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A Classification of Linear Controllable Systems*

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The concept of feedback (F-) equivalence of linear controllable systems is defined and the classification of such systems based upon this equivalence concept is discussed.

1. INTRODUCTION

This paper is concerned with linear control systems, which can be represented by systems of linear differential equations of the form

$$\dot{x} = Ax + Bu \tag{1}$$

where x and u are n- and m-vectors respectively, A and B are matrices of appropriate size, in general time-dependent.

Since the system (1) is uniquely determined by the pair of matrices A, B, we shall frequently call it $\langle A, B \rangle$.

The basic question, which leads to the classification studied in this paper, can be formulated as follows:

Having two systems $\langle A, B \rangle$ and $\langle A', B' \rangle$, is there a linear feedback which, added to $\langle A, B \rangle$, yields a system, which behaves like $\langle A', B' \rangle$?

We shall make this question more precise in the next section. At this point let us note that the feedback will be required to be constant, time varying or periodic in t according to the system itself.

We shall see, that this question gives rise to a classification of controllable autonomous systems into a finite number of classe seach of which can be represented by a very simple canonical form. For time-varying systems, such a classification will be given for an important subclass of controllable systems.

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Let us note that this paper is related to [1], [2], [3], where similar concepts of equivalence have been introduced, though for different purposes. For time-varying systems, this paper extends a result of [4]. The point of view of studying control system is to a certain extent related to that of [5].

2. AUTONOMOUS SYSTEMS

In this section, we shall assume that A, B are constant matrices and that the system is completely controllable, i. e.,

$$\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = n . \tag{2}$$

To formulate the question raised in the preceding section more precisely, we translate it into an algebraic form.

By adding a linear feedback to $\langle A, B\rangle$, we mean that in (1), we substitute $u = Qx + v$, where Q is $m \times n$ constant. As a result of this transformation, we obtain a system $\langle A'', B''\rangle$, with $A'' = A + BQ$, $B'' = B$.

By saying that $\langle A'', B''\rangle$ behaves like $\langle A', B'\rangle$ we mean that by nonsingular linear transformations of the state (output) and input variables, $\langle A'', B''\rangle$ can be brought into $\langle A', B'\rangle$ or, algebraically, there are nonsingular matrices C and D of type $n \times n$, $m \times m$ respectively, such that $A' = C^{-1}A''C$, $B' = C^{-1}BD$.

Summarizing, we find that the question of the preceding section asks, whether for given systems $\langle A, B\rangle$, $\langle A', B'\rangle$ there are matrices $C(m \times n)$, $Q(m \times n)$, $D(m \times m)$, C, D being nonsingular, such that

$$A' = C^{-1}(A + BQ)\,C\,, \quad B' = C^{-1}BD\,. \tag{3}$$

If the answer is positive, we shall say that $\langle A, B\rangle$ and $\langle A', B'\rangle$ are feedback (or, briefly, F-) equivalent.

By a straightforward computation it can be checked that F-equivalence is actually an equivalence relation, i. e., it is symmetric, reflexive and transitive. Moreover, the order of transformations, by which $\langle A', B'\rangle$ is obtained from $\langle A, B\rangle$, can be changed.

It will be frequently convenient to express that a system $\langle A', B'\rangle$ can be obtained from $\langle A, B\rangle$ by one of the transformations, occuring in the definition of F-equivalence, only.

Referring to our definition, $\langle A, B\rangle$ and $\langle A', B'\rangle$ will be called

– C-equivalent, if $Q = 0$, $D = E$,

– Q-equivalent, if $C = E_n$, $D = E_m$,

– D-equivalent, if $C = E$, $Q = 0$,

where E is the unity matrix, the index indicating its size. It is easy to check, that these relations are equivalence relations.

We associate with a given system $\langle A, B\rangle$ n numbers $r_0, r_1, \ldots, r_{n-1}$ as follows:

$$(4) \quad r_0 = \operatorname{rank} B\,,$$
$$r_j = \operatorname{rank}(B, AB, \ldots, A^jB) - \operatorname{rank}(B, AB, \ldots, A^{j-1}B)\,, \quad 1 \leqq j \leqq n-1\,.$$

Geometrically, if we denote by $L_j\langle A, B\rangle$ the linear subspace of R^n, spanned by the column vectors of $B, AB, \ldots, A^jB$, by Λ_j the orthogonal complement of L_{j-1} in L_j and by $\pi_j(f)$ the orthogonal projection of a vector f into Λ_j, then r_j is the dimension of Λ_j, which is equal to rank $(\pi_j(A^jB))$.

Obviously, $0 \leqq r_j \leqq m$ for $0 \leqq j \leqq n-1$ and, because of (2), $\sum_{j=0}^{n-1} r_j = n$. Moreover, since $A^jb_i = \sum_{\nu\in I} A^jb_\nu$, $I \in \{1, \ldots, m\}$ implies $A^{j+1}b_i = \sum_{\nu\in I} A^{j+1}b_\nu$, we have $r_0 \geqq r_1 \geqq, \ldots, \geqq r_{n-1}$ and we can choose a basis S of R^n from the column vectors of $(B, AB, \ldots, A^{n-1}B)$ in such a way that the vectors $\{\pi_j(A^jb_i) | A^jb_i \in S,\ j \text{ fixed}\}$ span Λ_j (consequently, their number is r_j) and if $A^jb_i \notin S$, than $A^{j+1}b_i \notin S$. Such a basis we shall call pyramidal.

Assuming that we have chosen a pyramidal basis S, we can associate with every column b_i a number p_i, such that $A^jb_i \in S$ for $0 \leqq j \leqq p_i - 1$, but $A^{p_i}b_i \notin S$. By re-ordering suitably the columns of B (this is a D-transformation) we can achieve that $p_1 \geqq p_2 \geqq, \ldots, \geqq p_m$. Consequently, the p-numbers can be uniquely determined by the r-numbers, associated with $\langle A, B\rangle$, as follows:

$$(5) \qquad p_i \text{ is the number of } r_j\text{'s, which are} \geqq i\,.$$

Conversely, the r-numbers are evidently uniquely determined by the p-numbers of $\langle A, B\rangle$.

