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On Multiple Grammars

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A modification of formal grammars, so called multiple grammars, in which rules are (in various manners) applied in groups are studied. It is shown that classes of languages generated by such grammars forms a hierarchy between the class of context-free sets and the class of context-sensitive sets. Many further properties of multiple grammars are shown.

1. PRELIMINARIES AND INTRODUCTION

We shall mainly use the notation from [9]. Alphabet V is an arbitrary finite set, elements of V are symbols, V^* is free semigroup of strings over V, Λ denotes an empty string, Λ is the unity element of V^*, $V^\infty = V^* - \{\Lambda\}$. If $x = x_1x_2 \dots x_s \in V^*$ and $y = y_1y_2 \dots y_t \in V^*$ then $xy = x_1x_2 \dots x_sy_1y_2 \dots y_t$ is a string formed by concactenation of x and y. For $A, B \subset V^*$ is $AB = \{xy \mid x \in A, y \in B\}$. Let $A \subset V^*$. Denoting $A^1 = A$, $A^{n+1} = A^nA$ for $n \geqq 1$ then obviously $V^\infty = \bigcup_{n=1}^{\infty} V^n$, $V^* = V^\infty \cup \{\Lambda\}$. Denote $A^{\infty n} = \bigcup_{j=1}^{n} A^j$, $A^{*n} = A^{\infty n} \cup \{\Lambda\}$. $|x|$ denotes for $x \in V^*$ the length of x. $\emptyset$ denotes an empty set.

Let A, B be arbitrary sets. Then $A \otimes B$ denotes the cartesian product of A and B i.e. $A \otimes B = \{(x, y) \mid x \in A, y \in B\}$. Denote further $\bigtimes_{i=1}^{n} A_i = \{(x_1, \dots, x_n) \mid x_i \in A_i$ for $i = 1, 2, \dots, n\}$, $A^{\otimes n} = \bigcup_{i=1}^{n} \bigtimes_{j=1}^{i} A$. The following convention will be broadly used: If A is a certain set then the set $\{\bar{a} \mid \bar{a}$ *is for* $a \in A$ *an abstract symbol*$\} = \{\bar{a} \mid a \in A\}$ denotes the set disjoint with all the sets discussed in the given proof and there is one to one correspondence between a's and $\bar{a}$'s.

Formal grammar is quartuple $G = (V_N, V_T, R, S)$ where V_N and V_T are nonterminal and terminal alphabets respectively $V_N \cap V_T = \emptyset$, $S \in V_N$ is the initial symbol and $R \subset V_N^\infty \otimes (V_N \cup V_T)^*$ is a finite binary relation. Elements of R are rules, R is called the set of rules of G. V will denote unless stated otherwise the set $V_T \cup V_N$.

The sequence $W = (w_0, w_1, \ldots, w_n)$ of strings over V^* is the derivation over G of the length n if it holds for $i = 0, 1, \ldots, n-1$, $w_i = puq$, $w_{i+1} = pvq$ where $p, q \in V^*$, $(u, v) \in R$. The string $y \in V^*$ is over G derivable from $x \in V^*$ (y is a consequence of x) if exists a derivation over G of y from x i.e. over G exists a derivation $W = (x, w_1, \ldots, w_{n-1}, y)$. A derivation W over G is nontrivial if the length of W is at least 1, a derivation over G is trivial if it is of the length 0. The rule $(u, v) \in R$ is applicable on $x \in V^*$ if $x = puq$, $x_1 \in V^*$ is a direct consequence of x if $x = puq$, $x_1 = pvq$, $(u, v) \in R$. Write $x \underset{G}{\Rightarrow} x_1$ if x_1 is over G a direct consequence of x. $x \underset{G}{\Rightarrow}^* y$ if y is a consequence of x. $x \underset{G}{\Rightarrow}^\infty y$ if it exists nontrivial derivation over G of y from x. The language (or the set) $L(G)$ generated by G is the set

$$L(G) = \{x \mid x \in V_T^*, S \underset{G}{\Rightarrow}^* x\}$$

a formal grammar $G = (V_N, V_T, R, S)$ is context-sensitive if $|u| \leqq |v|$ for every $(u, v) \in R$. We define context-sensitive grammars in other way as in [10]. Note, however, that a set is generated by a context-sensitive grammar G (in our sense) if and only if it is generated by a Chomsky's type 1 grammar.

A grammar G is context-free if $R \in V_N \otimes V^*$. A grammar G is Λ-free if $R \subset V_N \otimes \otimes V^\infty$. A set $A \subset V_T^*$ is a phrase-structure set (context-sensitive set, context-free set) if $A = L(G)$ for a formal (context-sensitive, context-free respectively) grammar G. If there will be no danger of misunderstanding we shall say the derivation instead of the derivation over G and write $\Rightarrow, \Rightarrow^*, \Rightarrow^\infty$ instead of $\underset{G}{\Rightarrow}, \underset{G}{\Rightarrow}^*, \underset{G}{\Rightarrow}^\infty$.

Now we can turn to the main topics of this paper. The very important feature of the grammar $G = (V_N, V_T, R, S)$ is the following property: If $W = (w_0, w_1, \ldots, w_n)$ is a derivation over G and $w_n = puq$ and moreover $(u, v) \in R$ then $W' = (w_0, w_1, \ldots \ldots, w_n, pvq)$ is a derivation over G. A rule can be therefore applied in the n-th step of derivation independently (in certain sense) on what was the rules applied in previous steps or indenpendently on that whether an another rule (u', v') can be applied on w_n. This assumption of indenpendency can be weakened in several ways. One way to realize this idea is discussed in [8]. Idea discused there can by roughly described in the following way. A partial ordering $<$ is defined on the set R of rules and a rule $(u, v) \in R$ is applicable on $x \in V^*$ if $x = puq$ (i.e. (u, v) is applicable on R in "normal sense") and no rule $(u', v') \in R$ for which it holds $(u', v') > (u, v)$ (i.e. which is "greater" than (u, v)) can be applied on x. Now we can define a derivations over such a grammar and a language generated by it similary as it is defined for "normal" grammars described above. It was shown that the indicated facility increases the generative power of context-free grammars (i.e. there exists a contexfree grammar $G^> = (V_N, V_T, R, >, S)$ with ordering of rules for which $L(G^>)$ is not a context-free language) but does not increase the generative power of context-sensitive or formal grammars.

The main feature of grammars with ordering of rules is that a rule can be applied only if another rules can not be applied. We shall go in another direction. We shall study grammars for which rules are applied in groups so that if a rule is applied in the given step of a derivation then, roughly speaking, some rules must be applied in the "following" steps. We shall show that formal (context-sensitive respectively) grammars with this facility generates phrase-structure (resp. contex-sensitive) sets meanwhile the classes of sets generated by context-free grammar with this facility forms a hierarchy between the class of context-free and class of context-sensitive sets.

2. DEFINITIONS AND BASIC PROPERTIES

Definition 1. Relational grammar G is the quintuple $G = (n, V_N, V_T, Q, S)$, where n is a positive integer, the multiplicity of G, $V_T = (V_{T_1}, V_{T_2}, \ldots, V_{T_n})$ is an n-tuple of terminal alphabets, $V_N = (V_{N_1}, V_{N_2}, \ldots, V_{N_n})$ is an n-tuple of nonterminal alphabets, $Q \subset \bigtimes_{i=1}^{n} R_i$ where $R_i \subset V_{N_i}^{\infty} \otimes (V_{N_i} \cup V_{T_i})^*$ is for $i = 1, 2, \ldots, n$ a finite binary relation, $S = (S_1, \ldots, S_n) \in \bigtimes_{i=1}^{n} V_{N_i}$. G is a context-sensitive or a context-free or a Λ-free grammar if R_i are for $i = 1, 2, \ldots, n$ context-sensitive or context free or Λ-free relations respectively. Let $V_i = V_{N_i} \cup V_{T_i}$. For $x, y \in \bigtimes_{i=1}^{n} V_i^*$ write $x \underset{G}{\Rightarrow} y$ if $x = (u_{11}u_1x_{12}, x_2u_2x_{22}, \ldots, x_{n1}u_nx_{n2})$, $y = (x_{11}v_1x_{12}, x_{21}v_2x_{22}, \ldots, x_{n1}v_nx_{n2})$ and $((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)) \in Q$. The sequence $(w_0, w_1, \ldots, w_m)$, $w_i \in \bigtimes_{j=1}^{n} V_j^*$, is a derivation over G if it holds for $i = 0, 1, 2, \ldots, m - 1$, $w_i \underset{G}{\Rightarrow} w_{i+1}$. We write $w_0 \underset{G}{\Rightarrow}^* w_m$ if there exists a derivation $W = (w_0, w_1, \ldots, w_{m-1}, w_m)$ over G. The relation $R(G)$ generated by G is the set

$$R(G) = \{x \mid x \in \bigtimes_{i=1}^{n} V_{T_i}^*, S \underset{G}{\Rightarrow}^* x\}$$

Denote further

$$L_R(G) = \{x \mid x \in (\bigcup_{i=1}^{n} V_{T_i})^*,$$

$$\textit{there exists } (x_1, x_2, \ldots, x_n) \in R(G) \textit{ so that } x = x_1x_2 \ldots x_n\}.$$

$A \subset (\bigcup_{i=1}^{n} V_{T_i})^*$ is a R-set (resp. a context-sensitive R-set resp. a context-free R-set) of the multiplicity n if there exists a relational (resp. context-sensitive relational resp. context-free relational) grammar G of the multiplicity n so that $A = L_R(G)$.

