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Criterion of the Correctness of an Analogue Model of a System of Differential and Algebraic Equations

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In the paper there are derived the general sufficient conditions for the correct analogue model of the ordinary differential and algebraic equations. The formulation of the sufficient conditions is based on the theory of the small parameter at the highest derivative for the systems of ordinary differential equations.

1. INTRODUCTION

In creating a model of a given system of differential and algebraic equations on an analogue computer we may, in certain cases, find that the solution obtained by means of this model differs principally from the correct one. Such an incorrect solution (i.e. the incorrect model) is caused by special features of the computer setup. Some authors tried to determine these special features and to formulate sufficient conditions for the given system in order to obtain its qualitatively correct solution on the analogue computer.

For example, in the years 1945–1947, I. S. Gradstein and B. A. Taft set the condition that the given system should contain no algebraic equations. In 1959, N. N. Leonov requested that the computer setup should not contain any closed loops composed of the operational units solving the algebraic relations only [1]. A similar condition is expressed by W. Giloi and R. Lauber in their book in 1963 [2] as well as by other authors.

These conditions and their formulation have a common defect. They are too strict and they do not show the dependence of the correctness of the solution from the dynamic qualities of the used computer's operational amplifier. The said conditions follow from purely physical considerations not based on a strict mathematical theory.

In this paper we shall try to formulate more general sufficient conditions for the correctness of the analogue model which would be based on the theory of the small parameter at the highest derivative for the systems of ordinary differential equations. The application of the above mentioned conditions will, in each individual case, depend on the choice of an appropriate approximation of the definition of the dynamic properties of the operational units.

The chosen examples are just intended to clarify the matter. The delimitation of the field where the model is correct is very closely connected with the dynamic properties of the operational units. Thus, as indicated in this paper, the procedure of the solution based on sufficient conditions depends on the type of the used computer. The question of the dynamic properties of the operational units that are given by their design is a very complicated problem from the general point of view and its detailed analysis would exceed the framework of this paper.

2. CLASSIFICATION OF OPERATIONAL UNITS

In the mathematical description of the action of analogue computer's operational units in the first approximation i.e. the ideal one (as described in text books dealing with computing on analogue computers), we find that they can be divided into two groups. In the first group there are the units the action of which has been described by a differential equation, usually of the 1st order (integrator, differentiator), in the second one the units the action of which has been defined by an algebraic equation (invertor, summing amplifier, multiplier, function generator). The dynamic properties (further called the dynamics) of the computer setup created by the connection of such units are described (in their ideal form) by a system of algebraic and differential equations corresponding with the respective operational units.

This dynamic is analogical to the dynamics of the problem solved by the computer setup. Therefore, let us call it "the given" or, more frequently, "the ideal" dynamics.

Nevertheless, certain phenomena, occuring in the computer setup, cannot be explained on the basis of the ideal dynamics. Finite deviations from the ideal solution (the so called solution errors), namely the quantitative phenomena will not be discussed in this paper. We shall concentrate on instances when the dynamic action, described by the ideal dynamics is stable, whereas the real dynamic action of the computer setup is quite different and always unstable. We shall speak on about this phenomena, as the qualitative phenomena.

While deriving the ideal equations describing the action of the operational units, certain phenomena are neglected. They are, altogether, undemanded and therefore called parasite phenomena.

For instance while deriving the ideal equations we suppose that the absolute value of amplification of the operation amplifier is infinite for all the frequencies. The amplification is, as a matter of fact, finite and dependent on the frequency and the action of the amplifier is described by a differential equation, i.e. the ampifier is a dynamic system. Similarly, the action of all the operational units is described by differential equations.

If all the parasitic influences are taken into account, the exact description of the computer setup, i.e. the real dynamics of the model, will be obtained. The dynamic phenomena induced by parasitic influences will be called the parasitic dynamics.

It is practically impossible to identify and to describe all the parasitic influences occuring in the computer setup. While analysing the model with the aim to determine its stability some influences have to be neglected.

Let us try to find the most simple approximation of the actual dynamics by which the qualitative phenomena can be identified.

As for the operational units of the first group, the output values of which will be designated by x_k, their parasitic dynamics (increasing the order of its differential

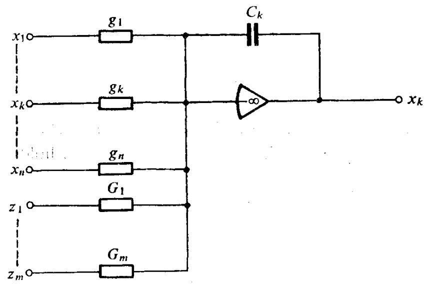

Fig. 1. Circuit of the integrator.

equation) does not influence the results of the solved problem from the qualitative point of view and can thus be neglected. The reasons of this assertion will be briefly given in the fourth paragraph of this paper. For example, the action of the k-th summing integrator connected into the computer setup will be described as a differential equation of the 1st order (see figure 1):

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = f_k(x_i, t, z_j)\,, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m\,, \tag{1}$$

$$f_k(x_i, t, z_j) = -\sum_{i=1}^{n} \frac{g_i(x_i, t)}{C_k}\, x_i - \sum_{j=1}^{m} \frac{G_j(z_j, t)}{C_k}\, z_j\,,$$

g_i and G_j are nonlinear and time-variable conductences.

With regard to the operational units of the second, group their output values will be designated by z_l, their dynamic properties being given by the parasitic elements only. Therefore, their parasitic dynamic cannot be neglected. In our basic considerations let us content ourselves to describe them approximately by differential equations of the 1st order. Further it will be demonstrated that this description will not be sufficient for many a purpose.

