k b_{k+j} = \mu_j$$

are the "autocorrelations" of $\{b_j\}$. For $j > h$, there is $\mu_j = 0$.

2. INVERSION IN PRESENCE OF DISTURBING SIGNALS

Let a finite set of signals $\{b_{lj}\}$ exist ($l = 0, 1, \ldots, m$), the signals having the same "length" $h + 1$. The sequence $\{b_{0j}\}$ will be considered as wanted signal, all other as disturbing ones. The signals can occur with the weights $w_0 > 0$, $w_1 \geqq 0, \ldots$

 $\ldots, w_m \geqq 0$. With the eventual normalization

$$\sum_{j=0}^{m} w_j = 1 \tag{13}$$

the weights may be interpreted as probabilities of occurrence.

Now, we define the generalized inversion filter as such satisfying

$$\frac{1}{2\pi i}\left\{w_0 \int_{C_1} |z^{-T} - A(z)\, B_0(z)|^2 \frac{dz}{z} + \sum_{j=1}^{m} w_j \int_{C_1} |A(z)\, B_j(z)|^2 \frac{dz}{z} = \min, \right. \tag{14}$$

where $B_j(z)$ is the Z-transform of $\{b_{jn}\}$.

Comparing (14) with (8), (5) and (10), it is clear that (14) can be expressed as

$$\begin{aligned} \Phi(a_0, \ldots, a_N) &= w_0(1 - 2c_{0T}) + w_0 \boldsymbol{a}' \boldsymbol{M}_0 \boldsymbol{a} + \\ &+ \ldots + w_m \boldsymbol{a}' \boldsymbol{M}_m \boldsymbol{a} = w_0(1 - 2c_{0T}) + \boldsymbol{a}'(w_0 \boldsymbol{M}_0 + \ldots + w_m \boldsymbol{M}_m)\, \boldsymbol{a} = \min . \end{aligned} \tag{15}$$

The meanings of c_{0T} and of the matrices $\boldsymbol{M}_j$ are clear.

Put

$$w_0 \boldsymbol{M}_0 + \ldots + w_m \boldsymbol{M}_m = w_0 \boldsymbol{M} . \tag{16}$$

Then from (15), (16) one gets the relation

$$\begin{aligned} &1 - 2c_{0T} + a_0(\mu_0 a_0 + \mu_{-1} a_1 + \ldots + \mu_{-h} a_h) + \\ &+ a_1(\mu_1 a_0 + \mu_0 a_1 + \ldots + \mu_{-h} a_{h+1}) + \ldots + a_N(\mu_h a_{N-h} + \ldots + \mu_0 a_N) = \min . \end{aligned} \tag{17}$$

In (17), μ_j are elements of the matrix $\boldsymbol{M}$ of (16).

The necessary and sufficient conditions of the minimum of (17) are given by the following system of linear equations:

$$\begin{array}{lllll} \mu_0 a_0 & + \mu_{-1} a_1 & + \ldots + \mu_{-h} a_h & = b_{0,T} , \\ \mu_1 a_0 & + \mu_0 a_1 & + \ldots + \mu_{-h} a_{h+1} & = b_{0,T-1} , \\ & & \vdots & \\ \mu_T a_0 & + \mu_{T-1} a_1 & + \ldots + \mu_{-h} a_{h+T} & = b_{00} , \\ \mu_{T+1} a_0 & + \mu_T a_1 & + \ldots + \mu_{-h} a_{h+T+1} & = 0 , \\ & & \vdots & \\ \mu_N a_0 & + & + \ldots + \mu_{-h} a_{h+N} & = 0 , \end{array} \tag{18}$$

where $\mu_j = 0$ for $j > h$, $b_{0j} = 0$ for $j > h$, $a_j = 0$ for $j > N$.

This system is formally the same as (15) in [1] and may be solved directly or, as it has been shown in [2], for N substantially greater than h, with advantage as a difference equation with boundary conditions.

Substituting from (18) in (17), one gets from (15)

$$\Phi(a_0, \ldots, a_N)_{\min} = w_0(1 - c_{0T}) . \tag{19}$$

From the right side, the value of the minimum can be easily computed.

