KYBERNETIKA — VOLUME 8 (1972), NUMBER 1

Zero Points of Impulse Characteristic

ZDENĚK VOSTRÝ

This paper contains the proof of the theorem about the number of zero points lying outside of the unit circle and besides it deals with the application for the estimation of the lenght of impulse characteristic and for the appreciation of the dependence of control circuit stability on the identification fidelity.

Synthetizing the control circuit containing digital computer we mostly start from discrete transfer functions in the form of rational fractional function. It is known, that the criterion value of optimal control process based on the minimum mean value of squared error depends only on the transportation delay and on zero points of the numerator of the transfer function lying inside of the unit circle.

In the following are given the statement and proof of a theorem about the number of zero points of impulse characteristic, lying inside of the unit circle, viz. those which are responsible for the control performace.

Be given the transfer function of a stable plant in the form

$$S = \frac{z^{-(T+1)}M}{N}, \tag{1}$$

where

$$M = m_0 + m_1 z^{-1} + \ldots + m_K z^{-K},$$

$$N = 1 + n_1 z^{-1} + \ldots + n_L z^{-L},$$

T is transformation delay.

Obviously, the zero points of the polynomial N are all situated inside of unit circle and at most one simple root $z = 1$.

The roots of the polynomial M can be arbitrary, but none of them may be $z = +1$. Zero point $z = +1$ would mean that for maintaining the output constant the controlling variable have to grow unceasingly. Other zero points on the unit circle indicate that the plant is capable to absorb certain periodical signal.

In the following text Γ denotes the circle $z = 1$. The impulse characteristic of the (1) is obviously of the form

(2) $$H' = z^{-(T+1)}(h_0 + h_1 z^{-1} + h_2 z^{-2} + \dots)\,.$$

We compute the ordinates $h, h_1, h_2, \dots$ of the impulse characteristic by developing (1) into infinite series in the powers of z^{-1}.

The identification of the plant S yields the impulse characteristic in form of a finite polynomial.

Suppose the identification is so exact that holds

(3) $$H = z^{-(T+1)}(h_0 + h_1 z^{-1} + h_2 z^{-2} + \dots + h_Q z^{-Q})\,.$$

Theorem 1. *Be given the plant* (1) *the zeros and poles of which does not lie on Γ, and its impulse characteristic* (3). *Then there exists a number Q so that* (3) *has as many zero points outside of Γ as many of them has the polynomial M outside of Γ.*

In order to prove this theorem we mention a theorem, known from the theory of functions of complex variable.

Theorem 2 (Rouche). *When $f(z)$ and $g(z)$ are regular in the bounded region $\bar{\mathcal{D}}$ and when there holds on the boundary of this region $|f(z)| > |g(z)| > 0$, then both function $f(z)$ and $f(z) + g(z)$ have the same number of zero points included the multiplicity.*

Proof 1. It is known that we get H from S by division. Write this fact in the form

(4) $$\frac{M}{N} = \sum_{i=0}^{Q} h_i z^{-i} + \sum_{i=Q+1}^{\infty} h_i z^{-i} = \tilde{H} + \frac{\Omega}{N}$$

where $\tilde{H} = z^{T+1}H$; so we exclude the transportation delay as it is of no influence on the course of the proof. Ω is the remainder of the division and has the form

(5) $$\Omega = z^{-Q}(\omega_1 z^{-1} + \omega_2 z^{-2} + \dots + \omega_L z^{-L})\,.$$

When Q increases, the ω_i obviously converge to zero. Effect now at least rough estimation of coefficients ω_i. Multiply (4) by the polynomial N and calculate Ω

(6) $$\Omega = N \sum_{i=Q+1}^{\infty} h_i z^{-i}\,.$$

Written in full (6) yields

$$z^{-Q}(\omega_1 z^{-1} + \omega_2 z^{-2} + \dots + \omega_L z^{-L}) = (1 + n_1 z^{-1} + \dots + n_L z^{-L}) \sum_{i=Q+1}^{\infty} h_i z^{-i}\,.$$

From here it follows by comparing the coefficients at particular powers

$$\begin{aligned} \omega_1 &= h_{Q+1}\,, \\ \omega_2 &= h_{Q+2} + h_{Q+1} n_1\,, \\ &\vdots \\ \omega_L &= h_{Q+L} + h_{Q+L-1} n_1 + \ldots + h_{Q+1} n_{L-1}\,. \end{aligned} \tag{7}$$

In order to make full use of theorem (5) we choose for the region $\mathscr{D}$ the unit circle $|z| < 1$.