Should the action of the l-th operational unit be described in the ideal case by the algebraic relation

(2) $$F_l(x_i, t, z_j) = 0\,, \quad i = 1, 2, \ldots, n\,, \quad j = 1, 2, \ldots, m$$

and should it be taken into account that the amplifier is the dynamic system (for brevity's sake) described by the differential equation of the 1st order (i.e. the frequency dependence of its gain can be expressed by the logarithmic asymptotic characteristics with the constant slope -20 dB/dec and the boundary frequency ω_0 for 0 dB) the action of such a unit can be described by the differential equation

(3) $$\mu_l \frac{\mathrm{d}z_l}{\mathrm{d}t} = \beta_l F_l(x_i, t, z_j)$$

where

$$\mu_l = \mu_l(\omega_0) > 0\,, \quad \beta_l = \beta_l(h_i, H_j) > \delta > 0$$

(h_i and H_j are nonlinear and time-variable conductences, δ being a positive constant). For the number μ_l reaching small values (the so called "small parameter") it holds

$$\mu_l(\omega_0) \to 0 \quad \text{if} \quad \omega_0 \to \infty\,.$$

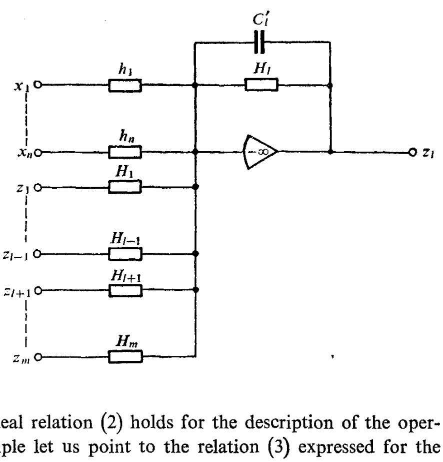

Fig. 2. Equivalent circuit of the summing amplifier. ($C_l' = 1/\omega_0 \,.\, H_l/\beta_l$. Infinite gain of the operational amplifier is supposed.)

Therefore, for $\omega_0 = \infty$ the ideal relation (2) holds for the description of the operational unit. By way of example let us point to the relation (3) expressed for the summing amplifier and the servomultiplier.

According to (3) the differential equation (4) holds for the equivalent circuit the scheme of which is illustrated in figure 2:

$$\mu_l \frac{dz_l}{dt} = -\beta_l \left(\sum_{i=1}^{n} \frac{h_i(x_i, t)}{H_l(x_l, t)} x_i - \sum_{j=1}^{m} \frac{H_j(z_j, t)}{H_l(x_l, t)} z_j \right), \tag{4}$$

where

$$\mu_l = \frac{1}{\omega_0}, \quad \beta_l = \frac{H_l}{\sum_{j=1}^{m} H_j + \sum_{i=1}^{n} h_i},$$

h_i and H_j are nonlinear and time-variable conductences.

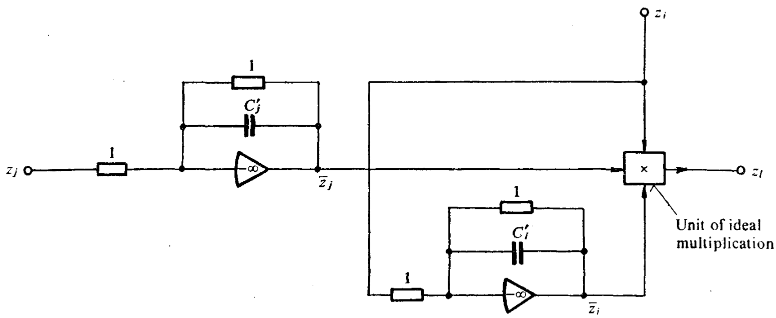

Fig. 3. Equivalent circuit of the servomultiplier.

The mathematical description in our approximation of the fourquadrant multiplier can be given according to the equivalent circuit in figure 3.

For $\bar{z}_j > 0$ the following equations hold:

$$z_l = z_i \bar{z}_j, \quad \mu_j \frac{d\bar{z}_j}{dt} = -0{\cdot}5(\bar{z}_j + z_j), \quad \mu_j = \frac{1}{\omega_s};$$

for $\bar{z}_j < 0$ it holds:

$$z_l = \bar{z}_i \bar{z}_j, \quad \mu_j \frac{d\bar{z}_j}{dt} = -0{\cdot}5(\bar{z}_j + z_j), \quad \mu_i \frac{d\bar{z}_i}{dt} = -0{\cdot}5(\bar{z}_i + z_i), \quad \mu_i = \frac{1}{\omega_0}.$$

The initial conditions of the differential equations describing the units of the second group will be discussed in paragraph 4.

3. THE GIVEN AND THE EXTENDED SYSTEMS OF DIFFERENTIAL EQUATIONS

Prior to more general considerations let us demonstrate the ideal and then the more accurate description of the simple computer setup on the following example.

A system of two differential equations of the 1st order is to be solved on an analogue computer:

$$\frac{dX_1}{dt} + A\frac{dX_2}{dt} + BX_1 + CX_2 = -v(t)\,, \tag{5}$$

$$D\frac{dX_1}{dt} + \frac{dX_2}{dt} + EX_1 + FX_2 = -w(t)\,.$$

The corresponding block diagram is shown in figure 4.

The ideal dynamics of the computer setup is described by the system of equations (5), the corresponding variables being indicated between the square brackets in figure 4.