3. INVERSION AND MATCHED FILTERS

Let us suppose specially $N = T = h$. Further, let the signals $\{b_{lj}\}$, $j = 0, 1, \ldots, h$, $l = 0, 1, \ldots, 2^{h+1} - 1$ be all 2^{h+1} sequences of 1's and -1's of the "length" $h + 1$, possesing equal probabilities of occurrence.

An arbitrarily selected sequence is considered as the wanted signal, all remaining sequences represent disturbance.

With these suppositions, it is easy to show that in the matrix $\boldsymbol{M}$ all μ_j with exception of μ_0 are zero and the solution of (18) is formally the conventional matched filter for the case of white noise.

This result may be generalized. For general discrete signals with weights considered as probabilities of occurrence, the matrix in (18) is a true correlation matrix representing a colored noise.

Denoting this matrix by $\boldsymbol{M}_0$ and the vector on the right side of (18) by $\boldsymbol{b}_0$, (18) gives the known result

$$\boldsymbol{a} = \boldsymbol{M}_0^{-1}\boldsymbol{b}_0 \tag{20}$$

and with c_{0T} from (4) the known result

$$c_{0T} = \boldsymbol{b}_0'\boldsymbol{M}_0^{-1}\boldsymbol{b}_0 . \tag{21}$$

From (21) and (19) there follows that the minimum value of the minimum Φ is obtained precisely if the signal $\boldsymbol{b}_0$ is an eigenvector belonging to the greatest eigenvalue of the matrix $\boldsymbol{M}_0^{-1}$, a result which may be found in [3].

The only difference is that in (20), (21) the wanted signal occurs in $\boldsymbol{M}_0$, whereas in the formulas derived from the signal/noise concept $\boldsymbol{M}_0$ is represented only by the disturbances.

Thus the usual matched filter may be considered as the "limiting case" of the filter from this section. For small relative weight of the wanted signal there is practically no difference.

Now, if the supposition $N = T = h$ is not valid, (18) still has solution which may be considered as a generalization of the matched filter concept.

4. INVERSION BY RECURSIVE FILTER

Suppose now that $A(z)$ in (14) is an infinite series representing a physically realizable stable filter. To find it, we will use the method of decomposition as described for the case of Kolmogorov-Wiener filters e.g. in [5].

Rewrite (14) as

$$\Phi[A(z)] = \frac{1}{2\pi i}\left\{w_0 \int_{C_1} \left|\frac{z^{-T}}{B_0(z)} - A(z)\right|^2 . |B_0(z)|^2 \frac{dz}{z} + \right. \tag{22}$$

$$\left. + \sum_{j=1}^{m} w_j \int_{C_1} |A(z)|^2 . |B_j(z)|^2 \frac{dz}{z}\right\} = \min .$$

The necessary conditions for (22) to be minimum are easily proved to be

$$\frac{1}{2\pi i}\int_{C_1} z^k A(z) \sum_{j=0}^{m} w_j B_j(z) B_j(z^{-1}) \frac{dz}{z} = \frac{1}{2\pi i}\int_{C_1} z^{T-k} w_0 B_0(z) \frac{dz}{z} . \tag{23}$$

The function

$$f(z) = \sum_{j=0}^{m} w_j B_j(z) B_j(z^{-1}) \tag{24}$$

for $z = \exp i\lambda$ represents the spectral density of the input signals and has the maximum degree term of degree $s \leqq h$.

Further,

$$\frac{1}{2\pi i}\int_{C_1} z^{T-k} B_0(z) \frac{dz}{z} = b_{0,T-k} , \tag{25}$$

so that (23) may be written with (24), (25) as

$$\frac{1}{2\pi i}\int_{C_1} z^k A(z) f(z) \frac{dz}{z} = b_{0,T-k} \tag{26}$$

($k = 0, 1, \ldots$ and $b_{0l} = 0$ for $l < 0$ and for $l > h$).

