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On Optimal Fault-Finding Strategy of Element-Measurement Method for Systems with Exactly One Failure

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Necessary and sufficient condition for the optimality of the strategy is given, provided the method used is confined to measurements of single elements and the system is known to contain exactly one failure.

Let the elements of the system be numbered by 1, 2, ..., n and let us denote by p_i ($p_i > 0$, $\sum_{i=1}^{n} p_i = 1$) the probability of the i-th element to be defective and by T_i ($T_i > 0$) the cost of the measurement of the i-th element.

A strategy δ is the n-tuple of indices 1, 2, ..., n that determines the order in which the elements are to be measured. Since only systems with exactly one defective element are considered, the fault-finding procedure ends whenever the defective element is determined. Thus, at most $n - 1$ measurements are performed.

For every strategy

(1) $$\delta = (i_1, i_2, \ldots, i_n)$$

the mean cost $V(\delta)$ is defined by

(2) $$V(\delta) = \sum_{j=1}^{n} p_{i_j} \sum_{k=1}^{j} T_{i_k} - p_{i_n} T_{i_n} = \sum_{j=1}^{n-1} T_{i_j} \sum_{k=j}^{n} p_{i_k} .$$

We say that the strategy δ dominates the strategies $\delta_1, \delta_2, \ldots, \delta_r$ if

(3) $$V(\delta) \leqq V(\delta_j) \quad \text{for} \quad 1 \leqq j \leqq r .$$

Further, we say that the strategy δ^* is optimal, if it dominates all other strategies, i.e. if

(4) $$V(\delta^*) = \min V(\delta) .$$

The problem of the present paper is to determine and characterize the optimal strategy by $p_1, p_2, \ldots, p_n$ and $T_1, T_2, \ldots, T_n$.

Such a characterization is given in [1], where the authors claim:

Kuznetsov-Ptchelintsev Theorem. *The necessary and sufficient condition for the strategy* $(1, 2, \ldots, n)$ *to be optimal is*

(5) $$\frac{p_1}{T_1} \geqq \frac{p_2}{T_2} \geqq \ldots \geqq \frac{p_{n-1}}{T_{n-1}}$$

and

(6) $$T_n \geqq T_k \quad for \quad 1 \leqq k \leqq n - 1 .$$

However, this theorem is valid for $n \leqq 2$ only; for $n \geqq 3$ the condition is neither necessary nor sufficient, what could be for $n = 3$ demonstrated by the following counter-example.

Example 1. Let

(7) $$p_a = \tfrac{3}{4}, \quad p_b = p_c = \tfrac{1}{8}, \quad T_a = 3, \quad T_b = T_c = 2 .$$

Then we have by (2)

(8) $$\begin{aligned} V(a, b, c) &= V(a, c, b) = \tfrac{14}{4}, \\ V(b, a, c) &= V(c, a, b) = \tfrac{37}{8}, \\ V(b, c, a) &= V(c, b, a) = \tfrac{15}{4}. \end{aligned}$$

Thus, setting $a = 1$, $b = 2$, $c = 3$, the necessity is contradicted and setting $a = 3$, $b = 2$, $c = 1$, the sufficiency is contradicted.

Though the characterization of the optimal strategy is simple and the proof of our result requires only elementary algebra, we have decided to state even trivial results as lemmas. It is hoped that such a detailed treatment will be appreciated by some readers.

We will call every interchange between two neighbour elements i_j and i_{j+1} the transposition, we will denote it by $\langle i_j \leftrightarrow i_{j+1} \rangle$ and will speak about a transposition of the type I if $1 \leqq j \leqq n - 2$ and about a transposition of the type II if $j = n - 1$.

The difference between the mean costs of the transposed and the original strategy will be denoted by $D(\langle i_j \leftrightarrow i_{j+1} \rangle)$, i.e.

(9) $$D(\langle i_j \leftrightarrow i_{j+1} \rangle) = V(i_1, \ldots, i_{j-1}, i_{j+1}, i_j, i_{j+2}, \ldots, i_n) - V(i_1, i_2, \ldots, i_n) .$$

Further, for the sake of brevity, we denote by δ_0 and ${}_k\delta_m$ the following strategies:

