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Solution of Simple Logical Problems by Colouring Graphs

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This method was invented as an application of graph theory and presented in a primordial form at the International Congress of Mathematicians held in Moscow, 1966. Later on, the didactical utilization of the method was developed and some papers concerning these questions were published by author in Czechoslovak and foreign journal for teachers. That work suggested some generalizations of the method as well as some simplifications of it. This paper gives a concise information on the mathematical and logical core of the graphical method in its present form.

1. INTRODUCTION

The symbolic logic knows different methods for solving such logical problems as testing the validity of arguments, establishing the truth-values of some propositions etc. Textbooks of logic show the application of the truth-table analysis, logical trees, punch-cards, propositional calculus or Boolean algebra. In this paper, the application of graphs will be explained, the core of the matter consists in graphical representation of a binary relation defined in a finite set of propositions. The truth-values of these propositions are represented by colours of the vertices, the rules for colouring the vertices express the rules of logical inference in the language of the graph theory. No special knowledge of graph theory is required, because of the construction of a graph is merely needed.

2. GRAPHICAL REPRESENTATION OF SETS OF PROPOSITIONS

In this paper, the sign $\mathrm{tv}(x)$ denotes the truth-value of the proposition x, the signs $\bar{x}$, $x \,\&\, y$, $x \vee y$, $x \Rightarrow y$, $x \Leftrightarrow y$ denote the negation *not* x, conjunction x *and* y, disjunction x *or* y, implication *if* x, *then* y and equivalence x *if and only if* y, respectively. The construction of the graphs will be described as the first step in solving the following problem:

Problem 1. Given a condition tv $(p_1 \& p_2 \& \ldots \& p_n) = 1$, where each proposition p_i is constructed from some of the basic propositions $b_1, b_2, \ldots, b_k$ by using the logical connectives mentioned above. The question we face, then, is to establish the tv (b_1), tv (b_2), ..., tv (b_k).

Example 1. Given tv $[(\bar{a} \vee b) \& (\bar{b} \vee \bar{c}) \& (c \Leftrightarrow d) \& (d \vee e) \& (e \Rightarrow a) \& (e \Rightarrow d)] = 1$. Establish the tv (a), tv (b), tv (c), tv (d) and tv (e).

Example 2. Given tv $[(a \Rightarrow \bar{d}) \& (b \Rightarrow c) \& (a \& c \Rightarrow d) \& (\bar{b} \& c \Rightarrow a \& d) \& (\bar{a} \& b \Rightarrow \bar{c})] = 1$. Establish the tv (a), tv (b), tv (c) and tv (d).

Let us take for granted that all propositions p_i are put down in one of the forms x, $\bar{x}$, $x \vee y$, $x \Rightarrow y$, where x, y are the propositions with the least number of logical connectives as possible. The collection of all these propositions x, y is the key set for our method and will be called the fundamental set $\mathscr{S}$ of propositions (with regard to the given problem). In the example 1 we find the set $\mathscr{S}_1 = \{a, b, c, d, e, \bar{a}, \bar{b}, \bar{c}\}$, whereas in example 2 we find $\mathscr{S}_2 = \{a, b, c, d, a \& c, \bar{b} \& c, a \& d, \bar{a} \& b, \bar{c}\}$.

Let us denote the set of all propositions p_i by the letter $\mathscr{P}$ and concentrate our attention to the connection between the sets $\mathscr{P}$ and $\mathscr{S} \times \mathscr{S}$ (cartesian product of the set $\mathscr{S}$ with itself). All disjunctions or implications included in $\mathscr{P}$ are constructed from ordered pairs of elements of the fundamental set $\mathscr{S}$. The collection of all p_i which are disjunctions may be regarded as a binary relation $\mathscr{D}$ defined in the set $\mathscr{S}$, as follows:

$$\mathscr{D} = \{(x, y) \in \mathscr{S} \times \mathscr{S} ; \quad x \vee y \text{ is a true proposition included in } \mathscr{P}\} .$$

The collection of all p which are implications may be regarded as a binary relation $\mathscr{J}$ defined in $\mathscr{S}$ as follows:

$$\mathscr{J} = \{(x, y) \in \mathscr{S} \times \mathscr{S} ; \quad x \Rightarrow y \text{ is a true proposition included in } \mathscr{P}\} .$$

Obviously, it will be useful to take into consideration the union-set $\mathscr{D} \cup \mathscr{J}$ of both binary relations defined above.