As a result of our discussion we have:

Lemma 1. *For every controllable system $\langle A, B\rangle$, the finite sequences of numbers $R\langle A, B\rangle = \{r_j\}_{j=0}^{n-1}, P\langle A, B\rangle = \{p_i\}_{i=1}^{m}$, defined respectively by* (4), (5), *have the following properties*:

(i) $0 \leqq r_j \leqq m\,, \quad r_0 \geqq r_1 \geqq \ldots \geqq r_{p_1-1} > 0\,, \quad r_j = 0$ *for* $j \geqq p_1, \sum_{j=0}^{n-1} r_j = n\,,$

(ii) $0 \leqq p_i \leqq n, p_1 \geqq \ldots \geqq p_{r_0} > 0\,, \quad p_i = 0$ *for* $i > p_{r_0}, \sum_{i=1}^{m} p_i = n\,,$

(iii) $P\langle A, B\rangle = P\langle A', B'\rangle$ *if and only if* $R\langle A, B\rangle = R\langle A', B'\rangle\,.$

(iv) *There is a system $\langle A, B'\rangle$ D-equivalent with $\langle A, B\rangle$ such that the vectors A^jb_i, $1 \leqq i \leqq r_0$, $0 \leqq j \leqq p_i - 1$ form a basis of R_n.*

Now we are able to formulate

Theorem 1. *$\langle A, B\rangle$ is F-equivalent with $\langle A', B'\rangle$ if and only if $R\langle A, B\rangle = R\langle A', B'\rangle$ (or, $P\langle A, B\rangle = P\langle A', B'\rangle$).*

Theorem 2. *Let $P\langle A, B\rangle = \{p_i\}_{i=1}^{m}$, $R\langle A, B\rangle = \{r_j\}_{j=0}^{n-1}$. Then, $\langle A, B\rangle$ is F-equivalent with a decoupled system of r_0 integrators:*

(6) $$\dot{y}_{k_i+1} = y_{k_i+2}, \ldots, \dot{y}_{k_{i+1}-1} = y_{k_{i+1}}, \quad \dot{y}_{k_{i+1}} = v_{i+1}, \quad i = 0, \ldots, r_0 - 1$$

where $k_i = \sum_{\nu=1}^{i} p_\nu$.

We first prove theorem 2, with the aid of

Lemma 2. *Let $P\langle A, B\rangle = \{p_i\}_{i=1}^{m}$, $R\langle A, B\rangle = \{r_j\}_{j=0}^{n-1}$. Then, $\langle A, B\rangle$ is C-equivalent with a system $\langle A', B'\rangle$ of the following form:*

$$A' = \begin{pmatrix} A'_{11}, \ldots, A'_{1r_0} \\ \ldots\ldots\ldots\ldots \\ A'_{r_01}, \ldots, A'_{r_0r_0} \end{pmatrix}, \quad B' = \begin{pmatrix} B'_1 \\ \vdots \\ B'_{r_0} \end{pmatrix}$$

where A'_{ij} are $p_i \times p_j$,

$$A'_{ij} = \begin{pmatrix} 0 & , \ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots \\ 0 & , \ldots, 0 \\ \alpha_{ik_{j-1}+1}, & \ldots, \alpha_{ik_j} \end{pmatrix}$$

if $i \neq j$,

$$A'_{ii} = \begin{pmatrix} 0 & , 1 & , \ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & , 0 & , \ldots, 1 \\ \alpha_{ik_{i-1}+1}, & \alpha_{ik_{i-1}+2}, & \ldots, \alpha_{ik_i} \end{pmatrix},$$

$k_i = \sum_{\nu=1}^{i} p_\nu$, *$B'_i$ are $p_i \times m$,*

$$B'_i = \begin{pmatrix} 0, \ldots\ldots\ldots\ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0, \ldots\ldots\ldots\ldots, 0 \\ 0, \ldots, 0, 1, \gamma_{ii+1}, \ldots, \gamma_{im} \end{pmatrix}$$

(1 is in the i-th column), $i, j = 1, \ldots, m$.

This lemma is proved in [6] and, in fact, is also a special case of lemma 8 of this paper.

Proof of theorem 2. In virtue of lemma 1, we can assume that $\langle A, B\rangle$ has the special form, given in its formulation. Denote by $\tilde{B}$ the submatrix of B, consisting of its k_i-th rows, $i = 1, \ldots, r_0$ (those are precisely the non-zero rows of B), by $\tilde{\tilde{B}}$ the submatrix of $\tilde{B}$, consisting of its first r_0 columns. $\tilde{\tilde{B}}$ is a nonsingular triangular $r_0 \times r_0$ matrix and, furthermore, the columns $\tilde{b}_i$, $i > r_0$ of $\tilde{B}$ are linear combinations of the columns of $\tilde{\tilde{B}}$, i. e., there are r_0-vectors d_i, such that $-\tilde{b}_i = \tilde{\tilde{B}}\tilde{d}_i$, $r_0 < i \leqq m$. Denote

$$D = \begin{pmatrix} \tilde{\tilde{B}}^{-1}, \tilde{d}_{r_0+1}, \ldots, \tilde{d}_m \\ 0 \quad , E_{m-r_0} \end{pmatrix}.$$

Then, $\tilde{B}D = (E_{r_0}, 0)$ and, consequently, $B' = BD$ has all elements zero except for $b'_{k_i i}$, which are equal 1; $\langle A, B' \rangle$ is D-equivalent with $\langle A, B \rangle$. To complete the proof, we define Q as follows: $q_{ij} = -\alpha_{ij}$, $1 \leqq i \leqq r_0$, $1 \leqq j \leqq n$, $q_{ij} = 0$, $r_0 < i \leqq m$, $1 \leqq j \leqq n$. Then, the system $\langle A', B' \rangle$ with $A' = A + B'Q$ is F-equivalent with $\langle A, B \rangle$ and has the required form.

Proof of theorem 1. The F-equivalence of the systems with the same R (or P) follows directly from theorem 2.

To prove the opposite implication, it suffices to show that none of the C-, D-, Q-transformations changes the r-numbers of a controllable system.

For C- and D-transformations, this statement is trivial. If $\langle A', B \rangle$ is a Q-transform of $\langle A, B \rangle$, $A' = A + BQ$, then we have $A'^j B = (A + BQ)^j B = A^j B + G$, where G is an $n \times m$ matrix, whose columns are contained in $L_{j-1}\langle A, B \rangle$ and the statement follows by induction in j.

The system (6) of theorem 2 can be considered as a canonical form for a particular class of systems.

