KYBERNETIKA ČÍSLO 5, ROČNÍK 5/1969

On the Synthesis of Stationary Dynamical Systems

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In this paper the Wiener-Hopf equation is examined from the point of view of the distribution theory. The application of the solution of this equation to the synthesis of the stationary dynamical systems is given.

Let $\mathscr{D}$ denote the set of all complex-valued infinitely differentiable functions on real line having compact supports and equipped with the topology of Schwartz [8]. A random distribution X, as defined by Ito [5], is a continuous linear map of $\mathscr{D}$ into the L^2 of some probability space $\{\Omega, \mathscr{K}, \mathbf{P}\}$. Let $\langle \varphi, f \rangle$ and $X\varphi$ denote the value of the scalar distribution f and the value of the vectorial distribution X respectively for the function $\varphi \in \mathscr{D}$. We put $\mathsf{E}[X\varphi]$ for the expectation (mean value) of the random variable $X\varphi$ with respect to the probability measure $\mathbf{P}$. Then the mean m_X and the covariance K_X of the random distribution X are defined by

$$\langle \varphi, \mathsf{m}_X \rangle = \mathsf{E}[X\varphi] ,$$

$$\mathsf{K}_X(\varphi, \psi) = \mathsf{E}[(X\varphi)(\overline{X\psi})]$$

for every $\varphi \in \mathscr{D}$, $\psi \in \mathscr{D}$. The random distribution X is stationary if for any real number h we have

$$\langle \tau_h\varphi, \mathsf{m}_X \rangle = \langle \varphi, \mathsf{m}_X \rangle ,$$

$$\mathsf{K}_X(\tau_h\varphi, \tau_h\psi) = \mathsf{K}_X(\varphi, \psi)$$

where the translation operator τ_h is defined by

$$\tau_h \, \varphi(t) = \varphi(t + h) .$$

The covariance $\mathsf{K}_X(\varphi, \psi)$ of a stationary random distribution X is a hermitian bilinear functional, continuous separately in each coordinate and invariant under all translations. Hence, by a consequence of the theorem of kernels of Schwartz, there exists (Gelfand, Vilenkin [4]) a distribution k_X on $\mathscr{D}$ such that

$$K_X(\varphi, \psi) = \langle \varphi * \bar{\hat{\psi}}, k_X \rangle$$

where $\hat{\psi}(s) = \psi(-s)$ and $*$ is the convolution. We shall call k_X also the covariance distribution or sometimes simply the covariance of X.

Now, let be two random distributions X and Y. We define the mutual covariance between X and Y, and we denote it by K_{XY}, the hermitian bilinear functional, continuous separately in each coordinate

$$K_{XY}(\varphi, \psi) = E[(X\varphi)\,(\overline{Y\psi})]\,,$$

where $\varphi \in \mathscr{D}$, $\psi \in \mathscr{D}$, and we suppose that this mutual covariance is invariant under all translations, i.e.

$$K_{XY}(\varphi, \psi) = K_{XY}(\tau_h\varphi, \tau_h\psi)$$

for all real numbers h. Then there exists a distribution k_{XY} on $\mathscr{D}$ such that

$$K_{XY}(\varphi, \psi) = \langle \varphi * \bar{\hat{\psi}}, k_{XY} \rangle\,.$$

A stationary dynamical system $\mathscr{W}$ is a mathematical object $\mathscr{W} = [X, w, Y]$ (see Fig. 1), where X is a stationary random distribution – the input of the dynamical

Fig. 1.

system –, Y is also a stationary random distribution – the output of the dynamical system –, and w is a scalar distribution with compact support – the characteristic of the dynamical system – such that the correspondence between output and input of the system is given by

$$Y = w * X\,.$$

We suppose that the random distributions X and Y have mean values equal to zero and that they are stationarily correlated, i.e. with mutual covariances K_{XY} and K_{YX} invariant under all translations.

The synthesis problem for dynamical systems is the following: to find the characteristic w, and sometimes the input X of a dynamical system $\mathscr{W} = [X, w, Y]$ such that the output Y would be "near" to the ideal output Z. In this direction we will prove two theorems.