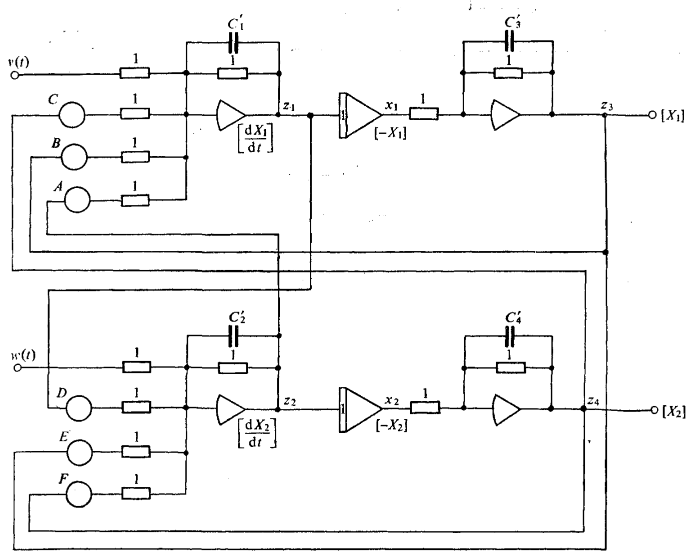

Fig. 4. Block diagram of the system (5).

The description including the parasitic dynamics in the approximation mentioned in the 2nd paragraph will be called *the extended system* which will have the following form for the computer setup shown in figure 4:

(6.1) $$\mu \frac{dz_1}{dt} = -\tfrac{1}{5}(z_1 + Az_2 + Bz_3 + Cz_4 + v),$$

(6.2) $$\mu \frac{dz_2}{dt} = -\tfrac{1}{5}(Dz_1 + z_2 + Ez_3 + Fz_4 + w),$$

(6.3) $$\mu \frac{dz_3}{dt} = -\tfrac{1}{2}(z_3 + x_1),$$

(6.4) $$\mu \frac{dz_4}{dt} = -\tfrac{1}{2}(z_4 + x_2),$$

(6.5) $$\frac{dx_1}{dt} = -z_1,$$

(6.6) $$\frac{dx_2}{dt} = -z_2.$$

The inner resistance of potentiometers realizing the coeficients A to F is, for the sake of simplicity, in the calculation of the value β, not accounted for. According to paragraph 2, it holds $\mu = 1/\omega_0$.

The above mentioned system is a six-order system which includes the parameters μ which, if compared with other coefficients, are lower by several orders. They may therefore be considered as "small parameters". The solution of the extended system will be a function of both time and the small parameters.

A question arises whether the extended system (6) will be stable if the given system (5) is stable. Or, more generally speaking, what will be the relation of the solution $x_1(t)$ and $x_2(t)$ of the system (5) to the corresponding solution $x_1(t, \mu)$ and $x_2(t, \mu)$ of system (6). Let us try to answer these questions.

First, let us introduce the description of the general computer setup in our approximation in the vector form:

(7) $$\frac{dx}{dt} = f(x, z, t),$$

$$\mu \frac{dz}{dt} = \beta(x, z, t)\, F(x,z, t)$$

with initial conditions

$$x(0) = x_0, \quad z(0) = z_0,$$

where

$$x = (x_1, \ldots, x_n), \qquad f = (f_1, \ldots, f_n),$$
$$x_0 = (x_{01}, \ldots, x_{0n}), \qquad z = (z_1, \ldots, z_m),$$
$$\beta \, . \, F = (\beta_1 F_1, \ldots, \beta_m F_m), \quad F = (F_1, \ldots, F_m),$$
$$z_0 = (z_{01}, \ldots, z_{0m}).$$

Let us suppose that the function $f, \beta \, . \, F, F$ are continuous, $\beta_j > \delta > 0, j = 1, 2, \ldots, m$, $\delta = \text{const}$, $\mu > 0$ is a constant. Further let us suppose that the system (7) has a unique solution $x(t, \mu)$ and $z(t, \mu)$.

If we put $\mu = 0$ in the equations (7), we obtain the singular system:

(8.1) $$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, z, t),$$

(8.2) $$0 = F(x, z, t).$$

This system describes the ideal dynamics of the computer setup and the computation process usually follows by means of the block diagram created on the basis of the said system.

The solution of the singular system is a function of time only:

$$x = X(t), \quad z = Z(t).$$

According to the supposition concerning system (7) its solutions, i.e. the function $x(t, \mu)$ and $z(t, \mu)$, are continuous functions of the variable μ for $\mu > 0$. If it holds

(9) $$\lim_{\mu \to 0^+} x(t, \mu) = X(t),$$
$$\lim_{\mu \to 0^+} z(t, \mu) = Z(t).$$

then for the sufficiently small μ, the solution of the extended system will slightly differ from the solution of the ideal system, maintaining its qualitative properties. In the opposite case only the computer setup may give the qualitatively different solution in comparation with the correct (ideal) solution.

As for the parasitic parameters, let us suppose that they are sufficiently small to keep the qualitative properties of the computer setup according to the upper paragraph. There remains the basic question as to what conditions have to be satisfied by the extended system, namely by the computer setup to fulfil the conditions (9).

A. N. Tikhonov [4] solves this question mathematically and the solution of our problem will be based on the results of this analysis.

First, let us introduce some concepts.

Let us designate by

$$z_k = \Phi_k(x, t) \tag{10}$$

one of the roots of the system of equations $F(x, z, t) = 0$. This root will be called the isolated root, if for the sufficiently small $\varepsilon > 0$ and each vector $\bar{z}$ fulfilling

$$\|\bar{z} - \Phi(x, t)\| < \varepsilon\,, \quad \bar{z} \neq \Phi(x, t)$$

the following relation holds:

$$F(x, \bar{z}, t) \neq 0\,.$$

If we choose an arbitrary firm moment $t_0 = \text{const}$ and carry out the substitution of the independent variable

$$\tau = \frac{t - t_0}{\mu}$$

in the system (7), we obtain a new form of system (7):

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mu\, f(x, z, t_0 + \mu\tau)\,,$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \beta(x, z, t_0 + \mu\tau)\, F(x, z, t_0 + \mu\tau)\,.$$

Let us call the system arising therefrom for $\mu \to 0^+$ (if the limit does exist) *the residual system*. It will have an important role in our further considerations. The system is the following (t_0 will be designated t again for better clearness):

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \beta(x, z, t_0)\, F(x, z, t_0) = \beta(x, z, t)\, F(x, z, t) \tag{11}$$

with initial conditions $z(0) = z(t_0)$ where x and t are the parameters. The root $z_k = \Phi_k(x, t)$ is a singular solution of the system.