Definition 2. The multiple grammar is a fourtuple $G = (V_N, V_T, Q, S)$ where V_N, V_T are terminal and nonterminal alphabets respectively; $V_N \cap V_T = \emptyset$; $Q \in R^{\otimes n}$, where $R \subset V_N^{\infty} \otimes V^*$ is a finite binary relation, Q is the set of multirules, the elements of Q are multirules; $S \in V_N$. The multiplicity of G is the least integer n such that $Q \subset R^{\otimes n}$. A grammar associated with G is the grammar $G^{(a)} = (V_N, V_T, R, S)$. A multiple grammar G is context-sensitive or context-free or Λ-free if $G^{(a)}$ is context-sensitive or context-free or Λ-free respectively. We shall write for $x, y \in V^* = (V_T \cup V_N)^*$:

(i) $x \underset{G}{\Rightarrow}_1 y$ if there exists a multirule $((u_1, v_1), (u_2, v_2), \ldots, (u_s, v_s)) \in Q$ and a derivation $(w_0, w_1, \ldots, w_s)$ over $G^{(a)}$ of the following properties: $x = w_0$, $y = w_s$ and for $i = 0, 1, 2, \ldots, s-1$ there exists $x_i \in V^*$ so that $w_i = x_i u_{i+1} y_i$, $w_{i+1} = x_i v_{i+1} y_i$;

(ii) $x \underset{G}{\Rightarrow}_2 y$ if there is a multirule $((u_1, v_1), \ldots, (u_s, v_s)) \in Q$ so that $x = x_1 u_1 x_2 u_2 \ldots x_s u_s x_{s+1}$, $y = x_1 v_1 x_2 v_2 \ldots x_s v_s x_{s+1}$;

(iii) $x \underset{G}{\Rightarrow}_3 y$ if there is a multirule $((u_1, v_1), \ldots, (u_s, v_s)) \in Q$ so that $x = x_1 u_1 x_2 u_2 \ldots x_s u_s x_{s+1}$, $y = x_1 v_1 x_2 v_2 \ldots v_s x_{s+1}$ and it holds for no $i = 1, 2, \ldots, s$, $x_i u_i = x_i' u_i x_i''$ where $x_i'' \neq \Lambda$.

A sequence $(w_0, w_1, \ldots, w_m)$ of strings over V^* is a derivation over G of the type i, $i = 1, 2, 3$, if it holds for $j = 0, 1, 2, \ldots, m-1$, $w_j \underset{G}{\Rightarrow}_i w_{j+1}$. For $x, y \in V^*$ write $x \underset{G}{\Rightarrow}_i^* y$ if over G there exist a derivation $W = (w_0, w_1, \ldots, w_{m-1}, w_m)$ of the type i such that $x = w_0$, $y = w_m$. Further

$$L_i(G) = \{x \mid x \in V_T^*, S \underset{G}{\Rightarrow}_i^* x\}$$

$A \subset V_T^*$ is a M-set of the type i $(i = 1, 2, 3)$ if $A = L_i(G)$ for some multiple grammar G.

Proposition 1. *If A, B are M-sets of the type i then $A \cup B$ is M-set of the type i. If A, B are R-sets of the multiplicity n then $A \cup B$ is a R-set of the multiplicity n.*

Proof. Proof can be obtained by a slight modification of the proof of the theorem that the union of context-free sets is a context-free set.

Definition 3. $CF(G_1, G_2)$ is an abbreviation of the following proposition. If G_1 is a relational or a multiple grammar which is context sensitive or a context-free or Λ-free then G_2 is a relational or a multiple grammar respectively which is context-sensitive or context-free or context-free Λ-free respectively.

Lemma 1. *To every relational grammar G there exists a multiple grammar G_1, the multiplicity of which being equal to the multiplicity of G, so that $CF(G, G_1)$ and $L_1(G_1) = L_R(G)$.*

Proof. Without loss of generality we may assume that it holds for $G = (n, (V_{N_1}, V_{N_2}, \ldots, V_{N_n}), (V_{T_1}, \ldots, V_{T_n}), Q, (S_1, \ldots, S_n))$ that $(V_{N_i} \cup V_{T_i}) \cap V_{N_j} = \emptyset$ for $1 \leq i \neq j \leq n$. Putting

$$G_1 = (\{S\} \cup \bigcup_j V_{N_j}, \bigcup_j V_{T_j}, Q \cup \{(S, S_1 S_2 \ldots S_n)\}, S)$$

 where S is a new symbol we can easily verify that it holds $CF(G, G_1)$ and $L_1(G_1) = L_R(G)$.

Remark 1. Let Q be a finite set of multirules (of rules). Index of a (multi)rule is a positive integer. There is one-to-one correspondence between (multi)rules in Q and their indexes. A multirule $((p, q))$ will be often denoted (p, q).

Lemma 2. *To every multiple grammar G exists a relational grammar G_1 of the multiplicity at most two such that $L_R(G_1) = L_1(G)$. In the case that $L_1(G) \subset V_T^\infty - V_T$ then there exists a relational grammar G_1 of the multiplicity two such that $L_1(G_1) = L_R(G)$ and $CF(G, G_1)$.*

Proof. Theorem obviously holds if the multiplicity of G is 1. Let us put for $G = (V_N, V_T, Q, S)$, $G_1 = ((V_N, \bar{V}), (V_T, \emptyset), Q^r, (S, \bar{S}))$ where $\bar{V} = \{[j, i] \mid [j, i]$ *is for* a multirule $((u_1, v_1), \ldots, (u_s, v_s)) \in Q$ with the index j and for $1 \leqq i \leqq s$ an abstract symbol $\} \cup \{\bar{S}\}$.

To every multirule $((u_1, v_1), (u_2, v_2), \ldots, (u_s, v_s)) \in Q$, Q^r contains a set of multirules of the form $((u_1, v_1), (\bar{S}, [j, 2]))$, $((u_2, v_2), ([j, 2], [j, 3])), \ldots, ((u_s, v_s), ([j, s-1], \bar{S}))$ and the multirule $((u_s, v_s)([j, s-1], \Lambda))$. Especially for $(p, q) \in Q$, Q^r contains the multirules $((p, q), (S, \bar{S}))$ and $((p, q), (\bar{S}, \Lambda))$. It is straightforward matter to verify that $R(G_1) = L_1(G) \otimes \{\Lambda\}$ so that $L_R(G_1) = L_1(G)$ and the first assertion of the lemma follows.

The proof of the second assertion of the theorem is rather cumbersome so the main ideas of it only will be given. Details can be found in [14]. Let $\overline{aC}$ be for $a \in V_N \cup V_T$ and $C \in V_N$ an abstract symbol. Let further B be any symbol from $\bar{V}$ and A any symbol from V_N. The grammar G_1 from the first half of the proof can be modified so that a grammar G_2 is obtained so that it holds.

$$(S, \bar{S}) \underset{G_1}{\Rightarrow}^* (A, B) \quad \text{if and only if} \quad (S, \bar{S}) \underset{G_2}{\Rightarrow}^* (A, B),$$

$$(S, \bar{S}) \underset{G_1}{\Rightarrow}^* (\gamma a C, B), \quad a \in V, \quad C \in V_N, \quad \text{if and only if}\, (S, \bar{S}) \Rightarrow (\gamma \overline{aC}, B),$$

i.e. a pair (x, B), $x \in V^\infty V_N$, is over G_1 derivable if and only if it is over G_2 derivable a pair $(\bar{x}, B)$, where $\bar{x}$ is x with two last symbols joined into one abstract symbol.

$$(S, \bar{S}) \underset{G_1}{\Rightarrow}^* (\gamma b, B), \quad b \in V_T, \quad B \in \bar{V}, \quad \text{if and only if} \quad (S, \bar{S}) \underset{G_2}{\Rightarrow}^* (\gamma, [b, B]),$$

where $[b, B]$, $b \in V_T$, $B \in \bar{V}$, is an abstract symbol. It gives the possibility to use the multirules $((p, q), ([b, B], b))$ instead of $((p, q), (B, \Lambda))$.

$$(S, \bar{S}) \underset{G_1}{\Rightarrow}^* (\gamma b, \Lambda) \quad \text{if and only if} \quad (S, \bar{S}) \underset{G_2}{\Rightarrow}^* (\gamma, b).$$

Obviously $L_R(G_2) = L_1(G)$. G_2 can be constructed so that it holds $CF(G_1, G_2)$. *QED.*

Corollary 1. *To every multiple grammar G there exists a multiple grammar G_1 of the multiplicity at most two such that $L_1(G) = L_1(G_1)$ and $CF(G, G_1)$.*

Proof. If the multiple grammar G is a general (a context-free respectively) multiple grammar the assertion of the theorem is a direct consequence of lemmas 1 and 2. Let G be a context-sensitive (Λ-free respectively). Then $L_1(G) \subset V_T^\infty$ and using the ideas used in the second part of the proof of lemma 2 it can be shown that there is a context-sensitive (Λ-free respectively) grammar G' so that $L_1(G') = L(G) - V_T$. But then $L_1(G) = L_1(G') \cup A$, where $A \subset V_T$. But $A = L_1(G_2)$ for a context-sensitive (respectively Λ-free) grammar and the theorem follows from the proof of proposition 1.

Lemma 3. Let $A = L_i(G)$ where i is equal to 2 or 3. Then there exists a multiple grammar G_1 of the multiplicity at most 2 so that $L_i(G) = L_i(G_1)$ and $CF(G, G_1)$.