Let $\mathscr{V} = [X, v, Z]$ be a stationary dynamical system with

$$Z = v * X$$

where v is a scalar distribution with compact support and X, Z are stationary random distributions, stationarily correlated. Then we prove

Theorem 1.

$$(1)\qquad \mathsf{k}_{ZX} = v * k_X .$$

Proof. For every $\varphi \in \mathscr{D}$, $\psi \in \mathscr{D}$ we have

$$(2)\qquad \langle \varphi * \bar{\tilde{\psi}}, \mathsf{k}_{ZX} \rangle = \mathsf{K}_{ZX}(\varphi, \psi) = \mathsf{E}[(Z\varphi)(\overline{X\psi})] =$$

$$= \mathsf{E}\{[(v * X)\varphi](\overline{X\psi})\} = \mathsf{E}[(X\varphi_1)(\overline{X\psi})] = \langle \varphi_1 * \bar{\tilde{\psi}}, \mathsf{k}_X \rangle$$

where

$$\varphi_1(s) = \langle \varphi(t + s), v(t) \rangle .$$

But

$$(3)\quad (\varphi_1 * \bar{\tilde{\psi}})(s) = \int_{-\infty}^{+\infty} \varphi_1(s - \xi)\, \bar{\tilde{\psi}}(\xi)\, \mathrm{d}\xi = \int_{-\infty}^{+\infty} \langle \varphi(t + s - \xi), v(t) \rangle\, \bar{\tilde{\psi}}(\xi)\, \mathrm{d}\xi =$$

$$= \left\langle \int_{-\infty}^{+\infty} \varphi(t + s - \xi)\, \bar{\tilde{\psi}}(\xi)\, \mathrm{d}\xi, v(t) \right\rangle = \langle (\varphi * \bar{\tilde{\psi}})(t + s), v(t) \rangle .$$

Then from (2) and (3) we obtain:

$$\langle \varphi * \bar{\tilde{\psi}}, \mathsf{k}_{ZX} \rangle = \langle (\varphi_1 * \bar{\tilde{\psi}})(s), \mathsf{k}_X(s) \rangle =$$

$$= \langle \langle (\varphi * \bar{\tilde{\psi}})(t + s), v(t) \rangle, \mathsf{k}_X(s) \rangle = \langle \varphi * \bar{\tilde{\psi}}, v * \mathsf{k}_X \rangle ,$$

i.e.

$$\mathsf{k}_{ZX} = v * \mathsf{k}_X ,$$

q.e.d.

Remark. If X and Y are usual stochastic processes and the distributions k_{ZX}, v and k_X are defined by the locally integrable functions $\mathsf{k}_{ZX}(z)$, $v(z)$, $\mathsf{k}_X(z)$, respectively, the equation (1) is the well known Wiener-Hopf equation (Lee [6]).

Let now $\mathscr{W} = [X, w, Y]$ be a stationary dynamical system, where the characteristic w has compact support and X, Y are stationary random distributions, stationarily correlated. Let also be the desired ideal output Z, a stationary random distribution, so that X and Z are stationarily correlated. Then

Theorem 2. *If the scalar distribution v with compact support is the solution of Wiener-Hopf equation*

$$(4)\qquad \mathsf{k}_{ZX} = v * \mathsf{k}_X$$

and if for every $\varphi \in \mathscr{D}$ we have

$$(5)\qquad |\langle \varphi, \mathsf{k}_Z - v * \mathsf{k}_{XZ} \rangle| < \varepsilon$$

then taking $w = v$ we obtain

$$\left|\langle \varphi, \mathsf{k}_{Z-Y} \rangle\right| < \varepsilon .$$

Proof. From the definition of covariance, for every $\varphi \in \mathscr{D}$, $\psi \in \mathscr{D}$ we can write

$$(6) \qquad \langle \varphi * \bar{\hat{\psi}}, \mathsf{k}_{Z-Y} \rangle = \mathsf{K}_{Z-Y}(\varphi, \psi) = \mathsf{E}[(Z\varphi - Y\varphi)\overline{(Z\psi - Y\psi)}] =$$
$$= \mathsf{E}[(Z\varphi)(\overline{Z\psi})] - \mathsf{E}[(Z\varphi)(\overline{Y\psi})] - \mathsf{E}[(Y\varphi)(\overline{Z\psi})] + \mathsf{E}[(Y\varphi)(\overline{Y\psi})] .$$