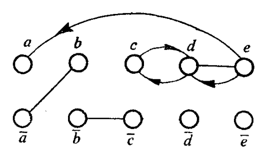

Fig. 1.

As soon as we have obtained a binary relation defined in a finite set $\mathscr{S}$ we can design an advantageous graphical representation of it. The elements of the set $\mathscr{S}$ will be represented by vertices of the graph (small circles on the fig. 1). Every element $x \vee y$ of the symmetric binary relation $\mathscr{D}$ will be represented by one edge xy of the graph, i.e. by a simple curve connecting the vertices x, y (see fig. 1). Every element $x \Rightarrow y$ of the relation $\mathscr{J}$ will be represented by a directed edge of the graph, i.e. by an edge $\overrightarrow{xy}$ provided with an arrow directed from the vertex x to the vertex y (see fig. 1).

Besides the disjunctions $x \vee y$ and implications $x \Rightarrow y$, the set $\mathscr{P} = \{p_1, p_2, \ldots, \ldots, p_n\}$ might include the propositions of the types x, $\bar{x}$, where $x \in \mathscr{S}$. If $p_i = x$ or $p_j = \bar{x}$, then the proposition x is an element of the set $\mathscr{S}$ and as such it is represented by a vertex of the graph. From the condition $\operatorname{tv}(p_1 \,\&\, p_2 \,\&\, \ldots \,\&\, p_n) = 1$ we can deduce that $\operatorname{tv}(x) = 1$, resp. $\operatorname{tv}(x) = 0$. The truth-value of a proposition $x \in \mathscr{S}$ may be marked by colour of the corresponding vertex. Because of the two-valued logic, we need two colours only. In this paper, we shall use a doubled circle for representing any true proposition $x \in \mathscr{S}$ and a black circle for representing any false proposition $x \in S$. From this reason, we shall work with a little strange colours of vertices – double, black.

Following the construction described above, we represent the elements of the set $\mathscr{P}$ by vertices of the graph and the elements of $\mathscr{P}$ by edges of the same graph or by colours of the vertices drawn before. The arrangement of the vertices is very important, therefore some remarks concerning this question will be made.

Each fundamental set of proposition may be embraced in a so called *complete fundamental set of proposition*. These sets form a certain hierarchy as it is obvious from their definition (the $b_1, b_2, \ldots, b_k$ are the basic propositions):

$$\begin{aligned}
\mathscr{F}_0 &= \{b_1, b_2, \ldots, b_k\}, \\
\mathscr{F}_1 &= \mathscr{F}_0 \cup \{\bar{x};\, x \in \mathscr{F}_0\}, \\
\mathscr{F}_2 &= \mathscr{F}_1 \cup \{x \,\&\, y;\, x \in \mathscr{F}_1 \,\&\, y \in \mathscr{F}_1\}, \\
\mathscr{F}_3 &= \mathscr{F}_2 \cup \{\bar{z};\, z \in \mathscr{F}_2\}, \quad \text{etc.}
\end{aligned}$$

It is evident that the set $\mathscr{S}_1$ mentioned above is a subset of the set $\mathscr{F}_1$ provided $\mathscr{F}_0 = \{a, b, c, d, e\}$. The figure 1 has shown a convenient graph representing the set $\mathscr{F}_1$ – the vertices of the graph are situated on two parallel lines with the pairs x, $\bar{x}$ accross. The fig. 2a illustrates the manner in which graphs of the set $\mathscr{F}_2$ will be represented. The elements of the set $\mathscr{F}_1$ included in $\mathscr{F}_2$ are situated on the main diagonal of a matrix scheme in order $b_1, b_2, \ldots, b_k, \bar{b}_k, \ldots, \bar{b}_2, \bar{b}_1$, where the pairs b_i, $\bar{b}_i$ lie symmetrically in regard to the other diagonal. Each conjunction $x \,\&\, y$ is represented by one small circle situated below the main diagonal in intersection of the row and column containing the vertex x or y. When the vertices on the main diagonal are designated by letters, we can easily say which conjunction is represented by any other vertex. It means that these other vertices need not be designated at all.