The isolated root (an isolated singular solution) will be called an asymptotic stable solution of the residual system if for arbitrary $\varepsilon > 0$ such $\delta(\varepsilon) > 0$ can be found so that two following relations hold:

$$\lim_{\tau \to \infty} \bar{z}(\tau) = z_k \quad \text{if} \quad \|\bar{z}(0) - z_k\| < \delta(\varepsilon)\,, \tag{12}$$

i.e. the trajectory of each solution $\bar{z}$ for initial conditions $\bar{z}(0)$ from the $\delta(\varepsilon)$ neighbourhood of the singular solution z_k converges to z_k for $\tau \to \infty$ and

$$\|\bar{z}(\tau) - z_k\| < \varepsilon \tag{13}$$

for an arbitrary $\tau > 0$ if $\|\bar{z}(0) - z_k\| < \delta(\varepsilon)$.

The isolated root $z_k = \Phi_k(x, t)$ will be called stable in a bounded region D of the space of the parameters x, t if for all the points of the closed region $\bar{D}$ (i.e. the region D

with its boundary) the proper values $z_k = \Phi_k(x, t)$ are asymptotically stable solutions of the corresponding residual systems (i.e. the residual systems with parameters x, t).

A set of initial values $z(0)$ of all the solutions of the residual system for which it holds

$$\lim_{\tau \to \infty} z(\tau) = z_k$$

will be called the region of contraction of the asymptotically stable root $z_k = \Phi_k(x, t)$ for $x = \text{const}$, $t = \text{const}$.

On the basis of these concepts the following theorem can be formulated.

The Tikhonov theorem. The solution of the extended system (7) converges for $\mu \to 0^+$ to the solution of the singular system (8), i.e. the relations (9) hold if:

1. the root $z_k = \Phi_k(x, t)$ of equations (8.2) on the basis of which the solution $X(t)$, $Z(t)$ of the system (8) was reached is an asymptotic stable solution of the residual system (11);
2. initial values z_0 of the system (7) lie in the region of contraction of the root $z_k = \Phi_k(x, t)$ for the values of the parameters $x = x_0$, $t = 0$ (i.e. for initial values of the corresponding variables).

The equation (9) holds for all t for which the solution z_k of the residual system lies within the stability region $D(x, t)$ of the root z_k.

The theorem is proved in [4].

This theorem answers the question on what conditions the relation (9) holds, namely when the solution of the extended system maintains the qualitative properties of the solution of the given (ideal) problem.

The computer setup described by the (extended) system of differential equations satisfying the conditions of the Tikhonov theorem will be called the correct model.

Now let us show in more details what these conditions mean in the computer setup.

4. THE RESIDUAL SYSTEM AND ITS INITIAL CONDITIONS

The residual system plays the main role in the Tikhonov theorem. Let us express the system as follows:

$$(14) \qquad \begin{aligned} \frac{dz_1}{d\tau} &= \beta_1 F_1(x_1, \ldots, x_n, z_1, \ldots, z_m, t), \\ &\vdots \\ \frac{dz_m}{d\tau} &= \beta_m F_m(x_1, \ldots, x_n, z_1, \ldots, z_m, t), \end{aligned}$$

$$x_i = \text{const}, \quad t = \text{const}.$$

Each equation holds for one operational unit of the second group, i.e. for the unit with the parasitic dynamics only. The parasitic dynamics of the units of the first group is not taken into account. As t and x_i appear as parameters in the residual system, we can say that the system (14) defines the parasitic dynamics of the model in the before mentioned approximation. According to the Tikhonov theorem the stability of the residual system is one of the conditions of the validity of the relation (9), i.e. the stability of the parasitic dynamics is one of the conditions of the correctness of the model.

As is often the practical case, the system (14) breaks down into several simultaneous systems or, more often, into both several simultaneous systems and a number of separate equations (i.e. simultaneous systems of the 1st order). For instance, in the example demonstrated in paragraph 3 regarded from the point of view of the residual system, the equations (6.1) and (6.2) form the simultaneous system of the 2nd order and the equations (6.3) and (6.4) two simultaneous systems of the 1st order (two separate equations). The solution of the residual system is stable if the solution of each of its simultaneous parts is stable.

Each of the simultaneous systems of the 1st order describes the parasitic dynamics of an operational unit corresponding to the proper equation. The stability of each operational unit is usually guaranteed by its design. Only in the case of nonlinear units it can occur that the operational unit is not stable in some region of the values of output variables.

Generally it holds: The action of the operational unit is described from the point of view of the residual system by the equations

$$\frac{dz_j}{d\tau} = \beta_j F_j(x_1, \ldots, x_n, z_1, \ldots, z_m, t) .$$

The l-th root of the singular equation $F_j = 0$ is designated:

$$z_{jl} = \varphi_{jl}(x_1, \ldots, x_n, z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m, t) .$$

In order to obtain a stable root, it is sufficient to have an $\varepsilon_0 > 0$ such that for each ε, $|\varepsilon| < \varepsilon_0$ it holds:

$$\operatorname{sgn} F_j(x_1, \ldots, x_n, z_1, \ldots, z_{j-1}, \varphi_{jl} + \varepsilon, z_{j+1}, \ldots, z_m, t) \neq \operatorname{sgn} \varepsilon .$$

Or else in the case of the differentiability of the functions β_j, F_j the following is to hold:

$$\left.\frac{\partial \beta_i F_i}{\partial z_j}\right|_{z_j=\varphi_{jl}} < 0 \quad \text{namely} \quad \left.\frac{\partial F_j}{\partial z_j}\right|_{z_j=\varphi_{jl}} < 0 .$$

If the order of the simultaneous system is higher than the 1st order, then the computer setup contains the feedback system of the units of the 2nd group, the so called algebraic loop. The analytical examination of its stability is possible (see 1st example of the 5th paragraph). Usually, however, it is mathematically very difficult, specially in the case of nonlinear systems.