(10) $$\delta_0 = (1, 2, \ldots, n) ;$$

(11) $${}_k\delta_m = (i_1, i_2, \ldots, i_n) ,$$

where for $0 \leqq k < m \leqq n$ we set

$$(12)\qquad \begin{aligned} i_j &= j && \text{for} \quad 1 \leqq j \leqq k\,, \\ &= n && \text{for} \quad j = k+1\,, \\ &= j-1 && \text{for} \quad k+2 \leqq j \leqq m\,, \\ &= j && \text{for} \quad m+1 \leqq j \leqq n-1\,, \\ &= m && \text{for} \quad j = n\,, \end{aligned}$$

and for $1 \leqq m \leqq k \leqq n-1$ we set

$$(13)\qquad \begin{aligned} i_j &= j && \text{for} \quad 1 \leqq j \leqq m-1\,, \\ &= j+1 && \text{for} \quad m \leqq j \leqq k-1\,, \\ &= n && \text{for} \quad j = k\,, \\ &= j && \text{for} \quad k+1 \leqq j \leqq n-1\,, \\ &= m && \text{for} \quad j = n\,. \end{aligned}$$

Thus, in particular

$$(14)\qquad \begin{aligned} {}_k\delta_m &= (1, \ldots, k, n, k+1, \ldots, m-1, m+1, \ldots, n-1, m) \quad \text{for} \quad k < m\,, \\ &= (1, \ldots, m-1, m+1, \ldots, k, n, k+1, \ldots, n-1, m) \quad \text{for} \quad m \leqq k\,, \end{aligned}$$

$$(15)\qquad {}_{n-1}\delta_m = (1, \ldots, m-1, m+1, \ldots, n, m) \quad \text{for} \quad 1 \leqq m \leqq n-2\,,$$

$$(16)\qquad {}_s\delta_s = {}_{s-1}\delta_s\,,$$

and

$$(17)\qquad {}_{n-1}\delta_n = \delta_0\,.$$

Now, let us state the difference between mean costs of the transposed and the original strategy for both types of transpositions.

Lemma 1. *For* $1 \leqq j \leqq n-2$ *we have*

$$(18)\qquad D(\langle i_j \leftrightarrow i_{j+1} \rangle) = p_{i_j} T_{i_{j+1}} - p_{i_{j+1}} T_{i_j}\,.$$

Proof. Relation (18) follows immediately form (9) and (2).

Lemma 2. *We have*

$$(19)\qquad D(\langle i_{n-1} \leftrightarrow i_n \rangle) = (p_{i_{n-1}} + p_{i_n})(T_{i_n} - T_{i_{n-1}})\,.$$

Proof. Relation (19) follows immediately from (9) and (2).

These two trivial lemmas enable us already to state a necessary condition for the strategy δ to be optimal.

Lemma 3. *If the strategy δ is optimal then*

$$\frac{p_{i_j}}{T_{i_j}} \geqq \frac{p_{i_{j+1}}}{T_{i_{j+1}}} \quad for \quad 1 \leqq j \leqq n-2 \tag{20}$$

and

$$T_{i_n} \geqq T_{i_{n-1}} . \tag{21}$$

Proof. If δ is optimal, then

$$D(\langle i_j \leftrightarrow i_{j+1} \rangle) \geqq 0 \quad \text{for} \quad 1 \leqq j \leqq n-1 . \tag{22}$$

However, (22) yields for $1 \leqq j \leqq n-2$ by Lemma 1 (20) and for $j = n-1$ by Lemma 2 (21).

Next lemma deals with strategies ${}_{n-1}\delta_m$ which are explicitly written out in (15).

Lemma 4. *For $1 \leqq m \leqq n$ we have*

$$V({}_{n-1}\delta_m) - V(\delta_0) = p_m \sum_{j=m+1}^{n} T_j - T_m \sum_{j=m}^{n} p_j + p_n T_n . \tag{23}$$

Proof. The strategy ${}_{n-1}\delta_m$ can be obtained from the strategy δ_0 by subsequent transpositions $\langle m \leftrightarrow m+1 \rangle$, $\langle m \leftrightarrow m+2 \rangle, \ldots, \langle m \leftrightarrow n-1 \rangle$, $\langle m \leftrightarrow n \rangle$, so that

$$V({}_{n-1}\delta_m) - V(\delta_0) = \sum_{j=m+1}^{n} D(\langle m \leftrightarrow j \rangle) . \tag{24}$$

All transpositions being of the type I except for the transposition $\langle m \leftrightarrow n \rangle$ which is of the type II, we have by Lemma 1 and Lemma 2

$$\sum_{j=m+1}^{n} D(\langle m \leftrightarrow j \rangle) = \sum_{i=m+1}^{n-1} [p_m T_j - p_j T_m] + (p_m + p_n)(T_n - T_m) \tag{25}$$

which can be rewritten in the form (23).

To demonstrate that the element with the maximal cost has practically no relation to the optimal strategy we give two examples, both of which can be used as counter-examples for the Kuznetsov-Ptchelintcev Theorem for arbitrary $n \geqq 3$.

The first example shows that the element with the maximal cost T can be on the arbitrary place in the optimal strategy with the only exception to be the last but one.

