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Algebraic Approach of the Root-Loci Method

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The correlation of the variations of coefficients and of the induced changes of root values in algebraic equations are studied. The results are aimed to the synthesis of linear feedback systems.

I. INTRODUCTION

The root-locus method, introduced by Evans [1; 2] and applied and developped by other authors (compare [3, 4, 5] and others) is a powerful tool in the synthesis of linear control circuits. Nevertheless its application remains relatively limited.

It seems to be two main reasons for it:

1. In its present form, the root-locus theory is based mainly on geometrical considerations, whereas the modern control theory prefers algebraic methods, especially those allowing numerical realization by iterative procedures, suitable for the solution on digital computers
2. The method remains limited to the root-loci derived from the alteration of a single parameter, influencing the coefficients of the characteristic equation. Therefore, it is not immediately applicable for the synthesis of linear control circuits with a more complex structure of feedback.

Further, an attempt is made to overcome these shortcomings. The solution is based on algebraic properties of the characteristic equation and allows the examination of the root trajectories for a linear transfer differential equation even under more complex variations of the parameters.

II. SYNTHESIS OF LINEAR FEEDBACK CONTROL CIRCUITS

Let us formulate, from this point of view, the problem of the synthesis of a PID feedback for the control of a linear system.

Let us assume, the system to be described by a linear differential equation with constant coefficients of the form

$$(1) \qquad \sum_{k=0}^{n} a_k \, . \, v^{(k)}(t) = \sum_{k=0}^{m} b_k \, . \, u^{(k)}(t) \, ,$$

$u(t)$ being the input signal, $v(t)$ the output signal, both expressed as functions of the time variable t.

Let us further introduce a feedback, the effect of which is defined by the integral setting r_i, the proportional setting r_p and the derivative setting r_d.

The transfer differential equation of the system with feedback control has the form:

$$(2) \qquad \sum_{k=0}^{n} a_k \, . \, v^{(k+1)}(t) + \sum_{k=0}^{m} b_k(r_d \, . \, v^{(k+2)}(t) + r_p \, . \, v^{(k+1)}(t) + r_i \, . \, v^{(k)}(t)) = \\ = \sum_{k=0}^{m} b_k \, . \, u^{(k+1)}(t) \, .$$

The problem of the synthesis consists in the determination of the operational settings r_i, r_p, r_d, so as to fulfil the requirements of the dynamics of the feedback controlled system. From the equation (2), it may be seen that any of the setting coefficients varies the ensemble of $(m + 1)$ coefficients on the left-hand side.

The corresponding characteristic equation of the initial differential equation (1)

$$(3) \qquad \sum_{k=0}^{n} a_k \, . \, p^k = 0$$

is altered by the introduction of the feedback to the form

$$(4) \qquad \sum_{k=0}^{n+1} a_k^* \, . \, p^k = 0$$

with the coefficients:

$$(5) \qquad \begin{aligned} a_0^* &= \qquad b_0 \, . \, r_i \, , \\ a_1^* &= a_0 + b_1 \, . \, r_i + b_0 \, . \, r_p \, , \\ a_2^* &= a_1 + b_2 \, . \, r_i + b_1 \, . \, r_p + b_0 \, . \, r_d \, , \\ &\vdots \\ a_{m+1}^* &= a_m + b_{m+1} \, . \, r_i + b_m \, . \, r_p + b_{m-1} \, . \, r_d \, , \\ &\vdots \\ a_{n+1}^* &= a_n \, . \end{aligned}$$

The synthesis, i.e. the choice of the operational setting of the feedback, is usually done with respect to the dominant roots of the characteristic equation, most exposed to the right-hand side of the complex plane of the roots. The setting must be performed in such a way that the dominant roots attain a position ensuring the desired character of the transient process in the system.

III. CHANGES OF ROOTS INDUCED BY COEFFICIENT VARIATIONS

Thus, the basic problem of the root-locus method is to find out the connection between the variations of the coefficients and between the changes of the roots in an algebraic equation with real coefficients of the form:

$$\sum_{k=0}^{n} c_k \, . \, p^k = 0 \, . \tag{6}$$

The examination should be performed in such a way, as to enable the use of suitable iterative proceedings, without the necessity of repeated solution of the algebraic equation with altered coefficients.