Estimate the maximum absolute value of function Ω for $|z| = 1$ by calculation

$$\max_{|z|=1} |\Omega(z)| \leqq |g(z)| \leqq \sum_{i=1}^{L} |\omega_i|\,.$$

We know that the coefficients of the stable polynomial N satisfy following inequalities

$$|n_i| \leqq \binom{L}{i}, \quad i = 1, 2, \ldots, L.$$

If we find in the same time $h_{\max} = \max\limits_{i>1} h_{Q+i}$ we get from (7) the estimate

$$|g(z)| \leqq h_{\max}\left[L + (L-1)\binom{L}{1} + (L-2)\binom{L}{2} + \ldots + 2\binom{L}{L-2} + \binom{L}{L-1}\right].$$

It can easily be seen that it holds

$$(L-k)\binom{L}{k} = L\binom{L-1}{k} \quad \text{and} \quad \sum_{k=0}^{L-1}\binom{L-1}{k} = 2^{L-1}\,,$$

therefore

$$|g(z)| \leqq h_{\max} L \,.\, 2^{L-1} \quad \text{for} \quad |z| = 1\,. \tag{8}$$

We know at the same time that $\lim\limits_{Q\to\infty} h_{\max} = 0$.

Multiply the equation (1.4) by N and arrange into the form

$$\overline{M} - \overline{\Omega} = \overline{\overline{H}N} \tag{9}$$

where $M(z) = \overline{M}(z^{-1})$.

It is evident that just those zero points of the polynomial $\overline{M}$ lie inside of Γ that are the zero points of M lying outside of Γ; their number be ξ.

Now we use Theorem 2:

$$f(z) = \overline{M}\,, \quad g(z) = \overline{\Omega}\,.$$

If there holds the assumption of this theorem – no point lies on Γ – then there exists number I such that for $Q > I$ and $|z| = 1$, $|\bar{M}| > |\bar{\Omega}|$ is valid and in accordance with theorem 2 polynomials $\bar{M}$ and $\bar{M} - \bar{\Omega}$ have just ξ zero points inside of Γ.

Considering that zero points of N in (9) lie only outside of Γ, then there are just ξ zero points of $\bar{\bar{H}}$ lying inside of Γ and just ξ zero points of H lying outside of Γ.

So the proof of theorem is completed.

Put $M = 1$ into (1); then (9) gets the form $1 - \bar{\Omega} = \overline{\bar{H}N}$ and we can state

Theorem 3. *Be $M = 1$ in (1); then the sufficient condition for the impulse characteristic to have zero points lying only inside of Γ is such length of impulse characteristic that it holds*

$$h_{\max} \leqq \frac{1}{L \, . \, 2^{L-1}} \tag{10}$$

where $h_{\max}$ is the maximum ordinate of the remainder of impulse characteristic, L is a number larger than or equal to the plant order.

Results following from (10) (see Table 1) must be used with caution because they give only very rough estimate of sufficient condition. The estimate holds well for very short sampling period as in this case the absolute values of zero points of polynomial N tend to 1.

Table 1.

L	$h_{\max}$
1	1
2	0·25
3	0·083
4	0·031
5	0·0125
6	0·0052

In the case of practical identification we can write

$$\frac{M}{N} = H + \Theta \, ,$$

where H is the measured impulse characteristic, Θ is the error impulse characteristic.

Theorem 4. *Be $|M| > |N\Theta|$ for $|z| = 1$; then the impulse characteristic H has as many zero points outside of Γ as M has outside of Γ.*

The proof of this theorem follows from Theorem 2.