Proof. Let $G = (V_N, V_T, Q, S)$. We put $\tilde{V} = \{\tilde{a} \mid \tilde{a}$ *is for* $a \in V_N \cup V_T = V$ *an abstract symbol*$\}$. Let further $\tilde{\Lambda} = \Lambda$ and for $x = x_1 x_2 \dots x_m$, $\tilde{x} = \tilde{x}_1 \tilde{x}_2 \dots \tilde{x}_m$. Let $\bar{V}$ have the same meaning as in the proof of the lemma 2. Put further $\bar{V}_1 = \{[j, i]_1, [j, i]_2 \mid [j, 1]_1, [j, 1]_2$ *are for* $[j, i] \in \bar{V}$ *abstract symbols*$\}$. Let $G_1 = (V_N^{(1)}, V_T, Q_1, \bar{S})$ where $V_N^{(1)} = \tilde{V} \cup \bar{V} \cup \bar{V}_1 \cup V_N \cup \{\bar{S}, \#, \#_1\}$ and let Q contain:

(a) rules $(\bar{S}, \# S), (\#, \Lambda)$

(b) to every multirule $((u_{11}A_1, v_1), \dots, (u_{s1}A_s, v_s)) \in Q$, where $A_i \in V_N$ for $i = 1, 2, \dots, s$, with the index j a sequence of multirules

$(j; 1)$ $\quad ((\#, \#_1), (u_{11}A_1, \tilde{u}_{11}[j, 1]))$
$(j; 2)$ $\quad (([j, 1], [j, 1]_1), (u_{21}A_2, \tilde{u}_{21}[j, 2]))$
....
$(j; s-1)$ $\quad (([j, s-2], [j, s-2]_1), (u_{s-1,1}A_{s-1}, \tilde{u}_{s-1,1}[j, s-1]))$
$(j'; s)$ $\quad (([j, s-1], [j, s-1]_2), (u_{s,1}A_s, v_s))$
$(j'; s-1)$ $\quad (([j, s-2]_1, [j, s-2]_2), (\tilde{u}_{s-1,1}[j, s-1]_2, v_{s-1}))$
....
$(j'; 1)$ $\quad ((\#_1, \#), (\tilde{u}_{11}[j, 1]_2, v_1))$

(c) to each rule $(u, v) \in Q$ the multirule $((\#, \#), (p, q))$.

It can be easily verified that if $\bar{S} \underset{G_1}{\overset{*}{\Rightarrow}}_2 R \underset{G_1}{\overset{\infty}{\Rightarrow}}_2 x \in V_T^*$ then $R = \xi Z$, $\xi \in \{\#, \#_1\}$. If a multirule of the type $(j; 1)$ is used in the i-th step of a derivation $W = (w_0, \dots, w_n)$, $w_n \in V_T^*$ of the type 2 or 3 then all the rules $(j; 2), \dots, (j; s-1), (j; s), \dots, (j'; 1)$ must be successively used in the following steps of W. The obviously $L_i(G) \subset L_i(G_1)$, $i = 2, 3$. As the reverse inclusion is obvious we have proved the first assertion of the theorem.

The proof of the second assertion is rather cumbersome so let again the main idea of it only will be given (see [14] for details). By a modification of the grammar G_1 a grammar $G_2 = (V_N^{(2)}, V_T, Q_2, [\#, S])$ can be obtained so that $\bar{S} \underset{G_1}{\overset{*}{\Rightarrow}}_i \alpha a y$, where $\alpha \in \{\#, \#_1\}$, $a \in V$, $\gamma \in V^*$ and $i = 2, 3$, if and only if $[S, \#] \underset{G_2}{\Rightarrow}_i [\alpha, a]\, \gamma$, $[\alpha, a]$ being for $a \in V$ and $\alpha \in [\#, \#_1\}$ an abstract symbol. It can be shown that G_2 can be constructed so that $CF(G, G_2)$ and $L_i(G) = L_i(G_2)$. Adding the rules $([\#, a], a)$ to G_2 we obtain the assertion of theorem.

It holds therefore

Theorem 1. *Let $A = L_R(G)$ where G is a relational grammar. Then there exists a relational grammar G_1 of the multiplicity at most two so that it holds $CF(G, G_1)$ and $L_R(G) = L_R(G_1)$. To every multiple grammar G and $i = 1, 2, 3$ exists a two-multiple grammar G_1 such that $CF(G, G_1)$ and $L_R(G_1) = L_R(G)$.*

Lemma 4. To every multiple grammar $G = (V_N, V_T, Q, S)$ there exists a multiple grammar G_1 such that $L_1(G) = L_2(G_1)$ and $CF(G, G_1)$.

Proof: Theorem 1 imply that it can be assumed without any loss of generality that G is of the multiplicity two. Put $G_1 = (V_N, V_T, Q_1, S)$ where $Q_1 = \{(u, v) \mid (u, v) \in Q\} \cup \{(u, v) \mid |u| \leq 3v$, there is a multirule $((u_1, v_1)(u_2, v_2)) \in Q$ so that $u = \alpha u_1 \beta$, $\alpha v_1 \beta = \alpha' u_2 \beta'$, $v = \alpha' v_2 \beta'\} \cup \{((u_1, v_1), (u_2, v_2)) \mid$ either $((u_1, v_1), (u_2, v_2)) \in Q$ or $((u_2, v_2), (u_1, v_1)) \in Q\}$, $v = \max \{|u| \mid (u, v) \in Q, ((u, v), (u_1, v_1)) \in Q\}$. It can be easily verified by induction that $L_1(G) = L_2(G_1)$. It is obvious that it holds $CF(G, G_1)$). QED.

Lemma 5. *To every multiple grammar $G = (V_N, V_T, Q, S)$ there exists a multiple grammar G_1 so that $L_2(G) = L_3(G_1)$ and $CF(G, G_1)$.*

Proof. It can be again assumed that G is two-multiple. Let us put $G_1 = (V_N \cup \bar{V}, V_T, Q_1, S_1)$ where $\bar{V} = \{[j, 1], \overline{[j, 1]}, [j, 2] \mid [j, 1], \overline{[j, 1]}, [j, 2]$ *are for a multirule with index j abstract symbols*$\}$.

Let further:

(a) Q_1 contain the following multirules
$(u, [j, 1]\, u')$, $(([j, 1], a), (u, [j, 1]\, u'))$, $([j, 1]\, u, v)$
for each $(u, v) \in Q$ with the index j, $u = au'$, $a \in V_N$,

(b) Let $r = ((au_1', v_1), (bu_2', v_2)) \in Q$ and let r have the index j. Then Q_1 contains the following multirules:
$(au_1', [j, 1]\, u_1')$,
$(([j, 1], a), (au_1', [j, 1]\, u_1'))$,
$(([j, 1], \overline{[j, 1]}), (bu_2', [j, 2]\, u_2'))$,
$(([j, 2], b), (bu_2', [j, 2]\, u_2'))$,
$((\overline{[j, 1]}\, u_1', v_1), ([j, 2]\, u_2', v_2))$.

It can be easily verified that if $xu_1yu_2z \underset{G}{\Rightarrow}_2 xv_1yv_2z$ where $u_1 = au_1$, $u_2 = bu_2'$ then $xu_1yu_2z \underset{G_1}{\overset{*}{\Rightarrow}}_3 x[j, 1]\, u_1'y[j, 2]\, u_2'z \underset{G_1}{\overset{*}{\Rightarrow}}_3 xv_1yv_2z$, i.e. $L_3(G_1) \supset L_2(G)$. By induction according to the length of derivation it can be shown that if $S \underset{G_2}{\Rightarrow}{}_3^* x \in (V_N \cup V_T)^*$ then $S \underset{G}{\Rightarrow}{}_2^* x$ and it follows $L_3(G_1) \subset L_2(G)$. Obviously it holds $CF(G, G_1)$, QED.

Lemma 6. *$L_3(G)$ is a context-sensitive set for arbitrary context-sensitive multiple grammar G.*

Proof. We can again assume that $G = (V_N, V_T, Q, S)$ is two-multiple, We shall construct a context-sensitive grammar $G_1 = (V_N', V_T, R, \bar{S})$ so that $L(G_1) = L_3(G)$. As the proof is rather cumbersome we shall describe the framework of it only. We put $\bar{V} = \{[u, j, i] \mid [u, j, i]$ is for $i = 1, 2, 3, 4$ and a multirule $r \in Q$ with index j where $r = (u, v)$ or $r = ((u, v_1), (u_2, v_2))$ or $r = ((u_1, v_1), (u, v_2))$ an abstract symbol$\}$.

$$\bar{V}_T = \{\bar{a} \mid \bar{a} \textit{ is for } a \in V_T \textit{ an abstract symbol}\},$$

$$V_N' = V_N \cup \bar{V}_T \cup \bar{V} \cup \{\#, \uparrow_0, \downarrow_0, \uparrow, \bar{S}\}.$$

Let us put further for $x = x_1 x_2 \ldots x_n \in V^*$, $\tilde{x} = \tilde{x}_1 \tilde{x}_2 \ldots \tilde{x}_n$ where $\tilde{\Lambda} = \Lambda$, $\tilde{x}_i = \bar{x}_i$ for $x_i \in V_T$ and $\tilde{x}_i = x_i$ for $x_i \in V_N$.

Let R contain the following rules.