The alteration Δc_m of any arbitrary coefficient c_m is connected with the variation Δp_i of any selected root p_i by the relation

$$\sum_{k=0}^{n} c_k \, . \, (p_i + \Delta p_i)^k + \Delta c_m \, . \, (p_i + \Delta p_i)^m = 0 \tag{7}$$

whence

$$\Delta c_m = \frac{- \sum_{k=0}^{n} c_k \, . \, (p_i + \Delta p_i)^k}{(p_i + \Delta p_i)^m} \, . \tag{8}$$

As the coefficients of the equation must be real, the increase Δc_m must be real, too. The relation (8) is valid for any value of the increase Δp_i.

IV. COMPLEX ROOTS

The relation (8) is applicable without difficulty in the domain of real roots. It allows to find the necessary alteration Δc_m, corresponding to the desired variation of a root Δp_i, chosen in advance.

The situation changes if the root is a complex one. Choosing in this case an arbitrary, real or complex, increase Δp_i, we get from (8) an increase Δc_m, which is in general complex, too. Thus, the altered equation will not fulfil any more the requirement of real values of the coefficients.

To fulfil the requirement of reality of the coefficients in the altered equation, the increase of the root, Δp_i, must be chosen in a quite definite way. To find out this characteristic variation of the root, corresponding to the real value of the increase

 Δc_m, let us examine the local properties of the relation

$$\Delta c_m = g_{mi}(\Delta p_i) . \tag{9}$$

Considering the left-hand side of the equation (6) as a function of two variables

$$f_{mi}(c_m, p_i) = \sum_{k=0}^{n} c_k \cdot p_i^k \tag{10}$$

we may bind both variables with a functional relation

$$p_i = F_{mi}(c_m) \tag{11}$$

in such a way that during a variation of c_m the initial equation (6) remains valid. By differentiating (10) we get

$$\frac{\mathrm{d}p_i}{\mathrm{d}c_m} \sum_{k=1}^{n} k \cdot c_k \cdot p_i^{k-1} + p_i^m = 0 \tag{12}$$

whence

$$q_{mi} = \frac{\mathrm{d}p_i}{\mathrm{d}c_m} = \frac{-p_i^m}{\sum_{k=1}^{n} k \cdot c_k \cdot p_i^{k-1}} . \tag{13}$$

The argument of the generally complex expression on the right-hand side of (13) indicates the direction of the infinitesimal variation of the complex root, or in other words the direction of the root trajectory, corresponding to the real value of the increase Δc_m.

For $m = 0$, i.e. for the alteration of the absolute term of the equation, we get

$$q_{0i} = \frac{\mathrm{d}p_i}{\mathrm{d}c_0} = \frac{-1}{\sum_{k=1}^{n} k \cdot c_k \cdot p_i^{k-1}} . \tag{14}$$

The alterations of other coefficients may be calculated from q_{0i} by the use of the relation

$$q_{mi} = q_{0i} \cdot p_i^m \tag{15}$$ (15)

or by the application of the recurrent formula

$$q_{m+1,i} = q_{mi} \cdot p_i . \tag{16}$$

V. GENERALIZED SETTING PARAMETERS

The proceeding indicated above may be generalized in the way that the root of the equation may be regarded as a function of several or all coefficients of the equation.

Thus, we get instead of (10):

(17) $$f_i(c_0, c_1, c_2, \ldots, c_n, p_i) = \sum_{k=0}^{n} c_k \, . \, p_i^k \, .$$

The alteration of singular coefficients yields then partial derivatives

(18) $$q_{mi} = \frac{\partial p_i}{\partial c_m} = \frac{-p_i^m}{\sum\limits_{k=1}^{n} k \, . \, c_k \, . \, p_i^{k-1}} \, .$$

A simultaneous alteration of several or all coefficients results in the total increase of the corresponding root

(19) $$\mathrm{d}p_i = \sum_{m=0}^{n} q_{mi} \, . \, \mathrm{d}c_m = q_{0i} \sum_{m=0}^{n} p_i^m \, . \, \mathrm{d}c_m \, ,$$

(19a) $$q_{0i} = \frac{\partial p_i}{\partial c_0} = \frac{-1}{\sum\limits_{k=1}^{n} k \, . \, c_k \, . \, p_i^{k-1}} \, .$$

If the coefficients are represented as given functions of several independent setting parameters $s_1, s_2, \ldots, s_j, \ldots, s_k$

(20) $$c_m = c_m(s_1, s_2, \ldots, s_k)$$

we obtain:

(21) $$\frac{\partial p_i}{\partial s_j} = \sum_{m=0}^{n} q_{mi} \, . \, \frac{\partial c_m}{\partial s_j} = q_{0i} \sum_{m=0}^{n} p_i^m \frac{\partial c_m}{\partial s_j}$$

with q_{0i} given by (19a).