The fig. 2b depicts a convenient scheme of graphical representation of the set $\mathscr{F}_3$. The trigonal graph of the set $\mathscr{F}_2$ included in $\mathscr{F}_3$ is completed by vertices situated above the main diagonal in intersections of rows and columns of the underlying matrix scheme. Each new vertex lying in the same row or column as the vertices

x, y represents the negation of conjunction $x \& y$. In this manner the negation of propositions is always connected with some symmetry in the graph. Each vertex situated on the main diagonal of the square graph may be comprehended as the representative of the conjunction $x \& x$; in regard to the equality $\text{tv}(x \& x) = \text{tv}(x)$ we use the symbol x only.

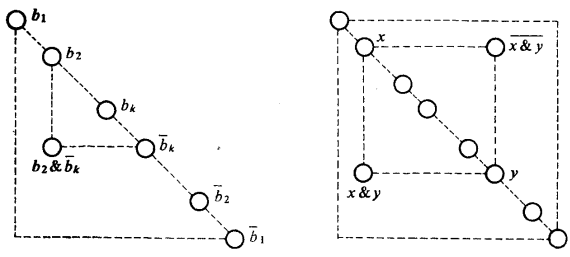

Fig. 2.

The simple logical problems mentioned in the title of this paper may be characterized as problems with such fundamental sets of propositions which are subsets of the complete fundamental sets $\mathscr{F}_0$, $\mathscr{F}_1$, $\mathscr{F}_2$ or $\mathscr{F}_3$. Having constructed the graph of the sets $\mathscr{S}$ and $\mathscr{P}$, we can solve the given problem by colouring vertices of this graph.

3. PROCEDURE OF THE GRAPH COLOURING

Let us start with the problem formulated in example 1 and recall the sets of propositions concerned with its solution:

set of all proposition p_i	$\ldots \mathscr{P} = \{\bar{a} \vee b, \bar{b} \vee \bar{c}, c \Leftrightarrow d, d \vee e, e \Rightarrow a, e \Rightarrow d\}$,
set of all basic propositons	$\ldots \mathscr{B} = \{a, b, c, d, e\}$,
fundamental set of propositions	$\ldots \mathscr{S} = \{a, b, c, d, e, \bar{a}, \bar{b}, \bar{c}\}$.

The graph $\mathfrak{G}$ constructed on the fig. 1 may be regarded as a structure on which we can formulate a problem isomorphic to the given problem. The transformation of the original problem into the new one may be described by rows of a vocabulary seen below. After recalling the correspondence of objects we shall transform the simple rules of inference into the rules for colouring the graph $\mathfrak{G}$:

proposition $x \in \mathscr{S}$	vertex $x \in \mathfrak{G}$
true disjunction $(x \vee y) \in \mathscr{P}$	edge $xy \in \mathfrak{G}$
true implication $(x \Rightarrow y) \in \mathscr{P}$	edge $\overrightarrow{xy} \in \mathfrak{G}$
true proposition $x \in \mathscr{S}$	doubled vertex $x \in \mathfrak{G}$
false proposition $x \in \mathscr{S}$	black vertex $x \in \mathfrak{G}$
to establish $\text{tv}(x)$	to establish colour of x

1. Each proposition $x \in \mathscr{S}$ has exactly one truth-value.	1. Each vertex x has exactly one colour.
2. The proposition $\bar{x} \in \mathscr{S}$ has another truth-value than the proposition $x \in \mathscr{S}$.	2. The vertex $\bar{x}$ has another colour than the vertex x.
3. If the disjunction $x \vee y$ is true and one of the propositions x, y is false, then the other is true.	3. If one of the vertices connected by the edge xy is black, then the other is doubled.
4. If an implication $x \Rightarrow y$ is true and the proposition x is true, then the proposition y is true.	4. If a directed edge $\overrightarrow{xy}$ has doubled initial vertex x then its terminal vertex y is also doubled.
5. If an implication $x \Rightarrow y$ is true and the proposition y is false, then the proposition x is false.	5. If a directed edge $\overrightarrow{xy}$ has a black terminal vertex y, then its initial vertex x is also black.