Let us also note that besides other things, theorem *1* justifies our restriction to controllable systems. Namely, it shows that there is no C-, D- or Q-transformation, which will make a controllable system from a non-controllable one and conversely.

It is apparent that for given m and n, there is only a finite number of equivalence classes. From lemma 1 and theorem 1 it follows that the number of classes is equal to the number ϱ_{mn} of ways in which n can be written as a sum of n nonnegative integers (or the number of ways, in which n can be written as a sum of n nonnegative integers, not exceeding m). However, there is no formula known for the computation of ϱ_{mn}. The problem of finding the numbers ϱ_{mn} is an old numbertheoretical problem called "partition problem". It goes back to Euler, who gave a "generating function" for ϱ_{mn}, which is

$$F_m(x) = \frac{1}{(1 - x)(1 - x^2)\ldots(1 - x^m)}.$$

This means that if we expand F formally into Taylor series, $F_m(x) = 1 + \sum_{n=1}^{\infty} \varrho_{mn} x^n$, than ϱ_{mn} are the numbers, defined above. For details, cf. [10].

The following two corollaries illustrate the usefulness of theorems 1 and 2. For other applications of theorem 2 (or, rather, lemma 2) the reader is referred to [6], [7].

Corollary *1* (cf. [1], [2], [7], [8], [9]). *To any n-th degree polynomial $P(\lambda)$ and any controllable system $\langle A, B\rangle$, there is a system $\langle A', B\rangle$, Q-equivalent with $\langle A, B\rangle$, such that the characteristic polynomial of A' is $P(\lambda)$. In particular, every controllable system can be stabilized by an appropriate linear feedback.*

Let us note, that also the converse is true, but we shall not prove it here (cf. [1], [2], [9]).

Proof. Since C- and D-transformations do not change the characteristic polynomial of A, we can assume that $\langle A, B\rangle$ is in the canonical form (6). Then, we define the Q-transformation by $v_i = x_{k_i+1} + w_i$, $1 \leqq i \leqq r_0 - 1$, $v_{r_0} = -(\beta_1 x_1 + \ldots + \beta_n x_n) + w_{r_0}$, where $P(\lambda) = \lambda^n + \beta_1 \lambda^{n-1} + \ldots + \beta_{n-1}$.

Corollary 2 (cf. [7]). *If $\langle A, B\rangle$ is controllable, then for any non-zero vector $f \in L_0\langle A, B\rangle$, there is an $m \times n$-matrix Q such that $\langle A + BQ, f\rangle$ is controllable.*

Proof. We can obviously again assume that $\langle A, B\rangle$ is in the canonical form (6). Let $f = \sum_{i=1}^{r_0} \lambda_i b_i$, and let $\lambda_l \neq 0$. Then, if $l \neq r_0$, we define

$$q_{ij} = \begin{cases} 1 \text{ if either } i \neq l, i \neq r_0, j = k_i + 1, \text{ or } i = r_0, j = 1, \\ 0 \text{ otherwise.} \end{cases}$$

If $l = r_0$, we define

$$q_{ij} = \begin{cases} 1 \text{ if } j = k_i + 1,\ i \neq r_0\,, \\ 0 \text{ otherwise}\,. \end{cases}$$

It can be easily verified that the matrix $H = (f, A'f, \ldots, A'^{n-1}f)$, where $A' = A + BQ$, has the following form:

$$H = \begin{pmatrix} H_{11}, 0 \\ H_{21}, H_{22} \end{pmatrix}$$

where H_{11} is $k_l \times k_l$, both H_{11} and H_{22} are supper triangular with λ_l on the diagonal. Consequently, H is non-singular, q. e. d.

3. TIME-VARYING SYSTEMS

In this section, we apply some of the ideas of the preceding section to time-varying systems.

We consider systems

$$\dot{x} = A(t)\,x + B(t)\,u \tag{7}$$

where $A(t)$ and $B(t)$ are defined and of continuity class $\mathscr{C}^\infty$ on some interval J.

For time-varying systems, we modify the concept of F-equivalence, by allowing the matrices C, Q, D to be time-varying. So, (7) and

$$\dot{y} = A'(t)\, y + B'(t)\, v \tag{8}$$

will be called F-equivalent on J, if there are matrices $C(t)$, $Q(t)$ and $D(t)$ on J, all of class $\mathscr{C}^\infty$, such that the transformations $u = Qx + Dv$ and $x = Cy$ bring (7) into (8). Translated completely into the language of time-varying matrices, defining the system, $\langle A, B\rangle$ will be said to be F-equivalent with $\langle A', B'\rangle$ on J, if there are $\mathscr{C}^\infty$-matrices C, Q, D on J such that $A' = C^{-1}[(A + BQ) - \dot{C}]\, C$, $B' = C^{-1}BD$ for $t \in J$.

It can again be easily verified that F-equivalence is actually an equivalence relation. In a similar way, the definitions of C-, Q-, D-equivalence can be modified.

There is no reason to expect that a classification of controllable time varying systems similar to that of autonomous ones can be obtained, whatever geometric definition of controllability we use. This is partly due to the fact, that for none of those concepts of controllability there is an equivalent algebraic condition, corresponding to (2).

However, there is a condition for time-varying systems, generalizing (2), which implies controllability in any reasonable geometric sense. This condition can be formulated as follows:

For F being a $\mathscr{C}^\infty$ $n \times s$-matrix function for some s, denote $\mathfrak{A}\, F(t) = A(t)\,(Ft) + \dot{F}(t)$. Then, for every t,

$$\operatorname{rank}(B, \mathfrak{A}B, \ldots, \mathfrak{A}^{n-1}B) = n\,. \tag{9}$$

Let us now ask the following question: When is a time-varying system F-equivalent to an autonomous one? Those systems, which are equivalent to autonomous ones, we are of course able to classify. It turns out, that the systems, F-equivalent to autonomous ones, are precisely those, which satisfy a somewhat strengthened condition (9).

For time-varying systems, we define $R\langle A, B\rangle$ as the n-tuple of functions $\{r_j(t)\}_{j=0}^{n-1}$ on J, where

$$r_j(t) = \operatorname{rank}(B(t), \ldots, \mathfrak{A}^j B(t)) - \operatorname{rank}(B(t), \ldots, \mathfrak{A}^{j-1} B(t))\,.$$

$P\langle A, B\rangle$ is the m-tuple of functions $p_i(t)$ defined for every t by (5). L_j, Λ_j and π_j are defined similarly as for autonomous systems with A replaced by $\mathfrak{A}$. They also depend on t.