The resulting variation of the root, caused by the alteration of the setting parameter s_j is

(22) $$\mathrm{d}p_i = \mathrm{d}s_j \sum_{m=0}^{n} \frac{\partial p_i}{\partial c_m} \frac{\partial c_m}{\partial s_j} \, .$$

The formulae indicated above are important from the practical point of view. The formula (18) indicates for any root its variation caused by the alteration of any singular coefficient of the characteristic equation. The generalized formula (21) gives the variation of any root, resulting from a simultaneous alteration of a complex of coefficients. The same situation has been met above in formulating the problem of the synthesis of a feedback control of a linear system.

It is important to emphasize that the trajectory of any root is examined separately, without taking into account the behaviour of other roots of the equation, or even without the knowledge of the values of these roots.

The foregoing considerations assumed the fulfilment of the condition

$$\sum_{k=1}^{n} k \cdot c_k \cdot p_i^{k-1} \neq 0 \tag{23}$$

equivalent to the assumption, the root, the trajectory of which is examined, to be simple.

Let us now pay attention to the case of multiple roots. For this purpose we transform the equation (7) to the form

$$\sum_{k=0}^{n} c_k \cdot \sum_{h=0}^{k} \binom{k}{h} p_i^{k-h} \cdot (\Delta p_i)^h = \tag{24}$$

$$= \sum_{h=0}^{n} (\Delta p_i)^h \sum_{k=0}^{n} c_k \binom{k}{h} p_i^{k-n} = -(p_i + \Delta p_i)^m \cdot \Delta c_m \, .$$

The change in the upper bound, performed in the summation by h, is based on the fact that

$$\binom{k}{h} = 0 \quad \text{for all} \quad h > k \, .$$

For polynomials of the form (10)

$$\sum_{k=0}^{n} c_k \cdot \binom{k}{h} p_i^{k-h} = \frac{1}{h!} \frac{\mathrm{d}^{(h)} f(p_i)}{\mathrm{d} p_i^h} \, . \tag{25}$$

Thus, if p_i is a l-fold root of the equation

$$f(p) = \sum_{k=0}^{n} c_k \cdot p^k = 0 \tag{26}$$

both sides in (25) are equal to zero for any

$$0 \leqq h \leqq l - 1$$

but are different from zero for all

$$l \leqq h \leqq n \, .$$

Keeping this in mind, we may arrange (25) to the form

$$\sum_{h=l}^{n} (\Delta p_i)^h \sum_{k=0}^{n} c_k \binom{k}{h} p_i^{k-h} = -\Delta c_m \cdot (p_i + \Delta p_i)^m \, . \tag{27}$$

Let us suppose the root to be a multiple one with $p_i \neq 0$. Neglecting increases of higher orders on both sides of (27) we get

$$\lim_{\Delta c_m \to 0} \frac{(\Delta p_i)^l}{\Delta c_m} = \frac{-p_i^m}{\sum_{k=0}^{n} c_k \binom{k}{l} p_i^{k-l}} \tag{28}$$

l indicating the multiplicity of the root.

The increase Δc_m is here of the same order as $(\Delta p_i)^l$.

From (28) results

$$\lim_{\Delta c_m \to 0} \Delta p_i = \lim_{\Delta c_m \to 0} \left(\frac{-p_i^m}{\sum_{k=0}^{n} c_k \binom{k}{l} p_i^{k-l}} \, . \, \Delta c_m \right)^{1/l} . \tag{29}$$

The expression on the right-hand side represents generally a complex value. To find its argument let us put:

$$\varphi_i = \arg \frac{-p_i^m}{\sum_{k=0}^{n} c_k \binom{k}{l} p_i^{k-l}} . \tag{30}$$

The value of Δc_m being real there is

$$\begin{aligned} \arg \Delta c_m &= 0 \quad \text{for} \quad c_m > 0 \, , \\ \arg \Delta c_m &= \pi \quad \text{for} \quad c_m < 0 \, . \end{aligned} \tag{31}$$

From (29), (30), (31) there results

$$\arg \Delta p_i = \frac{\varphi_i}{l} + \frac{h}{l} \pi \tag{32}$$

for $h = 0, 1, 2, \ldots, (2l - 1)$,
the even values of h corresponding to positive values of Δc_m,
the odd values of h corresponding to negative values of Δc_m.