To start colouring the graph we must know the colour of some vertex. Given colour of no vertex we shall choose the colour of the vertex a, but we must remember both possibilities for this choice. Let us suppose that the vertex a is doubled (see fig. 3a) and take steps based on the rules $1-5$. Applying the rule 2, we blacken the vertex $\bar{a}$; following the rule 3, we double the vertex b etc. After a few steps we get the state depicted on the fig. 3a – the two-coloured vertex e. This fact is in contradiction with the rule 1 and hence the presumption made about the colour of the vertex a is disproved.

Having pictured a new graph we mark the black colour of the vertex a (fig. 3b) and apply the rules $1-5$ again. In this case it is easy to colour the vertices $e, d, c, \bar{c}, \bar{b}, b, \bar{a}, \bar{d}, \bar{e}$ subsequently and gain the result as seen on the figure. No contradiction may be found, therefore the solution of the original problem is as follows: $\text{tv}(a) = \text{tv}(b) = 0$, $\text{tv}(c) = \text{tv}(d) = 1$, $\text{tv}(e) = 0$.

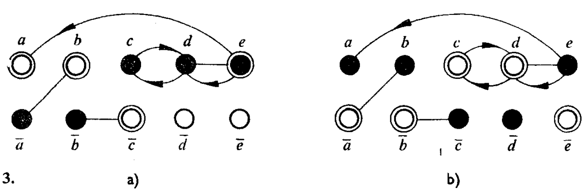

Fig. 3.

Let us solve the problem given in example 2. The following sets will participate in its solution:

set of all propositions p_i	$\ldots \mathscr{P} = \{a \Rightarrow \bar{d}, b \Rightarrow c, a \,\&\, c \Rightarrow d, \bar{b} \,\&\, c \Rightarrow a \,\&\, d, \bar{a} \,\&\, b \Rightarrow \bar{c}\}$
set of all basic propositions	$\ldots \mathscr{B} = \{a, b, c, d\}$
fundamental set of propositions	$\ldots \mathscr{S} = \{a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d}, a \,\&\, c, a \,\&\, d, \bar{a} \,\&\, b, \bar{b} \,\&\, c\}$

The graphical representation of the sets $\mathscr{S}$ and $\mathscr{P}$ is shown on the fig. 4 together with the chosen truth-value of the proposition a, $\text{tv}(a) = 1$. For colouring this triangular graph the afore-said rules may be used, but they are not sufficient. It is necessary to join some rules concerning the role of conjunctions in the logical inference and transform these rules into the rules for colouring the graph. For this purpose we shall use a few new terms.

A vertex of the triangular graph is said to be a diagonal vertex iff it is situated on the main diagonal of the scheme, otherwise it is said to be a subdiagonal vertex. The diagonal vertices x, y are said to be projections of the subdiagonal vertex $x \& y$. Using these words we can write:

6. The conjunction $x \& y$ is true iff both propositions x, y are true.
7. If a conjunction $x \& y$ is false and one of the propositions x, y is true, then the other is false.
8. If the proposition x is false, then every conjunction $x \& y$ is false.

6. The subdiagonal vertex is doubled iff both its projections are doubled.
7. If a subdiagonal vertex is black and one of its projections is doubled, then the other is black.
8. If a diagonal vertex is black, then all subdiagonal vertices situated in the same row or column are black, too.

The figure 4 shows the result of colouring the graph by application of the rules 1 – 8. The procedure may be marked briefly in the following comprehensible way: Choice: a doubled, 4: $\bar{d}$ doubled, 2: d black, 5: $a \& c$ black, 7: c black, 5: b black, 2: $\bar{b}$, $\bar{c}$ doubled, 8: $a \& d$, $c \& \bar{b}$, $b \& \bar{a}$ black. All vertices of the graph are coloured and no contradiction may be found, thus, we have got one solution of the given problem: $\text{tv}(a) = 1$, $\text{tv}(b) = \text{tv}(c) = \text{tv}(d) = 0$.