Example 2. Let $1 \leqq r \leqq n-2$ and let

$$\begin{aligned} p_i &= 2c , \quad T_i = 1 \quad \text{for} \quad 1 \leqq i \leqq r-1 , \\ p_i &= 5c , \quad T_i = 3 \quad \text{for} \quad i = r , \\ p_i &= c , \quad T_i = 2 \quad \text{for} \quad r+1 \leqq i \leqq n , \end{aligned} \tag{26}$$

where

(27) $$c = 1/(n + r + 3) .$$

Then the strategy $(1, \ldots, n)$ is optimal and the element with the maximal cost is in the r-th place.

Proof. Since

(28) $$2c/1 > 5c/3 > c/2$$

it follows from Lemma 3 that either δ_0 or ${}_{n-1}\delta_r$ must be optimal. However, using (26) we get by (23)

(29) $$V({}_{n-1}\delta_r) - V(\delta_0) = c[7(n - r) - 13] > 0 ,$$

which proves the optimality of δ_0.

The next example shows that on the last place of the optimal strategy can be any element except for the element with the minimal cost T.

Example 3. Let $1 \leqq r \leqq n - 1$ and let

(30) $$\begin{aligned} p_i &= 6c , \quad T_i = 4 \quad \text{for} \quad 1 \leqq i \leqq r - 1 , \\ p_i &= c , \quad T_i = 2 \quad \text{for} \quad r \leqq i \leqq n - 1 , \\ p_i &= c , \quad T_i = 3 \quad \text{for} \quad i = n , \end{aligned}$$

where

(31) $$c = 1/(n + 5r - 5) .$$

Then the strategy $(1, \ldots, n)$ is optimal and on the last place is the element with the r-th greatest cost.

Proof. Since

(32) $$6c/4 > c/2 > c/3$$

it follows from Lemma 3 that either δ_0 or ${}_{n-1}\delta_1$ must be optimal. However, using (30) we get by (23)

(33) $$V({}_{n-1}\delta_1) - V(\delta_0) = c[8(n - r) - 7] > 0 ,$$

which proves the optimality of δ_0.

As yet we have dealt with rather simple subclass of $\{{}_k\delta_n\}$ class of strategies, namely with $\{{}_{n-1}\delta_m\}$ strategies. Now we shall state two lemmas which give the mean cost of the strategy ${}_k\delta_m$ for arbitrary k and m.

Lemma 5. *For every $0 \leqq k < m \leqq n - 1$ we have*

(34) $$V({}_k\delta_m) - V(\delta_0) = \\ = \sum_{j=k+1}^{n-1} [p_j T_n - p_n T_j] + \sum_{i=m+1}^{n-1} [p_m T_i - p_i T_m] + [p_n T_n - p_m T_m] .$$

Proof. Strategy ${}_k\delta_m$ can be obtained from strategy δ_0 by subsequent transpositions $\langle n-1 \leftrightarrow n\rangle$, $\langle n-2 \leftrightarrow n\rangle, \ldots, \langle k+1 \leftrightarrow n\rangle$, $\langle m \leftrightarrow m+1\rangle$, $\langle m \leftrightarrow m+2\rangle, \ldots,$ $\langle m \leftrightarrow n-1\rangle$, so that

$$V({}_k\delta_m) - V(\delta_0) = \sum_{j=k+1}^{n-1} D(\langle j \leftrightarrow n\rangle) + \sum_{i=m+1}^{n-1} D(\langle m \leftrightarrow i\rangle)\,. \tag{35}$$

Since only the first and the last transposition is of the type II, all others being of the type I, we get by Lemma 1 and Lemma 2 after simple modification the relation (34).

Lemma 6. *For every* $1 \leqq m \leqq k \leqq n-1$ *we have*

$$V({}_k\delta_m) - V(\delta_0) = \tag{36}$$

$$= \sum_{j=k+1}^{n-1} [p_j T_n - p_n T_j] + \sum_{i=m+1}^{n-1} [p_m T_i + p_i T_m] + [(p_m + p_n)(T_n - T_m)]\,.$$

Proof. Strategy ${}_k\delta_m$ can be obtained from strategy δ_0 by subsequent transpositions $\langle m \leftrightarrow m+1\rangle$, $\langle m \leftrightarrow m+2\rangle, \ldots, \langle m \leftrightarrow n\rangle$, $\langle n-1 \leftrightarrow n\rangle$, $\langle n-2 \leftrightarrow n\rangle, \ldots$ $\ldots, \langle k+1 \leftrightarrow n\rangle$, so that

$$V({}_k\delta_m) - V(\delta_0) = \sum_{i=m+1}^{n} D(\langle m \leftrightarrow i\rangle) + \sum_{j=k+1}^{n-1} D(\langle j \leftrightarrow n\rangle)\,. \tag{37}$$

Since only transposition $\langle m \leftrightarrow n\rangle$ is of type II, all others being of type I, we get directly by Lemma 1 and Lemma 2 the relation (36).