A l-fold root is the starting point of $2l$ trajectories, forming in the proximity of the root a regular star. Indicating on the trajectories the direction of positive increase of Δc_m and marking the trajectories successively by numbers in such a way that the trajectory with the argument φ_i/l be marked by 0, we find that the odd trajectories aim towards the multiple root, the even ones run away from the root.

The case of a l-food zero root requires special examination. The corresponding algebraic equation has coefficients

$$c_k = 0$$

for all

$$0 \leqq k \leqq l - 1$$

Thus, the relation (7) has the form

$$\sum_{k=l}^{n} c_k \, . \, (\Delta p_i)^k + \Delta c_m \, . \, (\Delta p_i)^m = 0 \, . \tag{33}$$

Going over to the limit we get

$$\lim_{\Delta c_m \to 0} \frac{(\Delta p_i)^{l-m}}{\Delta c_m} = \frac{-1}{c_l} \tag{34}$$

valid for all $m < l$, whence

$$\lim_{\Delta c_m \to 0} \Delta p_i = \lim_{\Delta c_m \to 0} \left(\frac{-\Delta c_m}{c_l} \right)^{1/(l-m)} . \tag{35}$$

A l-fold zero root is the starting point of $2(l - m)$ trajectories, forming again in its proximity a regular star, one half of the trajectories aiming again towards the root, the other half running away from the root. The remaining m zero roots do not undergo any change with the alteration of Δc_m, keeping identically the zero value.

The alteration of the coefficients c_k with

$$n \geqq k \geqq l$$

does not influence the value of the zero roots. This fact is evident directly from the equation (33).

IX. VARIABILITY OF SIMPLE REAL ROOTS

Let us suppose the coefficient of the highest order

$$a_n > 0$$

and let us examine the variations of the simple real roots, if there are any. Let us put again

$$f(p) = \sum_{k=0}^{n} a_k \, . \, p^k \, . \tag{36}$$

It is evident that

$$\lim_{p \to \infty} f(p) = + \infty \, . \tag{37}$$

At first let us pay attention to the dominant simple real root. From topological considerations we find for this roots

(38)
$$\left(\frac{\mathrm{d}f(p)}{\mathrm{d}p}\right)_{p=p_i} > 0\,.$$

According to (18) there is

(39)
$$\operatorname{sign}\frac{\partial p_i}{\partial c_m} = \operatorname{sign}\left(-p_i^m\right).$$

In consequence we may formulate for the dominant simple real root the following rules:

1. If the root is positive it decreases with the increase of any coefficient
2. If it is negative it decreases with the increase of any coefficient with zero or even order number, but it increases with the increase of any coefficient with an odd order number.
3. If $p_i = 0$, it decreases with the increase of c_0 and remains unchanged under the alteration of any other coefficient of the equation.

Going over to the next lower simple real root we find again from topological considerations that the sign of the derivative has changed. From this fact we may conclude that neighbouring simple real roots are always moving in opposite directions under the alteration of any coefficient of the equation.

X. SENSITIVITY OF ROOTS VARIATIONS

The relations (15) and (16) make it possible to estimate the sensitivity of the variations of real and complex roots depending on the alteration of the coefficients. Let us define the sensitivity as the absolute values $|q_{mi}|$.

Then following relations are valid:

1. for $|p_i| > 1$

(40)
$$|q_{m+h,i}| > |q_{mi}|\,,$$

2. for $|p_i| < 1$

(41)
$$|q_{m+h,i}| < |q_{mi}|\,,$$

3. for $|p_i| = 1$

(42)
$$|q_{m+h,i}| = |q_{mi}|\,,$$

for any positive values of m, h.

Thus for $|p_i| = 1$ the roots are equally sensitive to the alterations of any coefficient of the equation. For $|p_i| > 1$, i.e. for roots situated far from the origin, alterations of coefficients of higher order show a more expressed influence on the root changes as compared with coefficients of lower order. For $|p_i| < 1$, on the contrary, the coefficients of lower order show a more expressed influence on the root variations as compared with coefficients of higher orders.

XI. EXAMPLES

In the following the application of the preceeding theoretical considerations is illustrated on several examples.