(1) $(a\downarrow, \downarrow a), (a\downarrow_0\downarrow, \downarrow_0\downarrow a), (a\downarrow_0\downarrow_0\downarrow, \downarrow_0\downarrow_0\downarrow a)$ for each $a \in V_N' - (\bar{V} \cup \{\#, \downarrow_0, \uparrow_0\})$;

(2) $(BA\uparrow a\gamma_1, aBA\uparrow \gamma_1)$ for each $A = [u_1, j, 1] \in \bar{V}$, $B = [u_2, j, 2] \in \bar{V}$, $|u_1| = |a\gamma_1|$, $a\gamma_1 \neq u_1$, $a\gamma_1 \in (V_N \cup \bar{V}_T)^*$;

(3) $(B\uparrow a\gamma_1, aB\uparrow\gamma_1)$ for each $B = [u_2, j, 2] \in \bar{V}$, $|a\gamma_1| = |u_2|$, $a\gamma_1 \in (V_N \cup \bar{V}_T)^*$, $a\gamma_1 \neq u_2$, $a \in V_N \cup V_T$;

(4) $(B[u, j, 1]\uparrow u, u[u, j, 3]B\uparrow), ([u, j, 2]\uparrow u, u[u, j, 4]\downarrow)$ for each $B \in \bar{V}$;

(5) $(u[u, j, 4]\downarrow, \downarrow_0\downarrow\tilde{v}), (\#\uparrow_0\uparrow_0\uparrow, \#\uparrow_0[u, j, 2]\uparrow)$ for each $(u, v) \in Q$ with index j;

(6) $(\#\uparrow_0\uparrow_0\uparrow, \#[u_2, j, 2][u_1, j, 1]\uparrow), (u_2[u_2, j, 4]\downarrow, \downarrow_0\downarrow\tilde{v}_2), (u_1[u_1, j, 3]\downarrow_0\downarrow, \downarrow_0\downarrow_0\downarrow\tilde{v}_1)$ for $((u_1, v_1), (u_2, v_2)) \in Q$ with index j;

(7) $(\bar{a}, a)$ for each $\bar{a} \in \bar{V}_T$;

(8) $(\#, \Lambda), (\uparrow_0, \Lambda), (\uparrow, \Lambda))$;

(9) $(\#\downarrow_0\downarrow_0\downarrow, \#\uparrow_0\uparrow_0\uparrow), (\#\uparrow_0\downarrow_0\downarrow, \#\uparrow_0\uparrow_0\uparrow)$,

(10) $(\bar{S}, \#\uparrow_0\uparrow_0\uparrow S)$.

If $S \underset{G}{\Rightarrow}{}_3^* x \underset{G}{\Rightarrow}{}_3 y$ where $x = xu_1yu_2z$, $y = xv_1yv_2z$ then $\bar{S} \underset{G_1}{\Rightarrow}{}^* \#\uparrow_0\uparrow_0\uparrow\tilde{x}u_1\tilde{y}u_2\tilde{z} \underset{G}{\Rightarrow}$ $\underset{G}{\Rightarrow} \#[u_2, j, 2][u_1, j, 1]\tilde{x}u_1\tilde{y}u_2\tilde{z} \underset{G_1}{\Rightarrow}{}^* \#\tilde{x}_1u_1[u_1, j, 3]\tilde{y}u_2[u_2, j, 4]\downarrow\tilde{z} \underset{G_1}{\Rightarrow}{}^* \#\uparrow_0\uparrow_0\uparrow \cdot$ $\cdot\, \tilde{x}\tilde{v}_1\tilde{y}\tilde{v}_2\tilde{z}$. where $\tilde{x}_i\tilde{u}_i$ is not expressible in the form $\tilde{x}_i'u_i\tilde{x}_i''$, $\tilde{x}_i'' \neq \Lambda$. It can be easily verified that if $\bar{S} \underset{G_1}{\Rightarrow}{}^* \#\uparrow_0\uparrow_0\uparrow\tilde{y}$, $\tilde{y} \in (\bar{V}_T, \cup V_N)^*$ then $S \underset{G}{\Rightarrow}{}_3^* y$, i.e. $L(G_1) = L_3(G)$. The more detailed discussion can be found in [14].

Lemma 6 in [8] implies that $L(G)$ is a context-sensitive set. QED.

Theorem 2. *Let $\mathfrak{M}_2(CS)$ be the class of R-sets generated by relational context sensitive grammars. Let $\mathfrak{M}_i(CS)$ be for $i = 1, 2, 3$ the class of M-sets of the type i*

 generated by context-sensitive multiple grammars. Let further CS be the class of context-sensitive sets. Then

$$CS = \mathfrak{M}_R(CS) = \mathfrak{M}_1(CS) = \mathfrak{M}_2(CS) = \mathfrak{M}_3(CS)$$

Proof. As every context-sensitive grammar is a relational grammar we have $CS \subset \mathfrak{M}_R(CS)$. Lemma 1 implies $\mathfrak{M}_R(CS) \subset \mathfrak{M}_1(CS)$. By lemma 4 $\mathfrak{M}_1(CS) \subset \mathfrak{M}_2(CS)$. By lemma 5 $\mathfrak{M}_2(CS) \subset \mathfrak{M}_3(CS)$. Lemma 6 implies $\mathfrak{M}_3(CS) \subset CS$. QED.

Theorem 3. *Let $\mathfrak{M}_R(G)$ be the class of R-sets, let $\mathfrak{M}_i(G)$ be for $i = 1, 2, 3$ the class of M-sets of the type i. Let* RE *denotes the class of recursively enumerable sets.*
Then RE $= \mathfrak{N}_1(G) = \mathfrak{N}_2(G) = \mathfrak{N}_3(G) = \mathfrak{N}_R(G)$.

Proof. It can be shown by a modification of the proof of lemma 6 that $\mathfrak{N}_3(G) \subset RE$. Using this fact the proof of the theorem is very similar to the proof of the previous theorem.

Remark 2. We shall use the following notation CF is the class of context-free sets not containing the emply string, CF^A is the class of context-free sets, $\mathfrak{N}_R(CF)$ is the class of sets generated by context-free relational grammars, $\mathfrak{M}_R(CF) = \{A \mid A = L_R(G)$ *for some relational context-free and Λ-free grammar*$\}$. Let further for $i = 1, 2, 3$, $\mathfrak{M}_i(CF) = \{A \mid A = L_i(G)$ *for a multiple context-free and Λ-free grammar G*$\}$,

$$\mathfrak{N}_i(CF) = \{A \mid A = L_i(G) \textit{ for a multiple context-free grammar } G\} .$$

Lemma 7. $CF \subsetneqq \mathfrak{M}_R(CF)$.

Proof. Obviously $CF \subset \mathfrak{M}_R(CF)$ because to every context-free set not containing Λ there exists a context-free Λ-free grammar G so that $A = L(G)$ (see [9]) and G is obviously a multiple grammar. The fact that $CF \neq \mathfrak{M}_R(CF)$ follows from the following example.

Example 1. The set $B = \{a^n b^n c^n \mid n \geqq 1\}$ belongs to $\mathfrak{M}_R(CF)$ because $B = L_R(G)$ for the grammar

$$G = (3, (\{A\}, \{B\}, \{C\}), (\{a\}, \{b\}, \{c\}), Q, (A, B, C))$$

where

$$Q = \{((A, aA), (B, bB), (C, cC)), ((A, a), (B, b), (C, c))\} .$$

Theorem 4.

$$CF \subsetneqq \mathfrak{M}_R(CF) \subsetneqq \mathfrak{M}_1(CF) \subset \mathfrak{M}_2(CF) \subset \mathfrak{M}_3(CF) \subset CS .$$

Proof. It follows from the above given lemmas that it suffices to prove that $\mathfrak{M}_R(CF) \neq \mathfrak{M}_1(CF)$. If G is a relational grammar which is Λ-free and have the multiplicity n then it holds for every $x \in L(G)$ that $|x| \geqq n$. The example 1 indicates that there is $A \in \mathfrak{M}_R(CF) - CF$ i.e. if $A = L_R(G)$ for a relational context-free and Λ-free grammar G then the multiplicity of G must be 2 at least. The set $\{a^n b^n c^n \mid n \geqq 1\} \cup \cup \{a, b, c\} = B$ is not a context-free set and it can be easily shown that $B \in \mathfrak{M}_1(CF)$. If B were generated by a relational grammar then it would be $|x| \geqq 2$ for any $x \in B$ — a contradiction. Therefore $B \notin \mathfrak{M}_R(CF)$. QED.

Theorem 5.

$$CF^\wedge \subsetneqq \mathfrak{N}_R(CF) = \mathfrak{N}_1(CF) \subset \mathfrak{N}_2(CF) \subset \mathfrak{N}_3(CF) \subset RS$$

where RS is the class of recursive enumerable sets.

Proof. Directly from above proved lemmas and the following remark.

Remark 3. If follows from the theorem 4 and from the lemma 2 that there exists $B \in \mathfrak{M}_R(CF)$ to every $A \in \mathfrak{M}_1(CF)$ so that $A \supset B$, $A - B \subset V_T$. The question how "great" are the classes $\mathfrak{M}_i(CF) - \mathfrak{M}_j(CF)$ for $2 \leqq i \neq j \leqq 3$ is, however, open.

3. SOME FURTHER PROPERTIES OF MULTIPLE GRAMMARS

In order to illustrate the properties of multiple grammars two examples will be given.

Example 2.

$$P_1 = \bigcup_{k=2}^{\infty} \{C_1^n C_2^n \dots C_k^n \mid n \geqq 1,\ C_{2j} = a,\ C_{2j+1} = b\} \in \mathfrak{M}_2(CF).$$

Proof. Let we have the multiple grammar $G = (V_N, \{a, b\}, Q, S)$ where Q contains the following multirules.

(A) $(S, \#AS_1), (S_1, AS_1), (S_1, K_a)$.