Theorem 3. *The system $\langle A, B\rangle$ is F-equivalent with a controllable autonomous system on J if and only if the functions $r_0(t), \ldots, r_{n-1}(t)$ from $R\langle A, B\rangle$ are constant on J and $r_0(t) + \ldots + r_{n-1}(t) = n$.*

For better orientation, we divide the proof of this theorem into several lemmas.

Lemma 3. *Denote* $S_j(t) = \{\mathfrak{A}^\nu b_i(t) | 0 \leqq \nu \leqq j, i \in M_\nu\}$, *where* M_ν *are subsets of* $\{1, \ldots, m\}$. *Let* $S_j(t_0)$ *be a basis of* $L_j(t_0)$. *Then,*

(i) $S_j(t)$ *is a basis of* $L_j(t)$ *in a neighbourhood* U *of* t_0.

(ii) *If* $f(t) \in L_j(t)$ *and* f *is* $\mathscr{C}^\infty$ *on* U, *then*

$$f(t) = \sum_{\substack{0 \leqq \nu \leqq j \\ i \in M_\nu}} \gamma_{i\nu} \mathfrak{A}^\nu b_i(t)$$

where $\gamma_{i\nu}$ *are* $\mathscr{C}^\infty$ *functions on* U *and* $b_i(t)$ *are the columns of* $B(t)$.

Proof. Let $\varrho_j = \sum_{\nu \leqq j} r_\nu$. Then, there is a $\varrho_j \times \varrho_j$ nonsingular submatrix $\tilde{S}_j(t_0)$ of $S_j(t_0)$. Since $\det \tilde{S}_j(t)$ is continuous, we have $\det \tilde{S}_j(t_0) \neq 0$ in some neighbourhood U of t_0. Since $\gamma_{i\nu}$ can be expressed as polynomials of the entries of $f(t)$ and $S_j(t)$ divided by $\det \tilde{S}_j(t)$, the lemma follows.

Chose a norm $\|.\|$ in R^n. As an immediate consequence of lemma 3 we obtain

Lemma 4. *Let the functions of* $R\langle A, B\rangle$ *be constant on* J. *Then, the subspaces* $L_j(t)$ *and* $\Lambda_j(t)$ *depend continuously on* t *on* J *in the following sense:*

To any ball $N_R = \{x | \|x\| \leqq R\}$ in R^n, *any* $t_0 \in J$ *and* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that for* $|t - t_0| < \delta$,

$$L_j(t) \cap N_R\,, \quad L_j(t_0) \cap N_R\,, \quad \Lambda_j(t) \cap N_R\,, \quad \Lambda_j(t_0) \cap N_R$$

are contained in the ε*-neighbourhoods of* $L_j(t_0)$, $L_j(t)$, $\Lambda_j(t_0)$, $\Lambda_j(t)$ *respectively.*

Lemma 5. *If* $\langle A, B\rangle$ *and* $\langle A, B'\rangle$ *are D-equivalent, then* $L_j\langle A, B\rangle(t) = L_j\langle A, B'\rangle(t)$ *and* $\Lambda_j\langle A, B\rangle(t) = \Lambda_j\langle A, B'\rangle(t)$ *for all* $t \in J$.

Proof. We have $\mathfrak{A}^j b'_i = \sum_{\mu=1}^m \mathfrak{A}^j(b_\mu d_{\mu i}) = \sum_{\mu=1}^m (\mathfrak{A}^j b_\mu)\, \mathrm{d}_{\mu i} - \sum_{\mu=1}^m (\mathfrak{A}^{j-1} b_\mu)\, \dot{d}_{\mu i}$. From this we obtain by induction $L_j\langle A, B'\rangle \subset L_j\langle A, B\rangle$, $0 \leqq j \leqq n - 1$. By symmetry of D-equivalence, we have the opposite inclusion and, thus, equality. The equality of Λ_j's follows trivially.

Lemma 6. *Let for some* t, $S_j(t)$ *be a pyramidal basis of* $L_j(t)$. *Then,* $S_j(t)$ *can be completed into a pyramidal basis of* $L_{n-1}(t)$.

Proof. We prove that $S_j(t)$ can be extended into a pyramidal basis $S_{j+1}(t)$ of $L_{j+1}(t)$. The rest follows then by induction.

Let $\mathfrak{A}^j b_i \in S_j(t)$, if and only if $i \in M_j$. Then we have for every b_k, $1 \leqq k \leqq m$, $\mathfrak{A}^j b_k = \sum_{i \in M_j} \lambda_i \mathfrak{A}^j b_i + g_k$, where $g_k \in L_{j-1}(t)$.

Thus,

$$\mathfrak{A}^{j+1} b_k = \sum_{i \in M_j} \lambda_i \mathfrak{A}^{j+1} b_i - \sum_{i \in M_j} \dot{\lambda}_i \mathfrak{A}^j b_i + \mathfrak{A} g_k = \sum_{i \in M_j} \lambda_i \mathfrak{A}^{j+1} b_i + f_k$$

where $f_k \in L_j(t)$. From this it is clear that to complete the basis for $S_{j+1}(t)$, we can add to $S_j(t)$ any $r_{k+1}(t)$ linearly independent vectors from $\mathfrak{A}^{j+1}b_i$, $i \in M_j$, q. e. d.

Corollary 3. $m \geqq r_0(t) \geqq r_1(t) \geqq \ldots \geqq r_{n-1}(t) \geqq 0$ *for all* t.

Lemma 7. *Let the functions of* $R\langle A, B\rangle$ *be constant on* J. *Then,* $\langle A, B\rangle$ *is D-equivalent with a system* $\langle A, B'\rangle$ *such that*

$S_i'(t) = \{\mathfrak{A}^\nu b_i'(t) | 0 \leqq \nu \leqq j,\ \ 1 \leqq i \leqq r_\nu\}$ *are bases of* $L_j(t)$ *for* $t \in J$.
Note that the theorem of [11] is a special case of lemma 7.

Proof. The matrix $D(t)$ will be constructed as a product of n matrices $D = D_0 \ldots \ldots D_{n-1}$ in such a way that for $\langle A, BD_0 \ldots . D_k\rangle$ the statement of the lemma will be valid for $j \leqq k$.