Example 1

For the equation

$$f(p) = \sum_{k=0}^{3} c_k \cdot p^k = 0 \tag{1A}$$

with the coefficients

$$c_3 = 1\,, \quad c_2 = 4\,, \quad c_1 = 9\,, \quad c_0 = 10$$

the variations of the root

$$p_1 = -1 + 2i$$

were examined for the alterations of singular coefficients c_0, c_1, c_2.

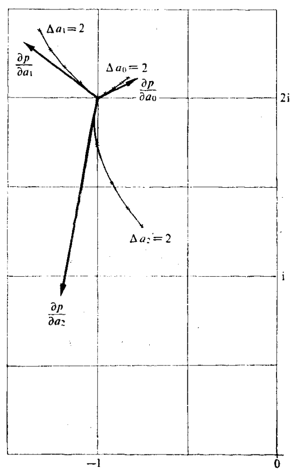

Fig. 1.

Table I.

c_0	p_1		q_{01}	
	Re	Im	Re	Im
10·0	−1·0000	+2·0000	+0·1000	+0·0500
10·1	−0·9901	+2·0050	0·0987	0·0504
10·2	−0·9803	2·0101	0·0972	0·0509
10·3	−0·9707	2·0152	0·0956	0·0514
10·4	−0·9612	2·0204	0·0941	0·0518
10·5	−0·9519	2·0256	0·0927	0·0521
10·6	−0·9427	2·0308	0·0912	0·0524
10·7	−0·9336	2·0361	0·0898	0·0527
10·8	−0·9247	2·0414	0·0885	0·0529
10·9	−0·9160	2·0467	0·0871	0·0531
11·0	−0·9073	2·0520	0·0858	0·0533
11·1	−0·8988	2·0573	0·0845	0·0534
11·2	−0·8905	2·0627	0·0833	0·0535
11·3	−0·8822	2·0680	0·0821	0·0536
11·4	−0·8741	2·0734	0·0809	0·0536
11·5	−0·8660	2·0787	0·0798	0·0536
11·6	−0·8581	2·0841	0·0786	0·0536
11·7	−0·8503	2·0895	0·0776	0·0536
11·8	−0·8426	2·0948	0·0765	0·0536
11·9	−0·8351	2·1002	0·0754	0·0536
12·0	−0·8276	2·1055	0·0744	0·0535

The examination was achieved on a digital computer by application of (13) with the choice of iterative steps $\Delta c_m = 0.02$. Intermediate results were fixed after every fifth step.

The results are given in Tables I, II, III and are shown graphically in Fig. 1. The results were not corrected during computation. In spite of it the relative error of the final roots after 200 iterative steps is of the order of 10^{-4}.

Example 2

The examination of root variations was done for the equation

(2A) $$g(p) = p \, . \, f(p) = 0$$

with $f(p)$ defined by (1A) and with the same values of coefficients, so that

(2B) $$g(p) = \sum_{k=0}^{4} c_k^* \, . \, p^k$$

Table II.

c_1	p_1		q_{11}	
	Re	Im	Re	Im
9·0	−1·0000	+2·0000	−0·2000	+0·1500
9·1	−1·0199	2·0152	−0·1990	+0·1547
9·2	−1·0397	2·0310	−0·1969	0·1604
9·3	−1·0592	2·0474	−0·1945	0·1657
9·4	−1·0784	2·0642	−0·1913	0·1710
9·5	−1·0972	2·0816	−0·1880	0·1760
9·6	−1·1158	2·0994	−0·1843	0·1807
9·7	−1·1339	2·1178	−0·1802	0·1850
9·8	−1·1516	2·1365	−0·1759	0·1892
9·9	−1·1689	2·1556	−0·1714	0·1927
10·0	−1·1857	2·1750	−0·1667	0·1961
10·1	−1·2020	2·1947	−0·1620	0·1989
10·2	−1·2179	2·2148	−0·1569	0·2013
10·3	−1·2333	2·2350	−0·1522	0·2035
10·4	−1·2482	2·2554	−0·1472	0·2052
10·5	−1·2626	2·2760	−0·1424	0·2068
10·6	−1·2765	2·2967	−0·1378	0·2078
10·7	−1·2900	2·3175	−0·1331	0·2085
10·8	−1·3030	2·3383	−0·1285	0·2091
10·9	−1·3155	2·3592	−0·1240	0·2095
11·0	−1·3276	2·3802	−0·1198	0·2097

with

$$c_0^* = 0\,, \quad c_k^* = c_{k-1}\,.$$

The same root as in Example 1 was chosen.