These rules generate the set $\{\#A^{n-1}K_a \mid n \geqq 2\}$ if $\{\#, A, K_a\}$ is assumed to be the terminal alphabet.

(B1) $((\#, \#), (A, a), (K_a, K_1 B K_b))$,
(B2) $((\#, \#), (A, a), (K_1, K_1), (K_b, BK_b))$,
(B3) $((\#, a), (K_1, \#_b))$,
(C1) $((\#, \#), (A, a), (K_a, K_4 K_5))$,
(C2) $((\#, \#), (A, a), (K_4, K_4), (K_5, bK_5))$,
(C3) $((\#, a), (K_4, b), (K_5, b))$,
(D1) $((\#_b, \#_b), (B, b), (K_b, K_6 A K_a))$,
(D2) $((\#_b, \#_b), (B, b), (K_6, K_6), (K_a, AK_a))$,
(D3) $((\#_b, b), (K_6, \#))$,
(E1) $((\#_b, \#_b), (B, b), (K_b, K_7 K_8))$,
(E2) $((\#_b, \#_b), (B, b), (K_8, aK_8))$,
(E3) $((\#_b, b), (K_7, a), (K_8, a))$.

It holds for any derivation of the type 2 over G:

(a) If $x = \gamma \# A^{n-1} K_a$, $x \underset{G}{\Rightarrow}{}_2^* y \in \{a, b\}^*$ then $y = \gamma\varphi$ where $\#A^{n-1}K_a \underset{G}{\Rightarrow}{}_2^* \varphi$ and moreover any derivation of the type 2 over G of the string φ from $\# A^{n-1} K_a$ contains either an element $a^n \#_b B^{n-1} K_b$ or it holds that $\varphi = a^n b^n$.

(b) If $x = \gamma \#_b B^{n-1} K_b$ then it holds similary for $y = \{a, b\}^*$: if $x \underset{G}{\Rightarrow}{}^*_2 y$ then $y = \gamma\varphi$, $\#_b B^{n-1} K_b \Rightarrow^*_2 \varphi$ and either $\varphi = b^n a^n$ or any derivation of φ from $\gamma \#_b B^{n-1} K_b$ contains the element $\gamma b^n \# A^{n-1} K_a$.

It follows from (a) and (b) that $P_1 = L_2(G)$.

Example 3. $P_2 = \{a^{n^2+1} \mid n \geqq 1\} \in \mathfrak{M}_2(CF)$.

Proof. $P_2 = L_2(G)$ for a grammar $G = (V_N, \{a\}, Q, S)$ where Q contains the following multirules:

(A) $(S, \# AS_1)$,
(S_1, AS_1),
(S_1, K_a);
(B1) $((\#, \#), (A, a), (K_a, K_1 B_0 K_b))$,
(B2) $((\#, \#), (A_0, a)\ (K_1, K_1 B_0))$,
(B3) $((\#, \#), (A, a), (K_1, K_1), (K_b, BK_b))$,
(B4) $((\#, a), (K_1, \#_1)$,
(C1) $((\#, \#), (A_0, a), (K_a, K_3))$,
(C2) $((\#, \#), (A_0, a), (K_3, K_3))$,
(C3) $((\#, a), (K_3, a))$,
(D1) $((\#_1, \#_1), (B, a), (K_b, K_4 A_0 K_a))$,
(D2) $((\#_1, \#_1), (B_0, a), (K_4, K_4 A_0))$,
(D3) $((\#_1, \#_1), (B, a), (K_4, K_4), (K_a, AK_a))$,
(D4) $((\#_1, a), (K_4, \#))$,
(E1) $((\#_1, \#_1), (B_0, a), (K_b, K_5))$,
(E2) $((\#_1, \#_1), (B_0, a), (K_5, K_5))$,
(E3) $((\#_1, a), (K_5, a))$.

Now if $W = (x, w_1, \ldots, w_{n-1}, y)$ is a derivation of the type 2 over G where $y \in \{a\}^*$ and $x = \beta \# A_0^m A^{n-m-1} K_a$ then the following conditions must be fulfilled:

(a) $y = \beta\varphi$,

(b) $\# A_0^m A^{n-m-1} K_a \underset{G}{\Rightarrow}{}^*_2 \varphi$,

(c) W must contain the member $\beta a^n \#_b B_0^{m+1} B^{n-m-2} K_b$. Similar conditions hold for $x = \beta \#_b B_0^{m+1} B^{n-m-2} K_b$ and it holds therefore $P_2 = L_2(G)$. (See also [14] for a more detailed discussion).

Definition 4. A string $x = x_1 x_2 \ldots x_n \in V^*$ where $x_i \in V^{*1}$ is equal to a $y \in V^*$ mod. permutation if $y = x_{i_1} x_{i_2} \ldots x_{i_n}$ for some permutation $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$. A set $A \subset V^*$ is equal to a set $B \subset V^*$ mod. permutation if to every $x \in A$ there is $y \in B$ equal to x mod. permutation and vice versa each $y \in B$ is equal to a $x \in A$ mod. permutation. A set $A \subset V^*$ is regular mod. permutation if it is equal mod. permutation to a regular set.

Corollary 2. *$\mathfrak{M}_2(CF)$ and therefore CS contains sets which are not regular modulo permutation.*

Proof. The set P_2 from example 3 is not regular modulo permutation.

Remark 4. It would be interesting to find some properties of functions f for which it holds that the set $\{a^{f(n)} \mid n \geqq 1\}$ belongs to CS.

Remark 5. It is interesting to study in more details the differencies between derivations of various types over multiple grammars. We can limit the considerations to grammars of the multiplicity two. It can be shown that to every multiple grammar $G = (V_N, V_T, Q, S)$ there exists a multiple grammar $G_1 = (V_N, V_T, Q_1, S)$ so that $L_1(G) = L_{\mathrm{I}}(G_1) = \{x \mid x \in V_T^*, S \underset{G_1\,\mathrm{I}}{\Rightarrow^*} x\}$. $x \underset{G_1\,\mathrm{I}}{\Rightarrow} y$ if and only if $x = tu_{i_1}wu_{i_2}z$, $y = tv_{i_1}wv_{i_2}z$ where $\{i_1, i_2\}$ is a permutation of $\{1, 2\}$ and $((u_1, v_1), (u_2, v_3)) \in Q_1$ or $x = tuz$, $y = tvz$ and $(u, v) \in Q$. The derivations of the type 2 differs from derivations of the type 1 in such a way that the rules are applied in the given order from left to right. This property was considerably used up in the examples 2 and 3. Therefore it seems that $P_1, P_2 \notin \mathfrak{M}_1(CF)$. In derivations of the type 3 is furthermore reguested to "use" the lef-most occurencies of the left hand sides of rules in multirules. It can be shown that a context sensitive grammar G_1 can be constructed to every context-sensitive grammar G such that $L_L(G_1) = L(G)$ where $L_L(G) = \{x \mid x \in V_T^*, S \underset{G\,L}{\Rightarrow^*} x\}$ is the set of terminal strings which are generated over G_1 by such derivations in which rules are applied on left most occurencies of their left hand sides only. Let us write for multiple context-free and Λ-free grammar $G = (V_N, V_T, Q, S)$, $x \underset{G}{\Rightarrow}_4 y$ if $x = tA_1yA_2z$, $y = tv_1yv_2z$, $((A_1, v_1), (A_2, v_2)) \in Q$, t does not contain A_1 and $y = \Lambda$ (we use parallel formulations to those from def. 2). It is a straighforward matter to construct to G_1 a multiple context free and Λ-free grammar G_2 such that $S \underset{G_1\,L}{\Rightarrow^*} x$ if and only if $S \underset{G_2\,4}{\Rightarrow^*} x$. We can therefore write $CS = \mathfrak{M}_4(CF)$ in an obvious notation. Applying some theorems from [11] we can show that the assumption $\mathfrak{M}_4(CF) = \mathfrak{M}_1(CF)$ implies that any contextsensitive set can be generated by an "almost context-free grammar" i.e. by a grammar the rules of which are context free but the rule r can be applied on a string w_i if a only if $w_i \in V^*A_rV^*$ where $A_r \subset V_N$ is a set associated with the rule r. It seems therefore that $\mathfrak{M}_1(CF) \subsetneqq CS$. Similar arguments can be stated for $\mathfrak{M}_2(CF)$ and $\mathfrak{M}_3(CF)$.

Theorem 6. *If $A, B \in \mathfrak{M}_R(CF)$ then $A \cup B \in \mathfrak{M}_R(CF)$ if $A, B \in \mathfrak{M}_i(CF)$ then $A \cup B \in \mathfrak{M}_i(CF)$, if $A, B \in \mathfrak{N}_i(CF)$ then $A \cup B \in \mathfrak{N}_i(CF)$ for $i = 1, 2, 3$.*

Proof. can be obtained by a modification of the proof that the union of context-free sets is a context-free set.

Remark 6. As any context-free grammar is a multiple or a relational grammar we obtain at once that many problems for relational and multiple grammars are not decidable (see [2]). For example there is not decidable for multiple Λ-free grammars whether $L_i(G_1) \cap L_i(G_2)$ is an empty, a finite or an infinite set, whether it holds for a multiple Λ-free grammar G $L(G) = V_T^*$ and so on (see [2]).

Remark 7. It is known that if $A, B \in CS$ then $A \cap B \in CS$. Theorem 4 implies that if $A, B \in \mathfrak{M}_j(CF)$ then $A \cap B \in CS$. The problem whether it must be $A \cap B \in \mathfrak{M}_i(CF)$ for some $i \geqq j$ is open.