The result obtained is usually a system of unequalities for variables x_i, t, with the values of variables x_i (excepting initial values) being unknown prior to creating a model (computer setup). Only the intervals of their position can be guessed.

Although we can succeed in determining the region of the stability of the residual system (14) by an analytical process, the theoretical computations will usually not correspond with practical results on the computer, this being caused by an only approximate description of the action of operational units of the 2nd group, as introduced in the 2nd paragraph.

A simplified description has enabled a clearer explanation aiming to a definition of a correct model of the system of both differential and algebraic equations. However, it is unsufficient for an exact quantitative determination of the regions of the correct model.

While deriving differential equations, describing the action of operational units of the 2nd group, we supposed that the frequency dependence of the operation amplifier was given by a logarithmic asymptotic frequency characteristics with a permanent slope -20 dB/dec. In fact, however, this dependence is far more complicated and dependent on the construction of the amplifier, i.e. of its type. The action of the amplifier (and of the operational units of the 2nd group as well) can be more exactly described by a differential equation of a higher order (for instance in dependence on the fact how many fractions of the asymptotic frequency characteristics must be taken into consideration). This description will differ for individual types of the amplifier. A common and important feature is, however, the fact that each equation of, let us say, the s-th order, can be decomposed into an s system of differential equations of the 1st order each of which has a small parameter μ_i. The "more exact" extended system thus obtained is again of the form (7). However, it is of a higher order, having a greater number of equations with small parameters. From the point of view of the residual system, however, the s equations belonging to one operational unit form a part of one simultaneous system, or else, they create an independent simultaneous system which, in the case of a linear operational unit, is always asymptotically stable.

Now we can outline the reasons for the assertion expressed in the 2nd paragraph, namely that the neglecting of parasitic dynamics of the units of the 1st group will not influence the analysis of the correctness of the computer setup. Each unit from the 1st group can be more exactly described, for instance, by a set of three differential equations, the first one describing the ideal dynamics, the two remaining ones with a small parameter at a derivative being dependent on the dynamic properties of the amplifier. From the point of view of the residual system, each of the remaining two equations will create an asymptotic stable simultaneous system of equations of the 1st order (for all the values of parameters x_l, t) without influencing the validity of the 1st condition of Tikhonov's theorem.

Thus, it can be summarized that the determination of the region of the asymptotic stability of the residual system, containing the simultaneous system of a higher order, will be very complicated from the analytical point of view. It will often be more complicated than the solution of the given problem. In conclusion let us point to the possibility of using the computer in order to solve the given problem.

The second condition of Tikhonov's theorem gives the demands for the initial conditions of the units of the 2nd group, namely of the units with parasitic dynamics only. There are no prescribed initial conditions for these units. The values of their output variables $z_j(t)$ in time $t = 0$, i.e. z_{0j} are given by the solution of the system (8.2), in which there are substituted x_i for given initial values x_{0i} of the operational units of the 1st group.

These values appear on the outputs of the corresponding units almost at the very moment of switching the computer into the regime "Solution". Only then, however, if the 1st condition of Tikhonov's theorem is satisfied, namely if the corresponding root of equation (8.2) is an asymptotically stable root of the residual system.

If the residual system is linear, then, generally speaking, it has one root

$$z_1 = \Phi_1(x, t)$$

which either is asymptotically stable or is not (we say that the system is either stable or unstable). If it is asymptotically stable, then all the points z_{0j} lie within the region of contraction of this root; therefore, in the linear case, the validity of the 1stTikhonov condition ensures that the 2nd Tikhonov condition is satisfied as well.

The situation is different in the nonlinear case: the system (8.2) can have more roots than one. Some of them are asymptotically stable and some are not. Each asymptotically stable root has its region of contraction. If we are interested in the root $z_k = \Phi_k(x, t)$ from the point of view of the solved problem, i.e. of the ideal dynamics, and if the root is an asymptotically stable root of a residual system, then several cases can occur: If the initial contions z_{0j} lie within the region of contraction of this root, the 2nd condition of Tikhonov's theorem is fulfilled, then relations (9) hold and the model is correct. In another case the initial conditions can lie in the region of contraction of another root; then, the requested solution cannot be obtained on this model. As it is often the case, the initial conditions lie between the regions of contraction of two (eventially more) asymptotically stable roots and it is impossible to determine beforehand which solution will be obtained. (It usually depends on the momentary condition of the circuits connecting the critical part of the computer setup.) Finally the initial values can lie in the region of instability. The solution corresponding to the root z_k cannot be obtained on this model (in spite of the 1st Tikhonov condition being fulfilled) and the model is incorrect. The questions concerning initial conditions will be demonstrated in the following paragraph in the example 3.

5. EXAMPLES

The presented text is illustrated by several examples.

Example 1. The following differential equation is to be solved on the analogue computer:

$$\frac{dX}{dt} = \frac{k_1}{k_2 - 1}[X - f(t)], \quad X(0) = 0 \tag{15}$$

where k_1 and k_2 are the coefficients which are to be set independently to each other during the computation process and f is a driving function. The solution of this equation is asymptotically stable if $k_1/(k_2 - 1) < 0$ holds, i.e. if inequalities either $k_1 < 0$, $k_2 > 1$ or $k_1 > 0$, $k_2 < 1$ hold simultaneously.