Now, decompose

$$f(z) = f_0 \frac{(z - z_1)\ldots(z - z_s)(1 - z_1 z)\ldots(1 - z_s z)}{z^s} \tag{27}$$

where it is supposed

$$|z_j| > 1 , \quad (j = 1, \ldots, s) \tag{28}$$

since we know that if $|z_j| = 1$ for some j, no stable recursive filter would result.

To satisfy the conditions (26), we suppose

$$A(z) = \frac{z^{-n} P(z)}{(1 - z_1 z)\ldots(1 - z_s z)} , \tag{29}$$

where n and the polynomial $P(z)$ are to be determined.

With (29) and with

$$f_0 \cdot (z - z_1) \dots (z - z_s) = q_0 z^s + \dots + q_s \tag{30}$$

one gets from (26)

$$\frac{1}{2\pi i} \int_{C_1} z^{k-n-s} P(z) (q_0 z^s + \dots + q_s) \frac{dz}{z} = b_{0,T-k} . \tag{31}$$

But $b_{0,-1} = b_{0,-2} = \dots = 0$, thus with $k = T + 1$ there must be $T + 1 - n - s = 1$, thus

$$n = T - s . \tag{32}$$

Further, for each b_{0j} $(j = 0, \dots, T)$ a coefficient in $P(z)$ is needed. Thus

$$P(z) = p_0 z^T + \dots + p_T . \tag{33}$$

Substituting from (33) in (31) for $k = T, T - 1, \dots, 0$, one gets the system of linear equations for p_j

$$\begin{aligned} q_s p_T \qquad\qquad\qquad\qquad\qquad &= b_{00} , \\ q_{s-1} p_T + q_s p_{T-1} \qquad\qquad\qquad &= b_{01} , \\ &\vdots \\ q_{s-T} p_T + q_{s-T+1} p_{T-1} + \dots + q_s p_0 &= b_{0T} , \end{aligned} \tag{34}$$

where $q_j = 0$ for $j < 0$. This system has precisely one solution. Thus the filter $\mathscr{A}$ with transfer function (29) is a uniquely determined stable recursive filter. That this solution gives the minimum of (22) is evident from the geometrical meaning of (5).

It remains to show that for the filter $\mathscr{A}$ the formula (19) still holds. To this end let us express (14) with the aid of Parseval's formula. One gets after easy calculation

$$\begin{aligned} \Phi[A(z)] &= \sum_{j=0}^{m} w_j(a_0 b_{0j})^2 + \sum_{j=0}^{m} w_j(a_0 b_{j1} + a_1 b_{j0})^2 + \\ &+ \dots + \sum_{j=0}^{m} w_j(a_0 b_{jh} + \dots + a_h b_{j0})^2 + w_0[1 - 2(a_{T-h} b_{0h} + \dots + a_T b_{00})] . \end{aligned} \tag{35}$$

Further, in the case of $\Phi_{\min}$ there is

$$\frac{a_0}{2} \frac{\partial \Phi}{\partial a_0} + \frac{a_1}{2} \frac{\partial \Phi}{\partial a_1} + \dots = 0 \tag{36}$$

and from this there follows

$$\begin{aligned} \sum w_j(a_0 b_{j0})^2 + \sum w_j(a_0 b_{j1} + a_1 b_{j0})^2 + \dots = \\ = w_0(a_{T-h} b_{0h} + \dots + a_T b_{00}) . \end{aligned} \tag{37}$$

From (35) and (37), (19) follows at once.

Example 1. Consider the Barker code $b_{00} = 1$, $b_{01} = -1$ as the wanted signal and the code $b_{10} = 1$, $b_{11} = 1$ as the disturbation, both signals with equal weights 1.

According to (16) one gets for $N = 1$

$$\boldsymbol{M} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \tag{38}$$

and from (18) with $T = 1$ one has

$$a_0 = -1/4, \; a_1 = 1/4, \; a_2 = a_3 = \ldots = 0 \, .$$

Thus the conventional matched filter is obtained.