Theorem 7. *The problem whether $L_2(G)$ is an empty, a finite or an infinite set is for multiple Λ-free grammars recursively unsolvable.*

Proof. We shall construct a multiple grammar G which generates a nonempty set if and only if some Post correspondence problem has a solution. Let $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be strings over an alphabet V_T containing at least two symbols. Let us form a multiple grammar $G = (V_N \cup \hat{V}_T \cup \overline{V}_T, V_T \cup \{\#\}, Q, S\}$ where $\hat{V}_T = \{\hat{a} \mid \hat{a}$ *is for* $a \in V_T$ *an abstract symbol*$\}$, $\overline{V}_T = \{\bar{a} \mid \bar{a}$ *is for* $a \in V_T \cup \{\#\}$ *an abstract symbol*$\}$, $V_N = \{S, A, B\}$. Q contains the following multirules:

(I) $(S, \overline{\#}A\overline{\#}B)$;

(IIa) $((A, \hat{a}_i A), (B, \hat{b}_i B))$ for $i = 1, 2, \ldots, n$ where for $x = x_1 x_2 \ldots x_s \in V_T^*$ is $\hat{\Lambda} = \Lambda$, $\hat{x} = \hat{x}_1 \hat{x}_2 \ldots \hat{x}_s$;

(IIb) $((A, \hat{a}_i), (B, \hat{b}_i))$ for $i = 1, 2, 3, \ldots, n$;

(IIIa) $((\bar{a}, a), (\hat{b}, \bar{b}), (\bar{a}, a), (\hat{b}, \bar{b}))$ for each $a, b \in V_T \cup \{\#\}$;

(IIIb) $((\bar{a}, a))\,(\bar{a}, a))$ for each a $\in V_T \cup \{\#\}$.

It holds obviously $S \underset{G}{\Rightarrow}{}_2^{\infty}\, y \underset{G}{\Rightarrow}{}_2^{*}\, x \in (V_T \cup \{\#\})^*$ if and only if $y = a\bar{b}\hat{y}_1 D_1 a\bar{b}\hat{y}_2 D_2$ where $a \in (V_T \cup \{\#\})^*$, $\bar{b} \in \overline{V}_T$, $(D_1, D_2) \in \{(A, B) \cup (\Lambda, \Lambda)\}$, $\hat{y}_1 \hat{y}_2 \in \hat{V}_T^*$. It follows that $L_2(G) = L_3(G) = \#x\#x$ where $x = a_{i_1} a_{i_2} a_{i_3} \ldots a_{i_k} = b_{i_1} b_{i_2} \ldots b_{i_k}$ i.e. $L_2(G) \neq \emptyset$ if and only if the Post correspondence problem for $a_1, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ has a solution. It is obvious that if $L(G) \neq \emptyset$ then $L(G)$ is infinite. QED.

Remark 8. An open question is whether the problem "is $L_1(G)$ an empty set?" is recursively decidable.

Theorem 8. *If* $A \in \mathfrak{M}_i(CF)$ $i = 1, 2, 3$ *then* A *is recursive. The problem* $x \in L_i(G)$ *is for multiple context-free grammars and* $i = 2, 3$ *recursively unsolvable.*

Proof. The first assertion of the theorem follows from the fact that $\mathfrak{M}_i(CF) \subset CS$ because context-sensitive sets are recursive. The second assertion of the theorem follows from the following observations. By a slight modification of the grammar G from the proof of the previous theorem a multiple grammar G_1 can be constructed so that $\#x\# \in L_2(G_1)$ if and only if a Post correspondence problem have a solution (see [14]). QED.

Theorem 9. *The problem whether* $V_T^* x V_T^* \cap L_2(G) = \emptyset$, $x \in V_T^{\infty}$, *is for multiple* Λ*-free grammars recursively undecidable.*

Proof. Let us have a grammar $G = (V_N \cup \hat{V}_T \cup \overline{V}_T, V_T \cup \{b_0, \#, \uparrow\}, Q, S)$ as in the proof of the theorem 7 with the only difference that instead of (IIb) Q contains the multirules $((A, \hat{a}_i \uparrow x \uparrow), (B, b_0))$ where $\uparrow$, b_0 are new terminal symbols and $x \in V_T^{\infty}$. Denote $V_T' = V_T \cup \{b_0, \uparrow, \#\}$. Obviously $(V_T')^* \uparrow x \uparrow (V_T')^* \cap L_2(G) = \emptyset$ if and only if there is a solution $y = a_{i_1} a_{i_2} \ldots a_{i_k} = b_{i_1} b_{i_2} \ldots b_{i_k}$ of the Post correspondence problem for $a_1, \ldots, a_n$; $b_1, \ldots, b_n$. QED.

Theorem 10. *Let* $A \in \mathfrak{M}_\alpha(CF)$, $B \in \mathfrak{N}_\alpha(CF)$, $\alpha \in \{R, 1, 2, 3\}$, *let* C *be a regular set. Then* $A \cap C \in \mathfrak{M}_\alpha(CF)$, $B \cap C \in \mathfrak{N}_\alpha(CF)$.

Proof. The idea of the proof is the same as the idea of the proof that the intersection of a context-free set and a regular set is a context-free set (see [2]). We prove theorem for the case that $\alpha \in \{1, 2, 3\}$ the proof for $\alpha = R$ is similar. Let $G = (V_N, V_T, Q, S)$ be a multiple context-free grammar such that $A = L_i(G)$. We can assume that G have the multiplicity two. Let $\mathscr{A} = (V_T, I, \Phi, s_0, F)$, where V_T is an input alphabet, I a set of states, Φ a transition function, s_0 an initial state and F a set of end states, be an automaton accepting A. Let us form the alphabets $\bar{V}_N = \{[s_1, A, s_2] \mid A \in V_N, s_1, s_2 \in I, s_1 = s_2 \textit{ or } s_2 \textit{ is accessible from } s_1\}$,

$$\bar{V}_T = \{[s_1, a, s_2] \mid a \in V_T, s_1, s_2 \in I, s_2 \in \Phi(a, s_1)\} \cup \{[s, \Lambda, s] \mid s \in I\}$$

Let us denote for $x = x_1 x_2 \dots x_m \in V^\infty$ ($x_i \in V$ for $i = 1, 2, \dots, n$) and for $s, s' \in I$

$$x(s, s') = \{[s_0, x_1, s_1][s_1, x_2, s_2] \dots [s_{m-1}, x_m, s_m] \mid s_0 = s, s_m = s' \textit{ and } [s_{i-1}, x_i, s_i] \in \bar{V}_T \cup \bar{V}_N \textit{ for } i = 1, 2, \dots, m\}$$

Let us consider the multiple grammar

$$\bar{G} = (\bar{V}_N \cup \bar{V}_T \cup \{\bar{S}\}, V_T, \bar{Q}, \bar{S})$$

where

$$\begin{aligned}\bar{Q} = {} & \{(\bar{S}, [s_0, S, s_1]) \mid s_1 \in F\} \cup \\ & \cup \{([s, a, s'], a) \mid [s, a, s'] \in \bar{V}_T\} \cup \\ & \cup \{(s, A_1, s'], \bar{g}_1, \mid (A_1, q_1) \in Q; s, s' \in I, \bar{g}_1 \in q_1\ (s, s')\} \cup \\ & \cup \{((([s_1, A_1, s_1'], \bar{q}_1), ([s_2, A_2, s_2'], \bar{q}_2)) \mid \\ & ((A_1, q_1), (A_2, q_2)) \in Q \textit{ and it holds } \bar{q}_i \in q_i(s_i, s_i'), s_i, s_i' \in I \\ & \textit{for } i = 1, 2\}.\end{aligned}$$

It can be shown in the same way as in [2] that $L_i(\bar{G}) = L_i(G) \cap A$. $CF(G, \bar{G})$ obviously holds. QED.

Theorem 11. *Let $A, B \in \mathfrak{M}_\alpha(CF)$ (resp. $A, B \in \mathfrak{N}_\alpha(CF)$) where $\alpha \in \{R, 1, 2, 3\}$ then*

(i) $AB \in \mathfrak{M}_\alpha(CF)$ (*resp.* $AB \in \mathfrak{N}_\alpha(CF)$)

(ii) for $\alpha \neq 3$, $A^R \in \mathfrak{M}_\alpha(CF)$ (*resp.* $A^R \in \mathfrak{N}_\alpha(CF)$), $A^R = \{x^R \mid x \in A\}$) *where for* $x = x_1 x_2 \dots x_t \in V^\infty$, $x_i \in V_T^{*1}$, $x^R = x_t x_{t-1} \dots x_1$.

(iii) $A^{\sim n} = \{\underbrace{xx \dots x}_{n-\text{times}} \mid x \in A\} \in \mathfrak{M}_3(CF)$ (*resp.* . $\mathfrak{N}_3(CF)$).

Proof. Proof of (ii) is a slight modification of the proof of the assertion that A^R is a context-free set if A is a context-free set.

Let $A = L_i(G_1)$, $B = L_i(G_2)$, $i = 1, 2, 3$, (the proof for $L_R(G)$ being similar). Let $G_j = (V_{N_j}, V_{T_j}, Q_j, S_j)$, $j = 1, 2$. We can assume that $V_{N_1} \cap V_{N_2} = \emptyset$. Let us form the grammar $G = (V_{N_1} \cup V_{N_2} \cup S, V_{T_1} \cup V_{T_2}, \bar{Q}, S)$ where S is a new symbol and

$$\bar{Q} = Q_1 \cup Q_2 \cup \{(S, S_1 S_2)\}$$

It is straighforward matter to verify that $L_i(G) = AB$.