Assume that we have already constructed the matrices $D_0, \ldots, D_{k-1}$. In virtue of lemmas 3, 6 we can cover J by a sequence of open intervals $J_\mu = (a_\mu, b_\mu)$, $-\infty < \mu < \infty$ in such a way that $\frac{1}{2}(a_\mu + b_\mu)$ is an increasing sequence, $b_\mu < a_{\mu+2}$ for all μ and for every μ there is a subset $M_\mu \subset \{1, \ldots, r_{k-1}\}$ such that $\{\pi_k(\mathfrak{A}^k b_i(t)) | i \in M_\mu\}$ span $A_k(t)$ for $t \in J_\mu$. D_k will again be constructed in steps. First we define $D_k(t)$ on $[b_{-1}, a_1]$ as follows.

We put $d_i = e_{\sigma i}$, $1 \leqq i \leqq r_k$, $d_i = e_i$ for $r_{k-1} < i \leqq m$ and the remaining d_i's we put equal to the remaining e_i's arbitrarily. There $M_0 = \{\sigma_1, \ldots, \sigma_{r_k}\}$, d_i, e_i are the columns of D_k, E_m respectively. By multiplication of B by D_k, the columns $\{b_i | i \in M_0\}$ are brought into the first r_k positions and the columns b_i, $i > r_{k-1}$ remain without change.

We proceed by introduction assuming that $D_k(t)$ has been constructed on $[b_{-\mu}, a_\mu]$ with following properties: (which are obviously satisfied for $\mu = 1$).

Denote $\tilde{D}_\mu(t) = D_k(t)$ for $t \in [b_{-\mu}, a_\mu]$, $\tilde{D}_\mu(t) = D_k(b_{-\mu})$ for $t < b_{-\mu}$, $\tilde{D}_\mu(t) = D_k(a_\mu)$ for $t > a_\mu$, $\tilde{B}(t) = B(t)\,\tilde{D}_\mu(t)$. Then:

(i) $D_\mu(t)$ is nonsingular and $\mathscr{C}^\infty$ for all t

(ii) The vectors $\{\mathfrak{A}^\nu \tilde{b}_i | 0 \leqq \nu \leqq k - 1,\ 1 \leqq i \leqq r_\nu\}$ span L_{k-1} for all t

(iii) The vectors $\{\pi_k(\mathfrak{A}^j b_i) | 1 \leqq i \leqq r_k\}$ span $A_k(t)$ for $a_{-\mu+1} \leqq t \leqq b_{\mu-1}$ and there are subsets M_μ, $M_{-\mu}$ of $\{1, \ldots, r_{k-1}\}$ such that

$$\{\pi_k(\mathfrak{A}^k \tilde{b}_i) | i \in M_\mu\},\ \{\pi_k(\mathfrak{A}^k \tilde{b}_i) | i \in N_\mu\} \text{ span } A_k(t) \text{ for } t \in [a_\mu, b_\mu]$$

and $t \in [a_{-\mu}, b_{-\mu}]$ respectively.

We show that $D_k(t)$ can be extended so the interval $[b_{-\mu+1}, a_{\mu+1}]$ in such a way that (i)–(iii) remains valid with μ replaced by $\mu + 1$. We show only the extension forwards, the extension backwards being entirely similar.

We divide $[a_\mu, b_{\mu-1}]$ into r_k subintervals of equal length $\tau = r_j^{-1}(b_{\mu-1} - a_\mu)$ and denote $\alpha_\zeta = a_\mu + \zeta\tau$. Again $D_k(t)$ will be extended by induction. We assume that $D_k(t)$ has been extended for $t \leqq a_{\zeta-1}$, $\zeta \leqq n$ in such a way that if we denote

$$\bar{D}_{\zeta-1}(t) = D_k(t) \text{ for } t \leqq \alpha_{\zeta-1},\ \bar{D}_{\zeta-1}(t) = D_k(\alpha_{\zeta-1}) \text{ for } t \geqq \alpha_{\zeta-1}\,,$$

$\bar{B}(t) = B(t)\,\bar{D}_{\zeta-1}(t)$, then we have

(a) $\bar{D}_{\zeta-1}(t)$ is $\mathscr{C}^\infty$, nonsingular for all t,

(b) The vectors $\{\mathfrak{A}^\nu \bar{b}_i | 0 \leqq \nu \leqq k-1,\ 1 \leqq i \leqq r_\nu\}$ span $L_{k-1}(t)$ for all t,

(c) The vectors $\{\pi_k(\mathfrak{A}^k \bar{b}_i) | 1 \leqq i \leqq r_k\}$ span $\Lambda_k(t)$ for $t \leqq b_{\mu-1}$,

(d) There is a subset N_ζ of $(\zeta, \ldots, r_{k-1}\}$ such that

$\{\pi_j(\mathfrak{A}^j \bar{b}_i) | 1 \leqq i < \zeta \text{ or } i \in N_\zeta\}$ is a basis of $\Lambda_k(t)$ for $\alpha_{\zeta-1} \leqq t \leqq b_\mu$.

We show that $D_k(t)$ can be extended for $t \leqq \alpha_\zeta$ so that (a)–(d) remain valid with $\zeta - 1$ replaced by ζ.

From (c) and (d) it follows that there is an $l \in N_\zeta$ such that $\mathfrak{A}^j \bar{b}_l$ is not contained in the subspace $\Delta(t)$ of $\Lambda_k(t)$, spanned by $\{\pi_k(\mathfrak{A}^k \bar{b}_i) | 1 \leqq i \leqq r_k,\ i \neq \zeta\}$ for any $t \in [\alpha_{\zeta-1}, b_{\mu-1}]$. If $l = \zeta$, we simply define $D_k(t) = \bar{D}_{\zeta-1}$ for $t \in [\alpha_{\zeta-1}, \alpha_\zeta]$, i. e. we extend $D_k(t)$ continuously as a constant. If $l \neq \zeta$ (note that then $l > r_k$) then $\pi_k(\mathfrak{A}^k \bar{b}_\zeta)$ and either $\pi_k(\mathfrak{A}^k \bar{b}_l)$ or its negative lie in the interior of the same one of the two halfspaces, into which $\Delta(t)$ divides $\Lambda_k(t)$, for all $t \in [\alpha_{\zeta-1}, \alpha_\zeta]$. In the first case, we define for $t \in [\alpha_{\zeta-1}, \alpha_\zeta]$, $D_k(t) = \bar{D}(t)\, Z(t)$, where $Z(t) = (z_{ij}(t))$ is defined as follows:

$$z_{\zeta\zeta}(t) = 1 - \psi(\alpha_{\zeta-1} + t)\,, \quad z_{\zeta l} = -\psi(\alpha_{\zeta-1} + t)\,, \tag{10}$$

$$z_{l\zeta}(t) = \psi(\alpha_{\zeta-1} + t),\ z_{ll} = 1 - \psi(\alpha_{\zeta-1} + t)\,,$$

$z_{ij}(t) = \delta_{ij}$ otherwise and $\psi(t)$ is a nonnegative $\mathscr{C}^\infty$ real function such that $\psi(0) = 0$ for $t \leqq 0$, $\psi(t) = 1$ for $t \geqq \tau$, $0 \leqq \psi(t) \leqq 1$ for $0 \leqq t \leqq \tau$. In the second case, we define $z_{\zeta l}$ and $z_{l\zeta}$ with opposite signs.