Now we have at our disposal all the auxiliary results for a rather simple proof of the following

Characterization Theorem. *The necessary and sufficient condition for the strategy* $(1, 2, \ldots, n)$ *to be optimal is the simultaneous fulfilment of*

$$\frac{p_1}{T_1} \geqq \frac{p_2}{T_2} \geqq \ldots \geqq \frac{p_{n-1}}{T_{n-1}} \tag{38}$$

and

$$\min_{\substack{0 \leqq k \leqq n-1 \\ 1 \leqq m \leqq n-1}} \left\{ \sum_{j=k+1}^{n-1} [p_j T_n - p_n T_j] + \sum_{i=m+1}^{n-1} [p_m T_i - p_i T_m] + [p_n T_n - p_m T_m] + \right. \tag{39}$$

$$\left. + (p_m T_n - p_n T_m) \max\left(\frac{k+1-m}{|k+1-m|}, 0\right)\right\} \geqq 0\,.$$

Proof. Necessity. Let δ_0 be optimal. Then by Lemma 3 for $\delta = \delta_0$ we get directly (38). Further δ_0 domainates all strategies ${}_k\delta_m$ for $0 \leqq k \leqq n-1$ and $1 \leqq m \leqq n-1$ so that (34) and (36) imply immediately (39).

Sufficiency. Let (38) and (39) hold and let us assume an arbitrary strategy δ. This strategy is dominated by the strategy ${}_k\delta_m$, where

$$k = \max_{p_j/T_j > p_n/T_n} j \tag{40}$$

and $m = i_n$, because ${}_k\delta_m$ can be obtained from δ by subsequent transpositions, whose differences D are all non-positive with respect of (38). However, by (34) or (36) and by (39) ${}_k\delta_m$ is itself dominated by δ_0.

Though the Characterization Theorem gives the necessary and sufficient condition for the strategy $(1, \ldots, n)$ to be optimal, it is not quite convenient for the construction of the optimal strategy. Therefore we will give another theorem, which requires to calculate more simple expressions than those in (39).

Determination Theorem. *Let*

$$\frac{p_1}{T_1} \geqq \frac{p_2}{T_2} \geqq \ldots \geqq \frac{p_n}{T_n} \tag{41}$$

and let m be such that

$$p_m \sum_{j=m+1}^{n} T_j - T_m \sum_{j=m}^{n} p_j = \min_{\substack{1 \leqq i \leqq n \\ T_i \geqq T_n}} \{p_i \sum_{j=i+1}^{n} T_j - T_i \sum_{j=i}^{n} p_j\} . \tag{42}$$

Then the strategy ${}_{n-1}\delta_m = (1, \ldots, m-1, m+1, \ldots, n, m)$ *is optimal.*

Proof. By Lemma 3 it is evident that the class $\{{}_{n-1}\delta_i : 1 \leqq i \leqq n\}$ contains the optimal strategy, therefore it suffices to compare among themselves the mean costs of these strategies only. By Lemma 4 we get directly from (23)

$$V({}_{n-1}\delta_i) = V(\delta_0) + p_i \sum_{j=i+1}^{n} T_j - T_i \sum_{j=i}^{n} p_j + p_n T_n . \tag{43}$$

However, the first and the fourth term of the right hand side of (43) being constant for all i, we get immediately (42).

Thus, to determine the optimal strategy one should proceed in the following way:

Arrange and number the elements so that (41) *holds. Calculate the expressions*

$$p_i \sum_{j=i+1}^{n} T_j - T_i \sum_{j=i}^{n} p_j \tag{44}$$

 for all i, for which

$$T_i \geqq T_n . \tag{45}$$

If the minimum is reached for $i = m$, *then* ${}_{n-1}\delta_m$ *is the optimal strategy.*

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VÝTAH

O optimální strategii vyhledávání poruch pro metodu měření prvků a systémy s právě jednou poruchou

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O systému s n prvky je známo, že právě jeden prvek je vadný. Měřením se dá zjistit stav jednotlivých prvků. Pravděpodobnost, že i-tý prvek je vadný, je p_i $(p_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} p_i = 1)$. Náklady na měření i-tého prvku jsou T_i $(T_i > 0)$.

Strategie, tj. pořadí, v kterém jsou prvky měřeny, je optimální, je-li odpovídající střední hodnota nákladů (2) minimální.

Jsou dokázány dvě věty, z nichž první ukazuje, že (38) a (39) je nutná a postačující podmínka pro to, aby strategie $(1, 2, \ldots, n)$ byla optimální, a druhá věta ukazuje, jak vypadá při uspořádání (41) optimální strategie.

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