Example 2. Consider the Barker code $b_{00} = 1$, $b_{01} = 1$, $b_{02} = -1$ as the wanted signal and the code $b_{10} = 1$, $b_{11} = 1$, $b_{12} = 1$ as the disturbation, both signals with equal weights 1.

According to (16) one gets for $N = 2$

$$\boldsymbol{M} = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix} \tag{39}$$

and from (18) with $T = 2$ one has

$$a_0 = -10/42 \doteq -0{,}238 \, , \quad a_1 = 9/42 \doteq 0{,}214 \, , \quad a_2 = 4/42 \doteq 0{,}095 \, .$$

Example 3. Consider the same signals and weights as in example 2.

For $N = 3$ one gets

$$\boldsymbol{M} = \begin{pmatrix} 6 & 2 & 0 & 0 \\ 2 & 6 & 2 & 0 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & 2 & 6 \end{pmatrix} \tag{40}$$

and from (18) with $T = 2$ one has

$$a_0 = -26/110 \doteq -0{,}236 \, , \quad a_1 = 23/110 \doteq 0{,}209 \, , \quad a_2 = 12/110 \doteq 0{,}109 \, ,$$

$$a_3 = -4/110 \doteq -0{,}036 \, .$$

Example 4. Consider the same signal and weights as in preceding two examples. Let $T = 2$ and let $N \to \infty$. We seek the "limiting" recursive filter.

From (24) and (27) one gets

$$f(z) \doteq 0{\cdot}764 \frac{(z + 2{\cdot}618)(1 + 2{\cdot}618z)}{z}. \tag{41}$$

From (29) and (32) one gets

$$A(z) = \frac{z^{-1} P(z)}{1 + 2{\cdot}618z}. \tag{42}$$

Now, from (30)

$$q_0 z + q_1 = 0{\cdot}764z + 2 \tag{43}$$

and since $T = h = 2$, one gets from the system (34)

$$p_2 = 0{\cdot}5\,, \quad p_1 = 0{\cdot}309\,, \quad p_0 = -0{\cdot}618\,,$$

so that finally

$$A(z) = \frac{z^{-1}(-0{\cdot}618z^2 + 0{\cdot}309z + 0{\cdot}5)}{1 + 2{\cdot}618z} \tag{44}$$

and from this

$$a_0 \doteq -0{\cdot}236\,, \quad a_1 \doteq 0{\cdot}208\,, \quad a_2 \doteq 0{\cdot}111\,, \quad a_3 \doteq -0{\cdot}042\,,$$

which compared with the values from the example 3 shows that the nonrecursive filter of example 3 is practically as good as the "limiting" filter from (44).

6. CONCLUDING REMARKS

The method described in this article makes it possible to analyze the influence of known disturbing signals on the least squares inversion filter and to synthetize the optimum one.

For unknown signals, their statistical properties must be measured experimentally and if they vary with time, then some sort of adaptivity in the computations of the filter must be provided, as has been shown for the matched filters and equalizers in the cited references.

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[1] Prouza, L.: On the Inversion of Moving Averages, Linear Discrete Equalizers and "Whitening" Filters, and Series Summability. Kybernetika *6* (1970), 3, 225—240.

[2] Di Toro, M. J.: Communication in Time-Frequency Spread Media Using Adaptive Equalization. Proc. IEEE (1968), 1653—1679.

[3] Kushner, H. J.: Adaptive Techniques for the Optimization of Binary Detection Systems. Int. Conv. Rec. IEEE (1963), 4, 107—117.

[4] Steiglitz, K., Thomas, J. B.: A Class of Adaptive Matched Digital Filters. Proc. Nat. Electron. Conf. (1964), 641—646.

[5] Яглом, А. М.: Введение в теорию стационарных случайных функций, Успехи матем. наук (1952). 5. 3—168,

VÝTAH

O zobecněných lineárních diskrétních inverzních filtrech

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V článku se zobecňují metody a výsledky práce [1] seznamu literatury na případ, kdy vedle žádoucího signálu se vyskytují ještě signály poruchové. Ukazuje se souvislost inverzního a tzv. přizpůsobeného filtru v tomto případě.

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