Now, the validity of (a) for ζ is obvious. For the proof of (b)–(d) denote $\bar{\bar{B}}(t) = B(t)\,\bar{\bar{D}}(t)$, where $\bar{\bar{D}}$ is defined as $\bar{D}(t)$ with $\zeta - 1$ replaced by ζ. We have

$$\mathfrak{A}^\nu \bar{\bar{B}} = \mathfrak{A}^\nu(\bar{B}Z) = (\mathfrak{A}^\nu \bar{B})\, Z + F$$

where $F \in L_{\nu-1}(t)$, which implies

$$\pi_\nu(\mathfrak{A}^\nu \bar{\bar{B}}) = (\pi_\nu(\mathfrak{A}^\nu \bar{B}))\, Z\,. \tag{11}$$

If we denote $\hat{\bar{B}}$, $\hat{\bar{\bar{B}}}$ the submatrices of $\bar{B}$, $\bar{\bar{B}}$ respectively, formed by their first r_ν columns and $\hat{Z}_\nu$ the submatrix of Z, formed by its first r_ν columns and r_ν rows then, because all elements of the first r_ν columns, not belonging to $\hat{Z}_\nu$, are zero, we have from (11)

$$\pi_\nu(\mathfrak{A}^\nu \bar{\bar{B}}_\nu) = (\pi_\nu(\mathfrak{A}^\nu \bar{B}_\nu))\, Z_\nu \text{ for } 0 \leqq \nu \leqq k-1$$

which implies (b).

(c) follows from the fact that by (10) and (11), $\pi_k(\mathfrak{A}^k\bar{b}_\zeta)$ is for all t a convex combination of two vectors, both of which lie in the interior of the same one of the half-spaces, into which $\Delta(t)$ divides $\Lambda_k(t)$ and the other vectors of $\{\pi_k(\mathfrak{A}^k\bar{b}_i)|1 \leqq i \leqq r_k\}$ are not affected by multiplication of $\bar{B}$ by Z.

(d) follows from the fact that for $t \geqq a_\zeta$, $\pi_k(\mathfrak{A}^k\bar{b}_l) = \pi_k(\mathfrak{A}^k\bar{b}_\zeta)$.

After extending $D_k(t)$ for $t \leqq b_{\mu-1}$ stepwise in the described way, we extend $D_k(t)$ for $t \leqq a_{\mu+1}$ by setting $D_k(t) = D_k(b_{\mu-1})$ for $t \in [b_{\mu-1}, a_{\mu+1}]$. Then, it is easy to verify that D_k satisfies (i)–(iii) for $t \leqq a_{\mu+1}$.

This, by introduction, proves the existence of $D_k(t)$ and, thus, of $D(t)$.

Remark 1. By a refinement of the above argument it can be shown that if the matrix $(B, \mathfrak{A}B, \ldots, \mathfrak{A}^{n-1}B)$ and all its p derivatives $(p = \max_{1 \leqq i \leqq m} p_i)$ are bounded on J and for each t there is an $n \times n$-subdeterminant of this matrix, the absolute value of which is bounded below by a positive constant, independent of t on J, then $D(t)$ can be constructed in such a way that $(B', \mathfrak{A}B', \ldots, \mathfrak{A}^{n-1}B')$ is bounded and $|\det S_{n-1}(t)|$ is bounded below by a positive constant, independent of t. This will be seen to be important for the stabilization problem.

The refinement is essentially based on the facts, that under the above boundedness assumptions the covering $\{J_\mu\}$ of J, which occurs in the proof of Lemma 7 can be for every k constructed in such a way that on every interval J_μ the absolute value of some pyramidal basis and also the length of the intersections of the consecutive intervals are bounded below by a positive constant, which is independent on μ.

Lemma 8. *Let for all $t \in J$, the vectors $\{\mathfrak{A}^\nu b_i|0 \leqq \nu \leqq n-1,\ 1 \leqq i \leqq r_\nu\}$ be linearly independent. Then, $\langle A, B\rangle$ is C-equivalent with a system $\langle A', B'\rangle$, where A', B' have the form of lemma* 1 *with α's and γ's time-dependent.*

In essence, this lemma is proved in [4], but we shall give an alternate proof, which is modeled after the proof of lemma 2 of this paper from [6].

Proof. Let $P\langle A, B\rangle = \{p_i|1 \leqq i \leqq m\}$, $k_i = \sum_{\nu=1}^{i} p_\nu$ and let $\tilde{C}$ be the submatrix of C, consisting of its k_i-th columns, G the submatrix of B', consisting of its k_i-th rows. Then, C, G have to satisfy

$$(12) \qquad AC = CA' + \dot{C}\,, \quad B = \tilde{C}G\,,$$

G being triangular, so is G^{-1}. We define $G^{-1} = (\gamma_{\nu i})$ as follows: $\gamma_{\nu i}$ for $1 \leqq \nu < r_{p_i}$ are the unique nunbers such that

$$\pi_{p_i}(\mathfrak{A}^{p_i}b_i + \sum_{\nu=1}^{r_{p_i}} \gamma_{\nu i}\mathfrak{A}^{p_i}b_\nu) = 0\,,$$

$\gamma_{ii} = 1$ and all remaining γ's are equal to zero.