But

$$Y = w * X$$

and if we put

$$\varphi_1(s) = \langle \varphi(t + s), w(t) \rangle , \quad \psi_1(s) = \langle \psi(t + s), w(t) \rangle$$

because

$$(\varphi_1 * \bar{\hat{\psi}})(s) = \langle (\varphi * \bar{\hat{\psi}})(t + s), w(t) \rangle ,$$
$$(\varphi * \bar{\hat{\psi}}_1)(s) = \langle (\varphi * \bar{\hat{\psi}})(t + s), \bar{\hat{w}}(t) \rangle$$

we have successively

$$(7) \qquad \mathsf{E}[(Z\varphi)(\overline{Z\psi})] = \langle \varphi * \bar{\hat{\psi}}, \mathsf{k}_Z \rangle ,$$

$$(8) \qquad \mathsf{E}[(Z\varphi)(\overline{Y\psi})] = \mathsf{E}\{(Z\varphi)[\overline{(w * X)\psi}]\} = \mathsf{E}[(Z\varphi)(\overline{X\psi_1})] =$$
$$= \langle \varphi * \bar{\hat{\psi}}_1, \mathsf{k}_{ZX} \rangle = \langle \varphi * \bar{\hat{\psi}}, \bar{\hat{w}} * \mathsf{k}_{ZX} \rangle ,$$

$$(9) \qquad \mathsf{E}[(Y\varphi)(\overline{Z\psi})] = \mathsf{E}\{[(w * X)\varphi](\overline{Z\psi})\} = \mathsf{E}[(X\varphi_1)(\overline{Z\psi})] =$$
$$= \langle \varphi_1 * \bar{\hat{\psi}}, \mathsf{k}_{XZ} \rangle = \langle \varphi * \bar{\hat{\psi}}, w * \mathsf{k}_{XZ} \rangle ,$$

$$(10) \qquad \mathsf{E}[(Y\varphi)(\overline{Y\psi})] = \mathsf{E}\{[(w * X)\varphi][\overline{(w * X)\psi}]\} = \mathsf{E}[(X\varphi_1)(\overline{X\psi_1})] =$$
$$= \langle \varphi_1 * \bar{\hat{\psi}}_1, \mathsf{k}_X \rangle = \langle \varphi * \bar{\hat{\psi}}_1, w * \mathsf{k}_X \rangle = \langle \varphi * \bar{\hat{\psi}}, \bar{\hat{w}} * w * \mathsf{k}_X \rangle .$$

From (6)–(10) we obtain for the covariance k_{Z-Y} of the error $Z - Y$ the expression

$$\mathsf{k}_{Z-Y} = \mathsf{k}_Z - w * \mathsf{k}_{XZ} - \bar{\hat{w}} * \mathsf{k}_{ZX} + \bar{\hat{w}} * w * \mathsf{k}_X .$$

Further, from (4) and by a simple calculation, we can write

$$(11) \qquad \mathsf{k}_{Z-Y} = \mathsf{k}_Z - v * \mathsf{k}_{XZ} + I$$

where

$$(12) \qquad I = (v - w) * \mathsf{k}_{XZ} + \bar{\hat{v}} * (w - v) * k_X +$$
$$+ (\bar{\hat{v}} * v - w * \bar{\hat{v}} - \bar{\hat{w}} * v + \bar{\hat{w}} * w) * \mathsf{k}_X .$$