The examination was done for the alterations of the coefficient c_0^* again on a digital computer by application of (13) and with iterative steps $\Delta c_0^* = 0.02$. The results are given in Tab. IV.

It was not necessary to calculate the variations of the roots, produced by alterations of other coefficients, as may be seen from the following consideration.

By differentiation of

$$g(p) = p\,.\,f(p) \tag{2C}$$

we get

$$g'(p) = f(p) + p\,.\,f'(p)\,. \tag{2D}$$

But for the roots of the equation (1A)

$$f(p_i) = 0\,,$$

Designating by q_m the variation of the root of equation (1A), defined by (13), and by r_m the

Table III.

c_2	p_1		q_{21}	
	Re	Im	Re	Im
4·0	−1·0000	+2·0000	−0·1000	−0·5500
4·1	−1·0084	1·9449	−0·0651	−0·5479
4·2	−1·0129	1·8898	−0·0247	−0·5445
4·3	−1·0132	1·8354	+0·0177	−0·5338
4·4	−1·0095	1·7823	+0·0546	−0·5185
4·5	−1·0020	1·7312	+0·0898	−0·4959
4·6	−0·9912	1·6827	+0·1201	−0·4686
4·7	−0·9777	1·6372	+0·1445	−0·4380
4·8	−0·9620	1·5949	+0·1624	−0·4061
4·9	−0·9449	1·5558	+0·1745	−0·3741
5·0	−0·9269	1·5199	+0·1816	−0·3438
5·1	−0·9084	1·4869	+0·1848	−0·3156
5·2	−0·8897	1·4567	+0·1848	−0·2901
5·3	−0·8712	1·4289	+0·1825	−0·2670
5·4	−0·8530	1·4032	+0·1790	−0·2465
5·5	−0·8352	1·3795	+0·1744	−0·2283
5·6	−0·8180	1·3576	+0·1692	−0·2122
5·7	−0·8013	1·3371	+0·1634	−0·1979
5·8	−0·7851	1·3180	+0·1577	−0·1852
5·9	−0·7696	1·3001	+0·1520	−0·1737
6·0	−0·7546	1·2832	+0·1464	−0·1636

variation of the root of equation (2A), we find out from (14), (15), (16) and (2D)

(2E) $$r_{0i} = \frac{1}{p_i \cdot f'(p_i)} = \frac{1}{p_i} q_{0i}$$

and further

(2F) $$r_{1i} = p_i \cdot r_{0i} = q_{0i} \, .$$

Generally

(2G) $$r_{mi} = q_{m-1,i} \, .$$

Example 3

Analysis and synthesis of feedback for the system with the characteristic equation

(3A) $$f(p) = \sum_{k=0}^{4} c_k \cdot p^k = 0$$

 Table IV.

c_0^*	p_1		r_{01}	
	Re	Im	Re	Im
0·0	−1·0000	+2·0000	0·0000	−0·0500
0·1	−1·0000	1·9950	0·0000	−0·0504
0·2	−1·0000	1·9899	0·0000	−0·0510
0·3	−1·0000	1·9848	0·0000	−0·0515
0·4	−1·0000	1·9796	0·0000	−0·0521
0·5	−1·0000	1·9743	0·0000	−0·0527
0·6	−1·0000	1·9690	0·0000	−0·0533
0·7	−1·0000	1·9637	0·0000	−0·0539
0·8	−1·0000	1·9582	0·0000	−0·0546
0·9	−1·0000	1·9527	0·0000	−0·0552
1·0	−1·0000	1·9472	0·0000	−0·0559
1·1	−1·0000	1·9416	0·0000	−0·0566
1·2	−1·0000	1·9359	0·0000	−0·0573
1·3	−1·0000	1·9301	0·0000	−0·0581
1·4	−1·0000	1·9242	0·0000	−0·0588
1·5	−1·0000	1·9183	0·0000	−0·0596
1·6	−1·0000	1·9123	0·0000	−0·0604
1·7	−1·0000	1·9062	0·0000	−0·0613
1·8	−1·0000	1·9000	0·0000	−0·0622
1·9	−1·0000	1·8937	0·0000	−0·0631
2·0	−1·0000	1·8874	0·0000	−0·0640

with

$$c_0 = 1\,, \quad c_1 = 4\,, \quad c_2 = 6\,, \quad c_3 = 4\,, \quad c_4 = 1\,,$$

i.e. with a quadruple real root.