The equation (15) may be written as a system of differential and algebraic equations:

$$\frac{dX}{dt} = k_1 Z, \quad (k_2 - 1) Z = X - f, \quad X(0) = 0.$$

This system describes the ideal dynamics of the computer setup illustrated in figure 5 (the variables of the ideal dynamics are designated there in the square brackets).

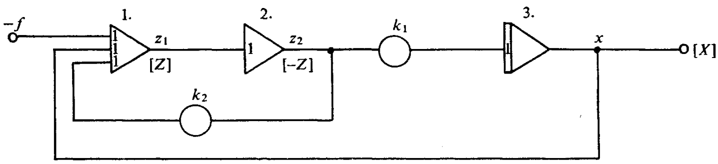

Fig. 5. Block diagram of the relation (15).

The extended system may be expressed by approximation in the form of a 1st order differential equation (in the calculation the inner resistance of the potentiometer k_2 was neglected, therefore $\beta = \frac{1}{4}$):

$$\frac{dx}{dt} = k_1 z_1, \tag{16}$$

$$\frac{1}{\omega_0}\frac{dz_1}{dt} = \tfrac{1}{4}(-z_1 - k_2 z_2 - x + f(t)),$$

$$\frac{1}{\omega_0}\frac{dz_2}{dt} = \tfrac{1}{2}(-z_1 - z_2).$$

The residual system has a linear form:

$$\frac{dz_1}{d\tau} = \tfrac{1}{4}(-z_1 - k_2 z_2 - x + f(t)), \quad \frac{dz_2}{d\tau} = \tfrac{1}{2}(-z_1 - z_2)$$

where t, x are parameters. When analysing the roots of the characteristic equation we find out that the system (and consequently the parasitic dynamics too) is stable for $k_2 < 1$.

When analysing the solution of the example 1, we get the results shown in table 1.

Table 1.

Analysis of the solution of the example 1.

Case			Ideal dynamics	Parasitic dynamics
1	$k_1 > 0$	$k_2 > 1$	unstable	unstable
2	$k_1 > 0$	$k_2 < 1$	stable	stable
3	$k_1 < 0$	$k_2 > 1$	stable	unstable
4	$k_1 < 0$	$k_2 < 1$	unstable	stable

The oscillogram of the solution of the case 2 for the driving function $f(t)$ in a form of step function is shown in figure 6 and the oscillogram of the case 4 is shown in figure 7. The model is correct in both mentioned cases (in the latter a greater error of the solution can be expected due to the instability of the ideal dynamics).

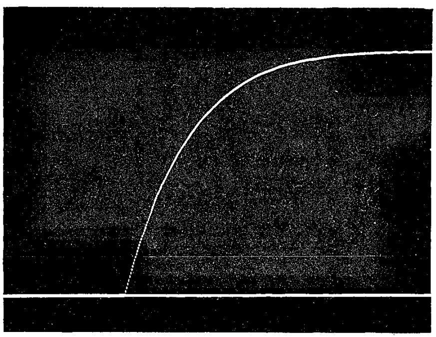

Fig. 6. Oscillogram of the solution of the example 1 for $k_1 > 0$, $k_2 < 1$.

The oscillogram of "the solution" of the case 3 is indicated in figure 8. In spite of the stable ideal dynamics Tikhonov's conditions are not satisfied and the model is not correct. The diagram indicated by figure 8 has no relation to the solution of equation (15). (The output voltages of the summing amplifiers reach limit values almost immediately after the computer was set to "Solution". The output voltage of the integrator 3 increases in time linearly.) The solution of the mentioned differential equations was carried out on the computer AP 3M (product of Tesla — Czechoslovakia).

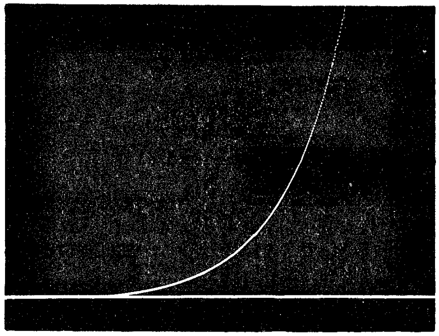

Fig. 7. Oscillogram of the solution of the example 1 for $k_1 < 0$, $k_2 < 1$.

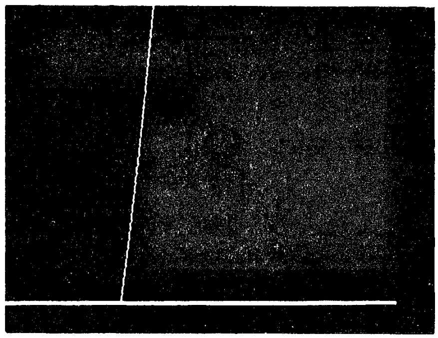

Fig. 8. Oscillogram of the incorrect "solution" of the example 1 ($k_1 < 0$, $k_2 > 1$).

Example 2. In the following example we shall show the relation between the stability boundary and the way of approximation. We shall investigate the greatest gain admissible for which a loop of three summing amplifiers is stable. See figure 9. The assumption is that all summing amplifiers possess the same dynamic properties.

a) Let us consider the case that the dynamics of the summing amplifier is described by a differential equation of the 1st order. On these conditions the action of the summing amplifier (see figure 10) can be described by the differential equation as follows:

$$\frac{1}{\omega_0} \frac{dz_j}{dt} = -\beta(z_j + kz_i)$$

where

$$k = \frac{G_v}{G_u} > 0\,, \quad \beta = \frac{1}{1 + k}\,.$$

The system of the three summing amplifiers will be described by the (extended) system of equations:

$$\frac{1}{\omega_0}\frac{\mathrm{d}z_1}{\mathrm{d}t} = -\beta(z_1 + kz_3)\,,$$

$$\frac{1}{\omega_0}\frac{\mathrm{d}z_2}{\mathrm{d}t} = -\beta(kz_1 + z_2)\,,$$

$$\frac{1}{\omega_0}\frac{\mathrm{d}z_3}{\mathrm{d}t} = -\beta(kz_2 + z_3)\,.$$

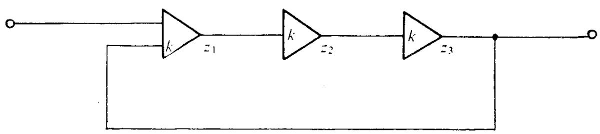

Fig. 9. Feedback loop of the three summing amplifiers.