We prove (iii) for $n = 2$, the proof for $n > 2$ is similar. By theorem 4 we can assume $A = L_3(G)$, $G = (V_N, V_T, Q, S)$. For $k = 1, 2$ put

$$V_{N,k} = \{[a, k] \mid [a, k] \text{ is for } a \in V_N \text{ an abstract symbol}\}$$

for $x = x_1 x_2 \dots x_m \in V^*$ write $[x, k] = \bar{x}_1 \bar{x}_2 \dots \bar{x}_m$ where $\bar{x}_j = x_j$ for $\bar{x}_j \in V_T$, $\bar{x}_j = [x_j, k]$ for $x_j \in V_N$. Let us put $G^{\sim 2} = (V_{N,1} \cup V_{N,2} \cup S, V_T, Q^{\sim 2}, S)$ where

$$\begin{aligned} Q^{\sim 2} = \{S, [S, 1]\,[S, 2]\} \cup \\ \cup \{((A, 1], [q, 1]), ([A, 2], [q, 2])) \mid (A, q) \in Q\} \cup \\ \cup \{(([A, 1], [u, 1]), ([B, 1), [v, 1]), ([A, 2], [u, 2]), ([B, 2], [v, 2])) \mid \\ \mid ((A, u), (B, v)) \in Q\} . \end{aligned}$$

By the inspection of possible applications of multirules in a derivation of the type three we can see that if a multirule is applied on $[x, 1]\,[x, 2]$ then the unique string $[x_1, 1]\,[x_1, 2]$ is obtained. QED.

Theorem 12. *The substitution theorem for multiple context-free grammars. Let $A \in \mathfrak{M}_i(CF)$ (resp. $A \in \mathfrak{N}_i(CF)$) $A \subset V_T^*$. Let τ be a substitution (see [2]) on V_T^* and let $\tau(a) \in \mathfrak{M}_i(CF)$ (resp. $\tau(a) \in \mathfrak{N}_i(CF)$) for all $a \in V_T$, then for $i = 2, 3$, $\tau(A) \in \mathfrak{M}_i(CF)$ (resp. $\tau(A) \in \mathfrak{N}_i(CF)$).*

Proof. Let $i = 2$, $A = L_i(G) \in \mathfrak{M}_i(CF)$ and $\tau(a) = L_i(G_a) \in \mathfrak{M}_i(CF)$. Let $G = (V_N, V_T, Q, S)$, $G_a = (V_{N,a}, V_{T,a}, Q_a, S_a)$. It can be assumed that all the nonterminal alphabets are mutually disjoint, that all the grammars have the multiplicity 2 and that no rule in Q_a contains S_a in its right-hand side. Let us form the grammar $G' = (V_N', V_T', Q', \bar{S})$ where

$$V_N' = V_N \cup \bigcup_{a \in V_T} V_{N,a} \cup \bar{V}_N \cup \bar{V}_T' ,$$

$$\bar{V}_N = \{\bar{a} \mid a \in V_N \cup \bigcup_{V_T} V_{N,a}\} \cup \{\Lambda\} ,$$

$$V_T' = \bigcup_{a \in V_T} V_{T,a} , \quad \bar{V}_T' = \{\bar{a} \mid a \in V_T'\} .$$

Let $A_1, A_2 \in V_N$, $A, B_1, B \in V$. Then Q' contains the following multirules:

(1a) $((\bar{A}, \bar{A}), (A_1, \bar{B}\tilde{q}_1)), (\bar{A}, \bar{\bar{B}}\tilde{q}_1)$ for each $(A_1, Bq_1) \in Q$ and every $A \in V$

(1b) $((\bar{A}, \bar{A}), (A_1, \bar{B}_1\tilde{q}_1), (A_2, \tilde{q}_2))$ and $((\bar{A}_1, \bar{\bar{B}}_1\tilde{q}_1), (A_2, \tilde{q}_2))$ for each $((A_1, B_1 q_1)$ $(A_2, q_2)) \in Q$ and each $A \in V$.

Here $\hat{A} = A$ and for $x = x_1x_2 \dots x_t \in (V_T \cup V_N)^*$ $\tilde{x} = \tilde{x}_1 \dots \tilde{x}_t$ where $\tilde{x}_i = x_i$ for $x_i \in V_N^*$, $\tilde{x}_i = S_{x_i}$ if $x_i \in V_T$. The multirules (Ib) and (Ia) generate the set $\{\bar{S}_{x_1}\bar{S}_{x_2} \dots \dots \bar{S}_{x_t} \mid x_1x_2 \dots x_t \in A\}$ if we assume that $\{S_a, \bar{S}_a \mid a \in V_T\}$ is a new terminal alphabet.

(II) Q' further contains the following multirules.

(IIa) $(\bar{A}_1, \bar{B}q_1)$ for each $(A_1, Bq_1) \in \bar{Q} = \bigcup_{a \in V_T} Q_a$, $\bar{B} \in \bar{V}_N \cup \bar{V}_T'$.

(IIb) $((\bar{A}, \bar{A}), (C, \bar{B}q_1))$ for each $(C, Bq_1) \in \bar{Q}$, $C \neq S_a$ and $\bar{A} \in \bar{V}_T' \cup \bar{V}_N$

(IIc) $((\bar{A}_1, \bar{B}q_1)(A_2, q_2))$ for each $((A_1, Bq_1), (A_2, q_2)) \in \bar{Q}$, $\bar{B} \in \bar{V}_N \cup \bar{V}_T'$

(IId) $((\bar{A}, \bar{A}), (A_1, Bq_1), (A_2, q_2))$ for each $\bar{A} \in \bar{V}_N \cup \bar{V}_T'$, $A_1 \notin \{S_a \mid a \in V_T\}$, $B \in \in V_N' \cup V_T'$ and $((A_1, Bq_1), (A_2, q_2)) \in \bar{Q}$.

(III) $(\bar{A}, A)$ and $((\bar{A}, A), (S_a, \bar{S}_a))$ belong to Q' for each $a \in V_T'$ and $\bar{A} \in \bar{V}_T' \cup \bar{V}_N$.

Obviously $CF(G, G')$. It can be verified that if $W = (w_0, w_1, w_2, \dots, w_n)$ is a derivation of the type 2 over G and $w_n \in (V_T')^*$ then W must contain a member w_j of the form $w_j = \bar{S}_a\tilde{x}$ where $S \underset{G}{\Rightarrow}_2^* ax$. Now if $\bar{S}_a$ is overwritten by some rule then from S_a only a string $\varphi_a \in L_2(G_a)$ can be derived. W therefore must contain a member $w_k = \varphi_a\bar{S}_b\tilde{y}$ where $S_a \underset{G_a}{\Rightarrow}_2^* \varphi_a$, $S \underset{G}{\Rightarrow}_2^* aby$. It follows that theorem for $\mathfrak{M}_2(CF)$ holds. By a slight modifications of the just given proof we can prove the assertion of the theorem for $\mathfrak{M}_3(CF)$, $\mathfrak{N}_2(CF)$ and $\mathfrak{N}_3(CF)$. More details can be found in [14].

Theorem 11a. *If* $A \in \mathfrak{M}_i(CF)$, $i = 2, 3$ *then* $A^\infty \in \mathfrak{M}_i(CF)$. *If* $A \in \mathfrak{N}_i(CF)$ *then* $A^* \in \mathfrak{N}_i(CF)$ $(i = 2, 3)$.

Proof. Let a be a symbol. Then $\{a\}^\infty \in \mathfrak{M}_2(CF)$ and $\{a\}^* \in \mathfrak{N}_2(CF)$ and the theorem follows from the theorem 10.

Remark 9: It is an open question whether the theorem 10 holds for $i = 1$. A string $v \in (V_N \cup V_T)^*$ is nonterminally k-bounded if it contains at most k nonterminal symbols. A derivation $W = (w_0, w_1, \dots, w_n)$ of the type i over a (multiple) grammar is nonterminally k-bounded if all its members are nonterminally k-bounded.

Theorem 13. *The set* $L_{i,k}(G) = \{x \mid x \in V_T^*$, *there exists a nonterminally k-bounded derivation* $W = (S, w_1, \dots, w_{n-1}, x)$ *over G of the type i\} is for every multiple grammar,* $k \geqq 1$ *and* $i = 1, 2, 3$ *a set regular mod. permutation.*

Proof. Let us put

$$\bar{V} = \{\bar{\alpha} \mid \bar{\alpha} \text{ is for } \alpha \in V^{\infty k} \text{ an abstract symbol}\}$$

and consider the grammar $\tilde{G} = (\bar{V}, V_T, R, \bar{S})$ where $R = \{(\bar{\alpha}, \xi\bar{\beta}) \mid \alpha \in \bar{V},\ \xi \in V_T^*,\ \alpha \underset{G}{\Rightarrow}_i w \text{ where } w = \xi\beta \text{ mod permutation}\} \cup \{(\bar{\alpha}, \xi) \mid \xi \in V_T^*,\ \alpha \underset{G}{\Rightarrow} \xi\}$. It can be easily verified that $\bar{S} \underset{\tilde{G}}{\Rightarrow}^* \xi\bar{\alpha}$ if and only if there is over G a derivation $(w_0, w_1, \dots, w_n)$ of the type i such that w_n is equal to $\xi\alpha$ mod permutation. But $\tilde{G}$ is a left-linear grammar. QED.

Corollary 2. *The problem whether* $L_{i,k}(G) = \emptyset$ is *algorithmically decidable for each* $k \geqq 1$.