The Hurwitz criterion gives for this case the stability condition [compare [6])

$$c_0 \leqq \frac{c_1}{c_3}\left(c_2 - \frac{c_4}{c_3}\,c_1\right). \tag{3B}$$

As far as we are concerned only with the alteration of c_0 in the realization of feedback setting, the admissible value of c_0 is limited by

$$c_0 \leqq 5\,.$$

For simultaneous setting of c_0, c_1, i.e. for PD feedback control, we get the admissible limit values from

$$\frac{dc_0}{dc_1} = \frac{1}{c_3}\left(c_2 - \frac{c_4}{c_3}\,c_1\right) - \frac{c_1\,.\,c_4}{c_3^2} = 0\,. \tag{3C}$$

whence

$$c_0 = 9\,, \quad c_1 = 12\,.$$

The setting range of c_1, given by (3C), may be too wide for most practical purposes. By limiting the range of c_1 to, let us say

$$4 \leqq v_1 \leqq 6$$

we get

$$c_0 \leqq 6{\cdot}75\,.$$

With respect of the quality of transients we are limited to about one half of this value, which gives for practical application

$$c_0 \leqq 3{\cdot}5\,.$$

This value is too low with respect to the static deviation. It is therefore advisible to use a feedback including the integral action, i.e. of the type PI or PID.

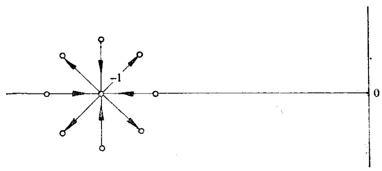

Fig. 2.

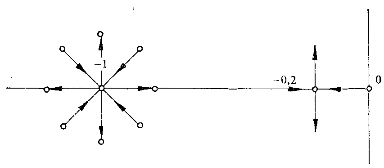

Fig. 3.

By the introduction of the integral action, the degree ot the characteristic equation is raised by a unit, so that the characteristic equation will get the form:

$$g(p) = \sum_{k=0}^{5} a_k \,.\, p^k = 0 \tag{3D}$$

with

$$c_k = a_{k+1}\,.$$

Let us consider a_0, a_1 (PI control) or a_0, a_1, a_2 (PID control) as setting parameters.

Let us examine the root trajectories in the proximity of the quadruple root

$$p_i = -1\,, \quad i = 1, 2, 3, 4\,,$$

connected with the alteration of the absolute term. The situation for the equation (3A) is shown in Fig. 2, for the equation (3D) in Fig. 3. In both cases the star has the same form, but with opposite sense of the movement. In equation (D3), a double root

$$p_d = -0{\cdot}2$$

is formed for

$$a_0 = 0{\cdot}08192$$

by coincidence of the former roots $p_1 = -1$ and $p_5 = 0$.

According to the theoretical considerations, the root trajectories show a high sensitivity in the proximity of the quadruple root. To make sure of it, the connection of the variation of the term c_0 in (3A) and of a_1 in (3D) with the variation of the roots was calculated.

Changing the roots to

$$p_1 = -1 + \Delta p\,, \quad p_2 = -1 - \Delta p\,, \quad p_{3,4} = -1 \pm i\,.\,\Delta p$$

we find that only the coefficients c_0, a_1 undergo variations. Values of these coefficients and of derivatives of the functions $f(p)$, $g(p)$ are given in Tab. V.

Table V.

p_1	c_0; a_1	Δc_0; Δa_1	$f'(p_1)$	$g'(p_1)$
0·1	0·9999	−0·0001	+0·004	−0·0036
0·2	0·9984	−0·0016	+0·032	−0·0256
0·3	0·9919	−0·0081	+0·108	−0·0756
0·4	0·9744	−0·0256	+0·256	−0·1536

As may be seen from Fig. 3, the initial root $p_5 = 0$ remains the dominant one up to the formation of the double root $p_d = -0{\cdot}2$. Afterwards, the pair of the complex conjugated roots, arising from this double root, represents again the dominant pair of roots.

For the absolute term a_0 reaching the value of

$$a_{0cr} = 0{\cdot}56853$$

the stability limit with the purely imaginary roots

$$p_{1,2} = \pm i\,.\,0{\cdot}41421$$

is attained.