The residual system has an analogue form. The analysis of stability will be carried out on the base of the analysis of the roots of its characteristic equation:

$$\left(\frac{1}{\beta}\lambda + 1\right)^3 + k^3 = 0\,.$$

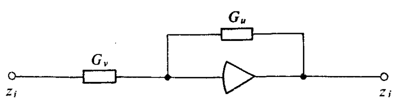

Fig.10. Circuit of the summing amplifier.

All the roots have negative real parts if it holds: $0 < k < 2$. The loop gain under the conditions mentioned above must fulfil $k^3 < 8$.

b) Now we suppose that the dynamics of the summing amplifier is described by the equation of the second order with critical damping:

$$\frac{1}{\omega_0}\frac{\mathrm{d}z_j}{\mathrm{d}t} = 4\beta w_j\,,$$

$$\frac{1}{\omega_0}\frac{\mathrm{d}w_j}{\mathrm{d}t} = -\beta(4w_j + z_j + kz_i)\,.$$

The extended system of the 6th order describing the action of the loop, composed of three summing amplifiers (see figure 9), contains three such couples of equations. The residual system has the analogue form to the extended system. By analysis of roots of the characteristic equation we obtain the condition $0 < k < 1{\cdot}33$ for the stable loop. The greatest admissible loop gain for the loop composed of three summing amplifiers is given by the following relation: $k^3 \doteq 2{\cdot}35$.

Evidently the greatest admissible loop gain depends to a great extent on the chosen approximation of the dynamic properties of the amplifier.

The values reached in practical cases differ considerably from each other according to different types of computers used. For instance, the authors Giloi - Lauber [2] state that the loop gain for the loop composed of three summing amplifiers is of the magnitude from 2 to 4. If the computer AP 3M is used, however, the mentioned loop is stable even for the values greater than 8.

Example 3. Let us solve the following system of differential equations:

$$\frac{dX}{dt} = -XY + 0{,}008\,, \quad X(0) = X_0 > 0\,, \tag{17.1}$$

$$X - Y^2 = 0\,. \tag{17.2}$$

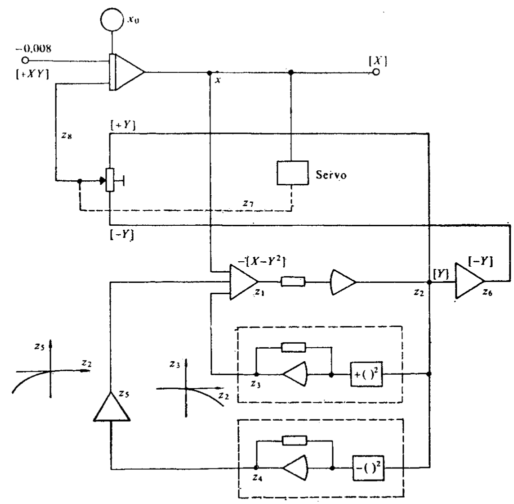

Fig. 11. Block diagram of the system (17).

The roots of the equation (17.2) are $Y_1 = +\sqrt{X}$, $Y_2 = -\sqrt{X}$ for $X > 0$. In the case of $X < 0$ no solution can be reached in the real domain. When the roots Y_1, Y_2 are substituted into equation (17.1), two solutions will be obtained. The singular solutions are $\sqrt{(X)} = +0{\cdot}2$ for the root Y_1

and $\sqrt{(X)} = -0{\cdot}2$ for the root Y_2. The solution $\sqrt{(X)} = +0{\cdot}2$ is stable, the solution $\sqrt{(X)} = -0{\cdot}2$ is unstable (it is easy to find out by the use of the second Liapunov's method).

The block diagram for the solution of equations (17) is shown in figure 11.

In describing the dynamics of the operational amplifier by an equation of the first order, the following system is obtained:

$$\frac{1}{\omega_0}\frac{dz_1}{dt} = -\tfrac{1}{3}(z_1 + z_3 + z_5 + x)\,, \tag{18.1}$$

$$\frac{1}{\omega_0}\frac{dz_2}{dt} = -z_1\,, \tag{18.2}$$

$$\left.\begin{aligned}\frac{1}{\omega_0}\frac{dz_3}{dt} &= \frac{-1}{1+|z_2|}(z_3 + z_2^2) \quad \text{for} \quad z_2 > 0\,,\\ z_3 &= 0 \quad \text{for} \quad z_2 < 0\,,\end{aligned}\right\} \tag{18.3}$$

$$\left.\begin{aligned}z_4 &= 0 \quad \text{for} \quad z_2 > 0\,,\\ \frac{1}{\omega_0}\frac{dz_4}{dt} &= \frac{-1}{1+|z_2|}(z_4 - z_2^2) \quad \text{for} \quad z_2 < 0\,,\end{aligned}\right\} \tag{18.4}$$

$$\frac{1}{\omega_0}\frac{dz_5}{dt} = -\tfrac{1}{2}(z_5 + z_4)\,, \tag{18.5}$$

$$\frac{1}{\omega_0}\frac{dz_6}{dt} = -\tfrac{1}{2}(z_6 + z_2)\,, \tag{18.6}$$

$$\frac{1}{\omega_0}\frac{dz_7}{dt} = -\tfrac{1}{2}(z_7 + x)\frac{\omega_s}{\omega_0}\,, \tag{18.7}$$

$$z_8 = z_7 z_2 \quad \text{for} \quad z_7 > 0\,, \tag{18.8}$$

$$z_8 = z_7 z_6 \quad \text{for} \quad z_7 > 0\,, \tag{18.9}$$

$$\frac{dx}{dt} = -z_8 + 0{\cdot}008\,. \tag{18.10}$$

The simultaneous system is made up by the equations (18.1). (18.2) and (18.3) if $z_2 > 0$ holds.