From lemma 3 it follows that $\gamma_{\nu i}$ are $\mathscr{C}^\infty$. Since $\tilde{C} = BG^{-1}$, we have $\pi_{p_i}(\mathfrak{A}^{p_i}c_{k_i}) = \pi_{p_i}(\mathfrak{A}^{p_i}b_i - \sum_{\nu=1}^{r_{p_i}} \gamma_{\nu i}\mathfrak{A}^{p_i}b_\nu) = 0$, or $\mathfrak{A}^{p_i}c_{k_i} \in L_{p_i-1}$. Moreover,

$$\{\mathfrak{A}^\nu c_{k_i} | 0 \leqq \nu \leqq j\,,\quad 1 \leqq i \leqq r_\nu\} \tag{13}$$

are bases of L_j.

Decomposing the first equality of (12) into columns, we obtain

$$c_{k_{i-1}+j} = \mathfrak{A}c_{k_{i-1}+j+1} - \sum_{\nu=1}^{r_0} \alpha_{\nu k_{i-1}+j+1} c_{k_\nu}\,, \quad j = 1, \ldots, p_i - 1, i = 1, \ldots, m\,,$$

$$0 = \mathfrak{A}c_{k_{i-1}+1} - \sum_{\nu=1}^{r_0} \alpha_{\nu k_{i-1}+1} c_{k_\nu}\,, \quad i = 1, \ldots, m\,.$$

Consequently

$$c_{k_{i-1}+j} = \mathfrak{A}^{p_i-j}c_{k_i} - \sum_{\mu=1}^{p_i-j} \mathfrak{A}^{p_i-j-\mu} \sum_{\nu=1}^{r_0} \alpha_{\nu k_i-\mu+1} c_{k_\nu}\,, \tag{14}$$

$$0 = \mathfrak{A}^{p_i}c_{k_i} - \sum_{\mu=1}^{p_i} \mathfrak{A}^{p_i-\mu} \sum_{\nu=1}^{r_0} \alpha_{\nu k_i-\mu+1} c_{k_\nu}\,. \tag{15}$$

Since for any $\mathscr{C}^\infty$ $\alpha(t)$ scalar and $b(t)$ n-vector functions

$$\mathfrak{A}^j(\alpha b) = \sum_{\mu=0}^{j} \binom{j}{\mu} (-1)^{j-\mu} \frac{\mathrm{d}^{j-\mu}\alpha}{\mathrm{d}t^{j-\mu}} \mathfrak{A}^\mu b \tag{16}$$

is valid, we can re write (15) as

$$\mathfrak{A}^{p_i}c_{k_i} = \sum_{j=0}^{p_i-1} \sum_{\nu=1}^{r_0} \sum_{\mu=0}^{j} \binom{j}{\mu} (-1)^{j-\mu} \alpha^{(j-\mu)}_{\nu k_{i-1}+j+1} \mathfrak{A}^\mu c_{k_\nu}$$

$$= \sum_{\mu=0}^{p_i-1} \sum_{\mu=1}^{r_0} \mathfrak{A}^\mu c_{k_\nu} \sum_{j=\mu}^{p_i-1} \binom{j}{\mu} (-1)^{j-\mu} \alpha^{(j-\mu)}_{\nu k_{i-1}+j+1}\,.$$

If we define $\varphi_{\nu\mu} \equiv 0$ for $\mu \geqq p_\nu$, then by (13) $\varphi_{\nu\mu}$, $1 \leqq \nu \leqq r_0$, $0 \leqq \mu \leqq p_i - 1$ are uniquely determined by the equation

$$\mathfrak{A}^{p_i}c_{k_i} = \sum_{\mu=0}^{p_i-1} \sum_{\nu=1}^{r_0} \mathfrak{A}^\mu c_{k_\nu} \varphi_{\nu\mu}$$

and they are $\mathscr{C}^\infty$ on J.

$\varphi_{\nu\mu}$ being known, $\alpha_{\nu\zeta}$ can be determined by solving r_0 triangular systems with 1's in the diagonal

$$\sum_{j=\mu}^{p_i-1} \binom{j}{\mu} (-1)^{j-\mu} \alpha^{(j-\mu)}_{\nu k_{i-1}+j+1} = \varphi_{\nu\mu}\,.$$

Obviously, the α's obtained from these equations are also $\mathscr{C}^\infty$ on J. The α's being known, we can determine C by (14) and it can be readily verified that (12) is satisfied.

Since by (14) and (16), $c_{k_{i-1}+j} = \mathfrak{A}^{p_i-1-j}c_{k_i} + f$, where $f \in L_{p_i-j}$, it follows from (13) that C is nonsingular.

Remark 2. It can be easily checked that if we want the α's and γ's to be merely continuous, it is sufficient to assume that $(B, \mathfrak{A}B, \ldots, \mathfrak{A}^{n-1}B)$ has $\max p_i - 1$ derivatives. Moreover, if these derivatives are bounded and $|\det S(t)|$, where $S(t)$ is the matrix, consisting of the columns $\{\mathfrak{A}^{\nu}b_i | 0 \leqq \nu \leqq n-1, 1 \leqq i \leqq r\}$ is bounded below by a positive constant, C, C^{-1} and A', B' are also obtained bounded.

Proof of theorem 3. If: By lemma 7 and 8, $\langle A, B \rangle$ can be brought to $\langle A', B' \rangle$ with A', B' having the special form of lemma 2 with α's and γ's time dependent. In the same way as in theorem 2, we construct Q (which will be, of course, time dependent) in such a way that $\langle A', B' \rangle$ will be Q-equivalent to the decoupled system of r_0 integrators (6), which is autonomous.

Only if: Lemma 5 proves that the r-numbers are invariants of a D-transformation. The theorem will be proved if we show that they are also invariant of the C- and Q-transformation.

For the C-transformation if follows from $\mathfrak{A}'f' = C^{-1}(AC - \dot{C})\, C^{-1}f - \dot{C}^{-1}f + C^{-1}\dot{f} = C^{-1}\,\mathfrak{A}f$ for every $\mathscr{C}^{\infty}$ vector function f, where $\langle A', B' \rangle$ is the C-transform of $\langle A, B \rangle$ and $\mathfrak{A}' = (A' - \mathrm{d}/\mathrm{d}t)$.

For the F-transformation we note that if $\langle A', B \rangle$ is a Q-transform of $\langle A, B \rangle$, then $\mathfrak{A}'f = (A + BQ)f - \dot{f} = \mathfrak{A}f + BQf$, from which it follows $L_j < A', B > \subset L_j\langle A, B \rangle$. From the symmetry of Q-equivalence it follows $L_j\langle A, B \rangle \subset L_j\langle A', B \rangle$ and, thus, $L_j\langle A, B \rangle = L_j\langle A', B \rangle$, q. e. d.