There exists a pair of dominant roots in (3D) formed by the alteration of the absolute term alone, fulfilling the condition

$$\text{(3E)} \qquad \operatorname{Re} p_i = \mp \operatorname{Im} p_i$$

They are

$$p_{1,2} = -0{\cdot}16580 \pm i\,.\,0{\cdot}16580$$

corresponding to the value of

$$a_0 = 0{\cdot}20891\,.$$

Let us choose this pair of complex conjugated roots as the starting point of the establishment of the feedback setting. We get for this pair of roots following characteristic values:

$$g'(p_{1,2}) = -0{\cdot}19568 \pm i\,.\,0{\cdot}48262\,,$$

$$\frac{\partial p_{1,2}}{\partial a_0} = +0{\cdot}72151 \pm i\,.\,1{\cdot}77950,$$

$$\frac{\partial p_{1,2}}{\partial a_1} = -0{\cdot}41467 \mp i\,.\,0{\cdot}17542\,,$$

$$\frac{\partial p_{1,2}}{\partial a_2} = +0{\cdot}09784 \mp i\,.\,0{\cdot}03966\,.$$

The setting diagram is shown in Fig. 4. It allows to find out the necessary alterations of the coefficients for any selected direction of an infinitesimal movement of the root. For example to obtain the movement of the root in such a direction as to keep up the validity of the relation

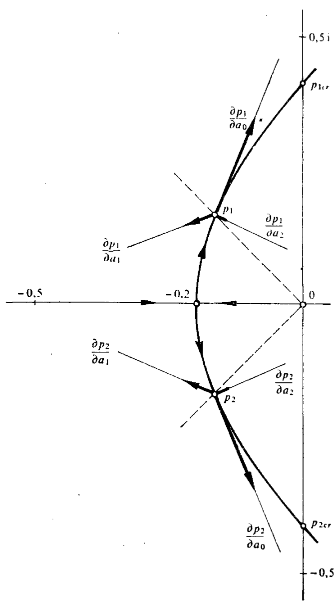

Fig. 4.

(3E), it is necessary to alter the coefficients of the equation so as to fulfil the relation

$$\frac{\Delta a_0}{\Delta a_1} = 0{\cdot}236$$

leaving a_2 unvaried.

Fig. 4 illustrates very clearly the relation of root sensitivity with respect to the variations of singular coefficients of the equation. As the absolute value of the chosen dominant roots

$$|p_{1,2}| = 0{\cdot}23448$$

is substantially smaller than unit, the considerations of Chap. X indicate an inferior sensitivity of the position of the roots to the variations of higher coefficients of the equation.

Actually, the absolute values

$$\left|\frac{\partial p_{1,2}}{\partial a_0}\right| = 1{\cdot}92955\,,$$

$$\left|\frac{\partial p_{1,2}}{\partial a_1}\right| = 0{\cdot}45027\,,$$

$$\left|\frac{\partial p_{1,2}}{\partial a_2}\right| = 0{\cdot}10557$$

show, for the case given, the sensitivity of the root position to the variations of the coefficient a_0 to be 4·27-times greater than the sensitivity to variations of the coefficient a_1, and 18·3-times greater than the sensitivity to variations of the coefficient a_2. Therefore, the PI type of feedback appears to be the most appropriate choice for achieving the setting the dominating roots considered in this example.

It should be emphasized that this fact is a local property of the root, dependent on its distance from the origin. The fact of the matter is considerably changed when considering other roots of the same equation, situated farther from the origin. For these roots, the influence of proportional, and especially of derivative feedback would be much more marked, as a result of the sensitivity properties described in Chap. X.

The diagram of Fig. 4 is rigidly valid only in the proximity of the root under consideration. For wider variations of the root position, an iterative procedure is to be used, taking into account the variations of the root as well as those of the coefficients along the trajectory of the root.

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Metoda kořenových trajektorií z algebraického hlediska

JINDŘICH SPAL

Článek se zabývá metodou kořenových trajektorií z algebraického hlediska. Formuluje problém syntézy zpětné vazby pro lineární systémy jako variaci součinitelů charakteristické rovnice. Rozebírá souvislost změn kořenů algebraické rovnice se změnami jejích koeficientů a podává tak algebraickou verzi vyšetřování kořenových trajektorií. Zvláštní pozornost je věnována současné proměnlivosti komplexů součinitelů pro případ, kdy každý součinitel komplexu je danou funkcí ladicího parametru. Vyšetřují se případy trajektorií v místě vícenásobných kořenů pro nulové i nenulové kořeny. Uvádějí se některé souvislosti vzájemných vztahů pohybu kořenů. Všeobecné zásady jsou znázorněny na příkladech.

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