By means of linearization in the neighbourhood of the singular point $z_1 = 0$, $z_2 = +\sqrt{x}$, $z_3 = -x$, we may find out that this point is the stable solution of the proper residual system within the limits of stability $0 < \sqrt{(x)} < 0{\cdot}5$ (on the computer AP 3M the upper limit of stability approaches $\sqrt{(x)} = 1$).

The other simultaneous system is constituted by the equations (18.1), (18.2), (18.4), (18.5) supposing that $z_2 < 0$. The proper singular solution $z_1 = 0$, $z_2 = -\sqrt{x}$, $z_4 = x$, $z_5 = -x$ is unstable for all values $-\sqrt{x}$. (The analysis is carried out again by means of linearization and by application of the Hurwitz criterion.)

The solution on the analogue computer shows that the region in which the residual system is stable is fully situated in the region of contraction of the root $Y_1 = +\sqrt{X}$. It means that the solution belonging to the root Y_1 (the steady state is $X = Y_1^2 = 0{\cdot}04$) is obtained, if the initial value of X is chosen within the interval $0 < X_0 < 0{\cdot}5$. The model is correct in this case.

The solution of the equation (17.4) proper to the root Y_2 satisfies none of Tikhonov's conditions. Thus, the model is not correct and the solution which can be reached by analytical calculation (the root $Y_2 = -\sqrt{X}$ is substituted in the equation (17.1)) is not obtainable on this model.

6. CONCLUSION

In our paper we tried to accomplish a new analysis of the computer setup dynamics – an analogue model of a system of differential and algebraic equations. We introduced the concept of the residual system. The idea is based on the Tikhonov's work about the systems of ordinary differential equations with small parameters at the highest order of derivatives. Further we demonstrated the stability of the residual system to be the fundamental condition for the correctness of the model. The case was also solved when the algebraic equations have more than one root and sufficient conditions were given to reach the solutions proper to individual roots.

The contents of the article indicate that for practical analysis of the residual system stability our interest may be limited to the operational unit of the second group only and further more to feedback systems of such units (i.e. to simultaneous parts of the residual system). The exact mathematical analysis is, as demonstrated by the examples, often too complicated. The result of the analysis may, in some cases, namely if an unsufficiently suitable approximation of the dynamics of an operational units is chosen, diverge from the practical results shown by the computer.

Practically the analogue computer can be used for the solution of the problem. As the first step we have to create the circuits described by simultaneous sets of equations with small parameters (such systems are, for instance, in example 3 – of paragraph 5 – the system of the equations (18.1), (18.2), (18.3) and the system (18.1), (18.2), (18.4), (18.5)), to realize the computer setup according to the block diagram. Let us note that the block diagram must also be created in the case of mathematical analysis in order to compose the residual system. Initial conditions of the units of the first group are parameters and they are realized as constant driving functions. Also all time dependent elements and functions are realized as constants equal to its values for time $t = 0$. Thus, for the values of the parameters $t = 0$, $x = X_0$ the model of the simultaneous parts of the residual system is created. As soon as the computer setup is put to "Solution", the stability of the residual system can be found out for $t = 0$ (it can be carried out even in a certain interval $t > 0$, as the solution is continuous in the relation to the parameters) as well as the fact whether the initial conditions lay in a region of contraction of the demanded root. It means the verification of the validity of the conditions of Tikhonov's theorem. If the conditions are satisfied, the computer setup may be filled up by necessary integrators connected according to a proper block diagram. This way, the correct model of a solved problem can be reached.

Unless Tikhonov's conditions are satisfied, a way must be searched how to manage it if possible. It may be done by a new formulation of the problem. For example a new variables can be introduced or some of algebraic operations can be carried out analytically (in the example 1 the case happens when the separate adjustment is not demanded and the expression $k_1/(k_2 - 1)$ is realized by one potentiometer).

Finally let us remark that the analysis of the computer setup dynamics shown in the article holds even in the case if either differential equations only or algebraic ones only are solved. A model of algebraic and transcendental equations contains units of the second group only. Its (parasitic) dynamics is given by the proper residual system. Practically, such a computer setup is stable only in some special cases. To warrant the stability of a solved problem, another way has to be chosen (creation of a different stable dynamic system whose steady state is described by the given algebraic equations [2], [3], [5] etc.). This problem has been much more discussed in the existing technical literature than the one which is dealt with in this article.

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Kritérium korektnosti analogového modelu soustavy diferenciálních a algebraických rovnic

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V článku jsou uvedeny obecné dostatečné podmínky pro „korektnost" modelu soustavy obyčejných diferenciálních a algebraických rovnic na analogovém počítači. Je zavedena tzv. „rozšířená soustava" diferenciálních rovnic, která popisuje dynamiku modelu při uvažování zpřesněného popisu počítacích jednotek. Na jejím základě je v článku definována tzv. „reziduální soustava", která popisuje v určitém přiblížení zavedenou „parazitní dynamiku" modelu. Je dokázáno, že dostatečnou podmínkou korektnosti modelu je asymptotická stabilita této parazitní dynamiky. Je rovněž rozebrána možnost získání požadovaného řešení v modelu nelineárních diferenciálních rovnic. Teorie je doplněna ukázkami řešených příkladů.

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