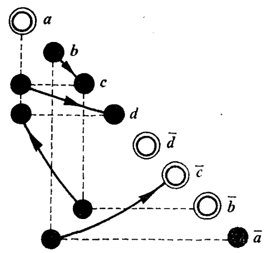

Fig. 4.

Supposing $\text{tv}(a) = 0$ we could apply the rules 2,8 and 5 and mark $\text{tv}(\bar{a}) = 1$, $\text{tv}(a \& c) = 0$, $\text{tv}(a \& d) = 0$ and $\text{tv}(c \& \bar{b}) = 0$, but further no other rule may be applied. Whenever we cannot apply the rules 1 – 8 for colouring the graph, we must choose colour of some vertex and remember the other possibility of the choice. In such cases it is usually necessary to draw a duplicate of the considered graph (with yet achieved state of its colouring, of course) and mark different colours of the discussed vertex on these two graphs. Then, the colouring procedure may be continued. In this way two other solutions of the problem may be found.

The rules 1 – 8 and the additional rule for the choice of colour of one vertex are sufficient for colouring the square graphs (see fig. 2b). This fact enables us to solve all simple logical problems of the type 1 (see the section 2 of this paper).

4. GRAPHICAL ALGORITHM FOR SOLVING SIMPLE ZEBRAS

A problem ending by the question "Who is breeding the zebra?" was published in many magazines all over the world. The name of this animal is often used for denoting problems similar to the above-mentioned famous one. Let us remember the main features of the mentioned problem without quoting its long text which gives some information about properties of five men living in five houses. These men differ each from other in nationality, favourite drink, sport, bred animal etc; every man has, of course, exactly one nationality, favourite drink, favourite sport, is breeding exactly one animal etc. The data are formulated in a little special way, for example by the following sentences: "The man drinking milk breeds a cat. The athlete is a French. The German does not drink wine". The solver's task lies in determinating all properties of the particular men as far as their nationalities, favourite drinks etc. are concerned.

The above examples of the "zebra" conditions show that a pair of men's properties occur in every simple sentence. Having denoted the set of five men by the letter $\mathscr{X}$, we could formulate the sentence "The athlete is a French" as follows: "There exists a man $X \in \mathscr{X}$ that is an athlete and a French simultaneously". Generally speaking, the "zebra" problems may be taken for problems concerning the set of all propositions expressed in the form "There exists a man $X \in \mathscr{X}$ that has the properties ξ and η simultaneously", where ξ, η are the elements of a set of properties.

To simplify the definition of the "zebras" we shall introduce some symbols. The letter $\mathscr{X}$ will denote a finite set of n persons, objects etc., the letters $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_m$ will denote disjoint sets of properties of the elements $x \in \mathscr{X}$, $\mathscr{P}_i = \{p_{i1}, p_{i2}, \ldots, p_{in}\}$. The basic proposition "There exists an element $x \in \mathscr{X}$ that has the properties ξ, η" will be denoted by the symbol $[\xi, \eta]$, where $\xi \in \mathscr{P}_i$, $\eta \in \mathscr{P}_j$; $i, j = 1, 2, \ldots, m$. The set of all basic propositions $[\xi, \eta]$ will be denoted by the letter $\mathscr{B}$.

The zebra of the type $m \times n$ will be defined as a logical problem concerning a set $\mathscr{X} = \{x_1, x_2, \ldots, x_n\}$ and a system $\{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_m\}$ of sets of properties of $x \in \mathscr{X}$. Given a set $\mathscr{T}$ of propositions constructed from the basic propositions $[\xi, \eta]$ by means of logical connectives, we have to determine the truth-values of all elements of $\mathscr{B}$ so that the three following conditions are satisfied:

(1) All propositions included in $\mathscr{T}$ are true.
(2) For every $\xi \in \mathscr{P}_i$, $1 \leqq i \leqq m$, and every j, $1 \leqq j \leqq m$ and $j \neq i$, there exists exactly one $\eta \in \mathscr{P}_j$, so that tv $[\xi, \eta] = 1$.
(3) ξ, η are elements of a set $\mathscr{P}_i$ $(i = 1, 2, \ldots, m)$, then tv $[\xi, \eta] = 0$ for all $\xi \neq \eta$, and tv $[\xi, \eta] = 1$ for all $\xi = \eta$.