 Now, let $\mathscr{F}$ be the Fourier transform and if we put

$$p = \mathscr{F}(v), \quad q = \mathscr{F}(w), \quad S_X = \mathscr{F}(\mathsf{k}_X), \quad S_{XZ} = \mathscr{F}(\mathsf{k}_{XZ})$$

we have from (12)

$$\mathscr{F}(I) = (p - q)\, S_{XZ} + \bar{p}(q - p)\, S_X + |p - q|^2\, S_X .$$

Obviously, if $w = v$, where v is the solution of Wiener-Hopf equation, then $p = q$ which implies that $\mathscr{F}(I) = 0$ and finally that $I = 0$. Now, from (11) and (5) we obtain that

$$|\langle \varphi, \mathsf{k}_{Z-Y} \rangle| < \varepsilon$$

for every $\varphi \in \mathscr{D}$. Q.e.d.

Remark. The Wiener-Hopf equation (4) and the condition (5) are non-contradictory. If we have a dynamical system $\mathscr{V} = [X, v, Z]$ and if there is the inverse dynamical system $\mathscr{V}^{-1} = [Z, v^{-1}, X]$, then from the theorem 1 we have

$$\mathsf{k}_{ZX} = v * \mathsf{k}_X$$

and

$$\mathsf{k}_{XZ} = v^{-1} * \mathsf{k}_Z \quad \text{i.e.} \quad \mathsf{k}_Z - v * \mathsf{k}_{XZ} = 0 .$$

Announcement. I would like to thank to Prof. R. Cristescu for general encouragement and specific advice.

(Received July 17th, 1967.)

REFERENCES

[1] Balagangadharan K.: The prediction theory of stationary random distributions. Memoirs of the College of Science University of Kyoto. Series A, Mathematics *33* (1960), 243—256.

[2] Chang S. S. L.: Synthesis of optimum control systems. McGraw-Hill Book Comp. Inc., New York—Toronto—London 1961.

[3] Cristescu R., Marinescu G.: Unele aplicatii ale teoriei distributiilor. Editura Academiei, Bucuresti 1966.

[4] Гельфанд И. М., Виленкин Н. Я.: Некоторые применения гармонического анализа. Оснащенные гильбертовы пространства. (Обобщенные функции. Выпуск 4). Гос. изд. физ.-мат. лит., Москва 1961.

[5] Ito K.: Stationary random distributions. Memoirs of the College of Science University of Kyoto *28* (1953), 209—223.

[6] Lee R. R. K.: Optimal estimation identification and control. Research Monograph No. 28, M. I. T. Press, Cambridge, Massachusetts.

[7] Mahadeorao Deo C.: Prediction theory of non-stationary processes. Sankhya, series A *27* (1965), 113—132.

[8] Schwartz L.: Théorie des distributions. I, II. Hermann, Paris 1950—1951.

[9] Солодовников В. В.: Статистическая динамика линейных систем автоматического управления. Гос. изд. физ. мат. лит., Москва 1960.

O syntéze stacionárních dynamických systémů

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Budiž stacionární dynamický systém $\mathscr{V} = [X, v, Z]$, kde vstup X a výstup Z jsou stacionární náhodné distribuce, které jsou stacionárně korelované a mají střední hodnoty rovné nule, a charakteristika v je skalární distribuce s kompaktním suportem, takže $Z = v * X$ ($*$ znamená konvoluci). Potom charakteristika v je řešením rovnice

$$k_{ZX} = v * k_X \,, \tag{1}$$

kde k_X je kovariance X a k_{ZX} je vzájemná kovariance X a Z. A opačně, je-li řešení v rovnice (1) charakteristikou dynamického systému $[X, v, Y]$ a je-li $|\langle\varphi, k_Z - v * k_{XZ}\rangle| < \varepsilon$ pro každé $\varphi \in \mathscr{D}$, potom $|\langle\varphi, k_{Z-Y}\rangle| < \varepsilon$ pro každé $\varphi \in \mathscr{D}$, kde $\mathscr{D}$ je množina všech komplexních neomezeně derivovatelných funkcí na reálné přímce, které mají kompaktní suporty.

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