The theorems mentioned enable us to at least roughly estimate the necessary lenght of impulse characteristic so as to maintain at least the number of zero points lying outside of Γ.

It appears that the accuracy of identification has no substantial influence on the stability of discrete control circuit but in return it has decisive influence on the control peformance.

We know that an absolute accurate identification cannot be carried out. The characteristic equation of all feedback control circuits is of the form

$$(11) \qquad N + (S - M) = 0\,,$$

where S is the actual transfer function of the plant, M is the model transfer function, N is the transfer function dependent on the model transfer function and the control criterion.

Usually only $N = 0$ is considered as the characteristic equation, i.e. $S = M$.

Suppose static plants in the following.

When the plant is exactly described by an infinite impulse characteristic

$$(12) \qquad S = z^{-(T+1)} \sum_{i=0}^{\infty} s_i z^{-i}\,,$$

the model by a finite impulse characteristic

$$(13) \qquad M = z^{-(T+1)} \sum_{i=4}^{Q} m_i z^{-i}$$

then N is a finite polynomial

$$(14) \qquad N = \sum_{i=0}^{L} n_i z^{-i}$$

zero points of which lie all inside of Γ.

Closed control loop will be stable, if all zero points of the characteristics equation (11) lie inside of Γ.

The sufficient condition for the characteristic equation to be stable is, according to Theorem 2, fulfilment of the inequality

$$(16) \qquad |N| > |S - M| \quad \text{for} \quad |z| = 1\,.$$

We will now find numbers $a \leqq \min_{|z|=1|} |N|$ and $b \geqq \max_{|z|=1} |S - M|$.

When $a > b$, then the inequality (15) is fulfiled and the circuit is stable.

Numbers b can be determined for example as follows

$$(15) \qquad b = \sqrt{\left[\sum_{i=1}^{\infty} (s_i - m_i)^2\right]}$$

or

(17) $$b = \sum_{i=0}^{\infty} |s_i - m_i|$$

where

$$m_i = 0 \quad \text{for} \quad i > Q .$$

Number a is lower estimate of polynomial modulus on the unit circle can be determined for example by putting $z^{-1} = e^{j\varphi}$ into (14) and finding minimum from the graph of the function $y = |N(e^{j\varphi})|$ for $0 \leqq \varphi \leqq \pi$.

As an illustration we issue an example.

Be given characteristic equation

(18) $$1 - 0{\cdot}1z^{-1} - 0{\cdot}22z^{-2} + 0{\cdot}04z^{-3} + (S - M) = 0 .$$

We find easily

$$a = \min_{0 \leqq \varphi \leqq \pi} |N(e^{j\varphi})| = 0{\cdot}72 .$$

Sufficient condition of stability of equation (16) is

$$\sqrt{\left[\sum_{i=0}^{\infty} (s_i - m_i)^2\right]} < 0{\cdot}72 .$$

This result is of great importance for the appreciation of sufficient accuracy of plant identification.

When the zero points of polynomial N in (15) lie in the vicinity of unit circle then the sufficient condition of stability is very strong and is insuitable for the appreciation of identification accuracy.

(Received July 7, 1970.)

Nulové body impulsních charakteristik

Zdeněk Vostrý

Při syntéze diskrétních regulačních obvodů z impulsních charakteristik se setkáváme s problémem délky konečné impulsní charakteristiky. V článku je uvedeno několik vět, které odhad délky impulsní charakteristiky umožňují. Odhad se provádí tak, aby byl zachován počet nulových bodů ležících vně jednotkové kružnice.

Na otázku jak přesně identifikovat soustavu, aby uzavřený regulační obvod byl stabilní, odpovídá závěrečná část článku. Je zde stanovena postačující podmínka pro stabilitu charakteristické rovnice regulačního obvodu a celý postup ukázán na příkladě.

Ing. Zdeněk Vostrý, Ústav teorie informace a automatizace ČSAV (Institue of Information Theory and Automation — Czechoslovak Academy of Sciences), Vyšehradská 49, Praha 2.