A zebra will be called the *simple zebra* iff all propositions included in $\mathscr{T}$ are propositions $[\xi, \eta]$, their negations or disjunctions, implications and equivalences constructed from two propositions $[\xi, \eta]$, $[\xi', \eta']$ only. The procedure of the graphical solution of simple zebras will be shown in solving the following problem.

Problem 2. Given four persons and three sets of their properties: $\mathscr{P}_1 = \{a_1, a_2, a_3, a_4\}$, $\mathscr{P}_2 = \{b_1, b_2, b_3, b_4\}$ and $\mathscr{P}_3 = \{c_1, c_2, c_3, c_4\}$. Every man has exactly one property included in $\mathscr{P}_1$ as well as in $\mathscr{P}_2$, $\mathscr{P}_3$. Determine all properties of each of these persons provided the following propositions are true: $[a_4, c_1]$, $[b_1, c_3]$, $\overline{[a_1, c_4]}$, $\overline{[a_3, b_2]}$, $[a_2, c_3] \vee [a_4, c_2]$, $[b_1, c_2] \vee [b_3, c_1]$, $[a_3, b_3] \Rightarrow [a_2, c_1]$.

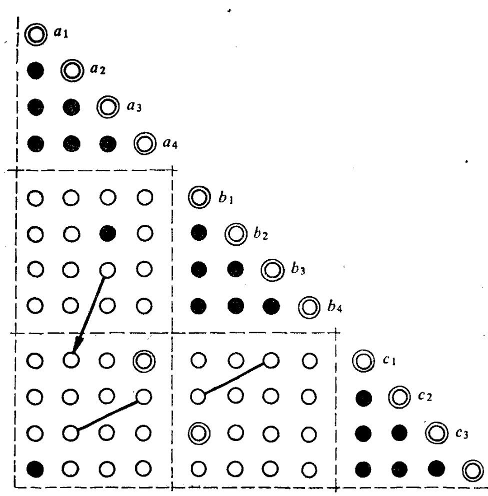

Fig. 5.

Let us show the graphical solution of this simple zebra. First, we have to draw a graph which could represent the set of all propositions $[\xi, \eta]$, where $\xi, \eta, \in \mathscr{P}_1 \cup \cup \mathscr{P}_2 \cup \mathscr{P}_3$. The presence of a conjunction in the formulation of the proposition $[\xi, \eta]$ suggests the using of a triangular graph (see fig. 5). The conditions (2) and (3) involved in the definition of zebras call for an outstanding representation of particular sets of properties. Therefore, the diagonal vertices will represent the elements of $\mathscr{P}_1$, $\mathscr{P}_2$ and $\mathscr{P}_3$ in such a way that all elements of one $\mathscr{P}_i$ $(i = 1, 2, 3)$, will be separated from others by lines. The horizontal and vertical lines shape square and triangular fields in the graph, these fields playing an important role in colouring the graph.

Now, the set $\mathscr{B}$ of all propositions $[\xi, \eta]$ is represented by vertices of the graph: the diagonal vertices represent the propositions $[\xi, \xi]$ which are true, whereas the other vertices situated in the triangular fields represent false propositions (see the last condition involved in the definition of zebras). The subdiagonal vertices may be coloured or connected by edges in order to represent the given true propositions (elements of $\mathscr{T}$) in the way described in the preceding sections of this paper. Fig. 5 shows the graphical representation of our simple zebra.

The colouring of the graph will be performed on the base of rules 1 – 5, but without any application of the rules 6 – 8 which refer to the case when the vertices represent the conjunctions $\xi \& \eta$ really. In our case, the vertices represent the propositions $[\xi, \eta]$, thus other rules must be used for colouring them. It is easy to see that the condition (2) may be expressed in the two following rules:

I. *If a vertex situated in a square field of the graph is doubled, then all vertices of this field lying in the same row or column with that doubled one, are black.*

II. *If all vertices of a row or column contained in a square field are black with the exception of one vertex only, then this remaining vertex must be doubled.*

Applying the rule I we can colour the vertices in two rows and columns of the graph and obtain an occasion for using the rules 3 and 5 immediately. Thereafter, the rule I may be applied again, but some vertices remain uncoloured yet. The most important rule for colouring the zebra graphs will be derived only.