Those systems, which satisfy the assumptions of theorem 3 we shall call autonomous-equivalent, or A-systems.

We have also proved

Theorem 4. *Two A-systems $\langle A, B \rangle$ and $\langle A', B' \rangle$ are F-equivalent if and only if $R\langle A, B \rangle = R\langle A', B' \rangle$. They are both equivalent to the canonical system* (6).

Corollary 4. *For autonomous systems, we obtain the same classification, whether we allow the transformation matrices to be time-dependent or not.*

Corollary 5. *To any n-th order polynomial $P(\lambda)$ and any A-system $\langle A, B \rangle$ there is an autonomous system $\langle A', B' \rangle$, F-equivalent with $\langle A, B \rangle$ such that $P(\lambda)$ is the characteristic polynomial of A'* (cf. [4]).

Remark 3. Corollary 5 is of use for the stabilization problem only if $J = [t_0, \infty)$ and the transformation matrices and the inverses of the C- and D-matrices are bounded. In virtue of remarks 1 and 2, this will be true if the assumptions of remark 1 are satisfied on some interval $J = [t_0, \infty)$. Let us also note that for the stabilization problem solely, it is sufficient to assume that the matrix $(B, \mathfrak{A}B, \ldots, \mathfrak{A}^{p-1}B)$, $p =$

$= \max_{1 \leq i \leq m} p_i$ has $n - 1$ bounded continuous dérivatives. Our results, therefore, improve the result of [4], where it is in essence assumed that the system satisfies the assumptions of Remark 2. We assume that A, B are $\mathscr{C}^\infty$ merely for the classification theory. Namely, if $\langle A, B \rangle$ is not assumed to be $\mathscr{C}^\infty$, then the system $\langle A', B' \rangle$ of lemma 8 is obtained with much fewer continuous derivatives, which is inconvenient.

Corollary 6. *Let $\langle A, B \rangle$ be an A-system and let $f(t)$ bea $\mathscr{C}^\infty$ vector function such that for all $t \in J$, $f(t) \in L_0\langle A, B \rangle (t)$, Then, there is a system $\langle A', B' \rangle$, F-equivalent with $\langle A, B \rangle$, such that $\langle A', f \rangle$ is an A-system.*

The idea of proof is the same as that of the proof of Corollary 2; we omit the details.

4. REMARKS ON DISCRETE AND PERIODIC SYSTEMS

For discrete systems, autonomous as well as time-dependent, a similar classification theory can be developed. We are not going to formulate the results, which can be drawn from those of $2-3$ by simple analogies.

Time-varying periodic systems with continuous time can be regarded as a particular case of general time-varying systems. It is natural to require from the transformation matrices to be periodic in this case but this requirement does not introduce new difficulties.

Sometimes, however, one does not need the information about the behaviour of the system for all t, but only about its behaviour in discrete moments, the period of the system apart. For instance the stability properties of the system are completely determined by the discretized system.

To make this point more precise, assume that the matrices of the system $\langle A, B \rangle$ are T-periodic and of class $\mathscr{C}^1$. The solutions of the system $\dot{x} = Ax + Bu$ can be expressed as

$$x(t) = Y(t)\, x(0) + \int_0^t Y(t)\, Y(-s)\, B(s)\, u(s)\, \mathrm{d}s$$

where $Y(t)$ is the fundamental matrix of $\dot{y} = Ay$ with initial condition $Y(0) = E$. Then, the corresponding discrete system is

$$\xi_{k+1} = Y(T)\, \xi_k + F(\eta_k)$$

where $\xi_k = x(kT)$, η_k is the piece of control function $u(s)$, $kT \leqq s \leqq (k+1)\, T$ and F is the linear operator, mapping η_k into

$$\int_0^T Y(T)\, Y(-s)\, B(0)\, u(kT + s)\, \mathrm{d}s\, .$$

It can be immediately seen that if we try to use the concept of D-transformation to this system, difficulties arise. Namely, a general linear transformation in the η-space would involve the future values of the control, which is physically unthinkable.

However, there is a result of the type of corrollary 1, which may be worthwhile to mention in this context:

Theorem 5. *A T-periodic system $\langle A, B\rangle$ with $\langle A, B\rangle$ bein $\mathscr{C}^1$ is controllable if and only if to every n-th degree polynomial $P(\lambda)$ with positive absolute term there is a T-periodic $m \times n$ matrix Q (which can be choosen piecewise constant or arbitrarily smooth) such that the characteristic multipliers of the system $\dot{y} = (A + BQ)\, y$ are exactly the roots of $P(\lambda)$. In particular, periodic controllable systems can always be stabilized by an appropriate periodic feedback.*

This theorem is in a sense stronger than corollary 4, since controllability in it is assumed only in its weakest geometrical sense, i. e. that we can join any two points in R^n by a trajectory of the system in sufficiently long time, with the aid of an appropriate control.

For the proof of theorem 5, see [7].

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Klasifikácia lineárnych riaditeľných sústav

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Vyšetruje sa relácia F-ekvivalencie a na nej založená klasifikácia lineárnych riaditeľných sústav. Dve lineárne sústavy riadenia $\dot{x} = Ax + Bu$ a $\dot{y} = Ay + Bv$ (A, B, A', B' konštantné), splňujúce predpoklad riaditeľnosti (2) sa nazývajú F-ekvivalentné, ak existujú matice Q, C, D (C, D regulárne) také, že je splnené (3). Odvozuje sa nutná a postačujúca podmienka ekvivalencie sústav, spočívajúce v rovnosti konečného počtu čísel, zviazaných s maticami A, B, resp. A', B'. Ukazuje sa, že v každej triede ekvivalencie existuje kanonická sústava, pozostávajúca z nezávislých integrátorov, ktorých počet a rády úplne charakterizujú triedu ekvivalencie.

Pojem F-ekvivalencie sa zovšeobecňuje pre sústavy závislé od času a to tak, že sa povoľujú matice Q, C, D závislé od času. Vyšetruje sa trieda sústav, F-ekvivalentných s časovo nezávislými sústavami – sú to práve tie, ktoré splňujú podmienku (9).

Nakoniec sa stručne diskutuje prípad sústav periodických a diskrétnych v čase.

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