Let us concentrate our attention to the rectangles sketched on the fig. 6a,b. The unordered 4-tuples of vertices (small circles) may be taken for vertices of a rectangle in the usual geometrical sense. The truth-values of the propositions $[\alpha, \gamma]$, $[\alpha, \delta]$, $[\beta, \gamma]$, $[\beta, \delta]$, resp. $[\alpha, \beta]$, $[\beta, \beta]$, $[\alpha, \delta]$, $[\beta, \delta]$, are to some extent mutually dependant.

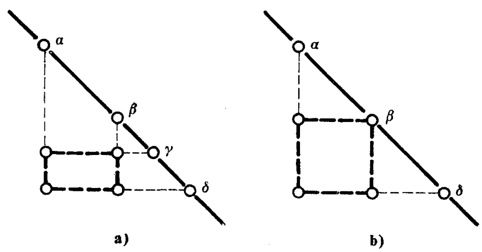

Fig. 6.

The truthfulness of three propositions from the considered 4-tuple makes it sure that there exists an object $x \in \mathscr{X}$ which has all four, resp. three, properties simultaneously, i.e. all four considered propositions are true. In other words, supposing two from four considered propositions are true, we can assert that the remaining two propositions have the same truth-value. The following rule will express thic fact in the language of graphs.

III. Rectangle's rule. *If two of four vertices of a rectangle (with horizontal and vertical sides) are doubled, then the two remaining vertices have the same colour, i. e. both are doubled or both are black.*

This rule is mostly applied in the case depicted on the fig. 6b; supposing $[\alpha, \beta]$ is doubled, we can vary the position of δ and pass through the whole columns below α and β. Doing so, we transfer the known colours from one column into the other; when β, δ belong to the same set $\mathscr{P}_i$ of properties, then the rule III gives the same results as the rule I. Of course, many occasions exist there for applying the rule III; the systematical use of it enables us to mark a considerable number of consequences of the given true propositions.

The final state of the graph colouring should show that there exists a person having the properties a_1, b_2 and c_2 simultaneously. Analogically, the triples of properties a_2, b_1, c_3, resp. a_3, b_4, c_4, resp. a_4, b_3, c_1, belong to a person mentioned in the text of the problem 2. These results may be read immediately from the four longest columns of the coloured graph.

Obviously, the method described above appears to be more advantageous for solving the simple zebras of higher degrees, for example the zebras of type 6×6, 6×7, 8×8 etc. The conditions concerning the arrangement of the considered persons or objects are very popular in the zebra texts (the man X is sitting on the left side of the man Y, the man X is sitting face-to-face with the man Y etc.). Such conditions may usually be expressed by means of a series of equivalences and negations and graphically represented, too.

5. FINAL REMARKS

The graphical method provides an accessible tool for solving simple logical problems which are solvable in the frame of propositional logic. It is no need to use the symbolic language, the graphs may be drawn on the base of the text directly. The colouring of the graph is a more concrete activity for beginners than the truth-table analysis or the propositional calculus. For this reason, the graphical method is regarded as a device for teaching the elements of logic.

Of course, the method itself may be generalized in many ways. Some problems solvable in the frame of the logic of classes may be solved by graphs, too. The inclusion of classes iI a binary relation which is closely connected with implications; this fact enables us to make use of colouring suitable graphs. The method may be used in many-valued logics, too. The construction of electrical devices working on the base of the rules mentioned above seems to be quite justified.

Nevertheless, the graphical method should be considered as an introductory method which has to prepare the users to the understanding the more powerfull methods.

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VÝTAH

Řešení jednoduchých logických úloh pomocí vybarvování grafů

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Některé logické úlohy lze řešit pomocí grafů, jež znázorňují binární relace definované na konečných množinách výroků. Pravdivostní hodnoty výroků jsou vyjádřeny barvou uzlů grafu; pravidla pro vybarvování grafu vyjadřují pravidla logického vyvozování založená na základních vlastnostech negací, disjunkcí, implikací a konjunkcí výroků.

V článku jsou řešeny dva typy problémů:

1. *Je dána pravdivá konjunkce* $p_1 \& p_2 \& \dots \& p_n$ *výroků utvořených z několika základních výroků* $b_1, b_2, \dots, b_k$ *pomocí logických spojek. Máme určit pravdivostní hodnoty výroků* $b_1, b_2, \dots, b_k$.

Výroky p_i $(i = 1, 2, \dots, n)$ lze upravit tak, aby měly jednu z forem x, $\bar{x}$, $x \vee y$, $x \Rightarrow y$, kde každý z výroků x, y obsahuje minimální počet logických spojek. Množina všech těchto výroků x, y je nazvána fundamentální množinou $\mathscr{S}$, zatímco množina všech výroků p_i je označena $\mathscr{P}$. Množiny výroků $x \vee y$, resp. $x \Rightarrow y$, obsažených v $\mathscr{P}$ lze považovat za binární relace $\mathscr{D}$, resp. $\mathscr{I}$, které jsou podmnožinami kartézského součinu $\mathscr{S} \times \mathscr{S}$. Prvky relace $\mathscr{D}$ jsou znázorněny neorientovanými hranami, prvky relace $\mathscr{I}$ orientovanými hranami grafu $\mathfrak{G} = [\mathscr{S}, \mathscr{D} \cup \mathscr{I}]$. Jsou-li prvky množiny $\mathscr{S}$ vyjádřeny pouze v jedné z forem u, $u \& v$, $u \vee v$, $u \Rightarrow v$, $u \Leftrightarrow v$, kde u, v jsou základní výrobky nebo jejich negace, nazveme příslušnou logickou úlohu *jednoduchou logickou úlohou*. Algoritmus grafického řešení těchto úloh se skládá z osmi pravidel a z úmluvy o volbě pravdivostní hodnoty jednoho výroku v případě, že žádné z osmi pravidel nelze aplikovat.

2. *Zebra typu* $m \times n$ je definována jako logický problém týkající se množiny $\mathscr{X} = \{x_1, x_2, \dots, x_n\}$ a systému $\{\mathscr{P}_1, \mathscr{P}_2, \dots, \mathscr{P}_m\}$ množin vlastností prvků $x \in \mathscr{X}$.

Je dána množina $\mathscr{T}$ výroků, jež jsou sestrojeny ze základních výroků tvaru „existuje $x \in \mathscr{X}$, které má vlastnosti ξ, η" pomocí logických spojek. *Máme určit pravdivostní hodnoty všech základních výroků (značených stručně $[\xi, \eta]$) tak, aby byly splněny tyto podmínky:*

(1) *Všechny prvky množiny $\mathscr{T}$ jsou pravdivé výroky.*

(2) *Ke každému $\xi \in \mathscr{P}_i$, $1 \leqq i \leqq m$, a ke každému j, $1 \leqq j \leqq m$* a $i \neq j$, existuje *právě jedno $\eta \in \mathscr{P}_j$, tak, že výrok $[\xi, \eta]$ je pravdivý.*

(3) *Jsou-li ξ, η prvky téže množiny $\mathscr{P}_i$ $(i = 1, 2, \ldots, m)$, pak výrok $[\xi, \eta]$ platí při $\xi = \eta$ a neplatí pro žádná dvě různá ξ, η.*

Za *jednoduchou zebru* je považována ta zebra, jejíž množina $\mathscr{T}$ obsahuje jen základní výroky, jejich negace a disjunkce, implikace a ekvivalence vytvořené ze základních výroků. Algoritmus pro řešení jednoduchých zeber zahrnuje pravidla $1-5$, úmluvu o volbě a tři specifická pravidla I, II, III. Pravidlo III (obdélníkové pravidlo) se týká čtveřic výroků znázorněných na obr. 6a,b.

Grafickou metodu lze zobecnit pro složitější úlohy, pro vícehodnotové logiky a bylo by možno modelovat její algoritmy elektrotechnicky. Její užitečnost však spočívá v názornosti řešení úloh, která přispívá k osvojení základních logických poznatků.

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