Proof. It can be shown by the direct inspection of the proof of the previous theorem that a grammar $\tilde{G}$ generating the regular set equal to $L_{i,k}(G)$ mod permutation can be effectively found. But $L_{i,k}(G) = \emptyset$ if and only if $L(\tilde{G}) = \emptyset$ and the problem $L(\tilde{G}) = \emptyset$ is, as it is known, decidable.

Definition 5. The multiple grammar $G^<$ with ordering of multirules is the quintuple $G^< = (V_N, V_T, Q, <, S)$ where $G = (V_N, V_T, Q, S)$ is a multiple grammar and $<$ some partial ordering of the set Q. The derivation $W = (w_0, w_1, \ldots, w_n)$ of the type i over $G^<$ is such derivation W of the type i over $G = (V_N, V_T, Q, S)$ satisfying for $i = 0, 1, 2, \ldots, n-1$ the following condition: If $w_i = x_{i1}p_1x_{i2}p_2 \ldots x_{is}p_sx_{i,s+1}$, $w_{i+1} = x_{i1}q_1x_{i2}q_2 \ldots x_{is}q_sx_{i,s+1}$ then there is no $\bar{r} = ((\bar{p}_1, \bar{q}_1), \ldots, (\bar{p}_k, \bar{q}_k)) \in Q$ such that

$$\bar{r} > r = ((p_1, q_1), \ldots, (p_s, q_s)), \tag{1}$$

$$w_i = \bar{x}_{i1}\bar{p}_1\bar{x}_{i2} \ldots \bar{x}_{ik}\bar{p}_k\bar{x}_{i,k+1}, \tag{2}$$

$$w_{i+1} \neq \bar{x}_{i1}\bar{q}_1\bar{x}_{i2} \ldots \bar{x}_{ik}\bar{q}_k\bar{x}_{i,k+1}, \tag{3}$$

Here x_{ij} denotes arbitrary strings over the alphabet V. Write $x \underset{G^<}{\Rightarrow}{}_i^* y$ if there is a derivation $W = (x, w_1, \ldots, w_{m-1}, y)$ over $G^<$ of type i. Let further

$$L_i(G^<) = \{x \mid x \in V_T^*, S \underset{G^<}{\Rightarrow}{}_i^* x\}$$

A grammar $G^< = (V_N, V_T, Q, <, S)$ with ordering of multirules is CF (resp. $CF(\Lambda)$) if $G = (V_N, V_T, Q, S)$ is a multiple context-free grammar (resp. G is a multiple Λ-free grammar).

Theorem 14. *Let* $\mathfrak{M}_i^<(CF)$ and $\mathfrak{N}_i^<(CF)$ *be the following classes:* $\mathfrak{M}_i^<(CF) = \{A \mid A = L_i(G^<)$ *for a multiple Λ-free grammar with ordering of multirules*$\}$, $\mathfrak{N}_i^<(CF) = \{A \mid A = L_i(G^<)$ *for a multiple context-free grammar* $G^<$ *with ordering multirules*$\}$. *Then for* $i = 2, 3$

$$\mathfrak{M}_i^<(CF) = CS\,, \quad \mathfrak{N}_i^<(CF) = RE$$

where RE is the class or recursively enumerable sets.

Proof. We shall prove the assertion of the theorem for $\mathfrak{M}_2^<(CF)$, the proof for $\mathfrak{N}_2^<(CF)$ is similar. Let $G = (V_N, V_T, R, S)$ be a context-sensitive grammar. According to [10] we can assume that $R \subset V_N^{\infty 2} \otimes V^\infty$. Let us have the context-sensitive grammar $\bar{G} = (\bar{V}_N, V_T, \bar{R}, S)$ where $\bar{V}_N = V_N \cup \tilde{\bar{V}}$, $\tilde{\bar{V}} = \{[A_1, 1, j], [A_2, 2, 1] \mid [A_1, 1, j], [A_2, 2, j]$ *are for a rule* $(A_1A_2, q) \in R$ *with the index j abstract symbols*$\}$, $A_1, A_2 \in V_N$,

$\bar{R} = \{(A, q) \mid (A, q) \in R\} \cup$
$\cup \{(A, [A, 1, i)], (A, [A, 2, i])) \mid A \in V_N$ *and* i *is index of some rule from* $R\} \cup$
$\cup \{([A_1, 1, i) [A_2, 2, i], q) \mid (A_1 A_2, q) \in R, (A_1, A_2 q)$ *has the index* $i\}$.

It can be shown that $S \underset{\bar{G}}{\Rightarrow}^* y \in (V_N \cup V_T)^*$ if and only if $S \underset{G}{\Rightarrow}^* y$ i.e. $L(G) = L(\bar{G})$. $\bar{G}$ is obviously a context-sensitive grammar. Form the multiple context-free grammar $G' = (V'_N \cup \{x\}, V_T, Q, \tilde{S})$ where

$$V'_N = \bar{V}_N \cup \tilde{V} \cup \hat{V}_T ,$$

x is a new symbol, $\tilde{V} = \{\tilde{a} \mid a \in V_T \cup \bar{V}_N\}$, $\hat{V}_T = \{\hat{a} \mid a \in V_T\}$. Denote for $y = a_1 a_2 \dots a_s \in (\bar{V}_N \cup V_T)^\infty$ $\bar{y} = \bar{a}_1 \bar{a}_2 \dots \bar{a}_s$ where $\bar{a}_i = a_i$ for $a_i \in \bar{V}_N$, $\bar{a}_i = \hat{a}_i$ for $a_i \in V_T$, $\bar{\bar{A}} = A$. Then

$$Q = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$$

where

$Q_1 = \{(\tilde{B}, \bar{a}\bar{\bar{q}}), ((\tilde{a}_1, \tilde{a}_1), (B, \bar{a}\bar{\bar{q}})) \mid (B, aq) \in \bar{R}, \tilde{a}_1 \in \tilde{V}, a \in V\}$,
$Q_2 = \{((\tilde{B}_1, \tilde{a}), (B_2, \bar{\bar{q}})), ((\tilde{a}_1, \tilde{a}_1), (B_1, \bar{\bar{a}}), (B_2, \bar{\bar{q}})) \mid (B_1 B_2, aq) \in \bar{R}, a \in V, B_1, B_2 \in V_N, \tilde{a}_1 \in \tilde{V}\}$,
$Q_3 = \{(\tilde{a}, a), ((\tilde{a}, a), (\tilde{b}, \tilde{b})) \mid a, b \in V_T\}$,
$Q_4 = \{((\tilde{B}_1, x), (A, x), (B_2, x)), ((\tilde{a}_1, \tilde{a}_1), (B_1, x), (A, x), (B_2, x)) \mid$ *there exist* $(B_1 B_2, aq) \in \bar{R}$, A *is an arbitrary symbol from* V_N *and* $B_1, B_2 \in V_N$, $a \in V\}$.

Define the partial ordering $<$ on Q in the following way

(I) If $((\tilde{B}_1, \tilde{a}), (\bar{B}_2, \bar{\bar{q}})) = r \in Q_2$ then $r < ((\tilde{B}_1, x), (A, x), (B_2, x))$ for each $A \in \bar{V}_N \cup \tilde{V} \cup \hat{V}_T$.
(II) If $((\tilde{a}_1, \tilde{a}_1), (B_1, \bar{\bar{a}}), (B_2, \bar{\bar{q}})) = r \in Q$ then $r < ((\tilde{a}_1, \tilde{a}_1), (B_1, x), (A, x), (B_2, x))$ for each $A \in \bar{V}_N \cup \hat{V}_T$.
(III) There is no other pair satisfying the relation $<$.

It holds for the multiple grammar $G^<$ with partial ordering of multirules where

$$G^< = (V'_N \cup \{x\}, V_T, Q, <, \tilde{S})$$

that

$$L_2(G^<) = L(G) .$$

For if $\tilde{S} \underset{G^< 2}{\Rightarrow}^* z \in V_T^*$ then

(a) No multirule $r \in Q_4$ can be used.
(b) Each member w_j of the derivation $W = (\tilde{S}, w_1, \dots, w_{n-1}, z)$, $z \in V_T^*$, over G of the type 2 has the form $\gamma \tilde{a} \beta$ where $\gamma \in V_T^*$, $\tilde{a} \in \tilde{V}$, $\beta \in (V'_N)^*$, $z = \gamma y$ and $\tilde{a}\beta \underset{G^< 2}{\Rightarrow}^* y$.

(c) A multirule $((B_1, q_1), (B_2, q_2)) \in Q_2$ can be applied on w_i if and only if w_i cannot be expressed in the form $w_i = x_{i1}B_1x_{i2}B_2x_{i3}$ where $x_{i2} \neq \Lambda$ because otherwise a multirule from Q_4 can be applied.

(d) The rules from Q_1 are applied in the same way as the corresponding rules from Q_1.

It follows that $L_2(G^<) \subset L(G)$. As the reverse statement is obvious we have $L_2(G^<) = L(G)$. Because it can be shown by the methods similar to those used in proofs of lemma 6 and 5 that

$$\mathfrak{M}_2^<(CF) \subset \mathfrak{M}_3^<(CG) \subset CS\,,$$

$$\mathfrak{N}_2^<(CF) \subset \mathfrak{N}_3^<(CF) \subset RS$$

the theorem is proved.

4. SOME SUBCLASSES OF RELATIONAL GRAMMARS

Definition 6. Relational grammar $G = (k, V_N, V_T, Q, S)$ where $V_N = (V_{N_1}, \ldots, V_{N_k})$, $V_T = (V_{T_1}, \ldots, V_{T_k})$, $S = (S_1, \ldots, S_k)$ is kT-regular if each multirule r of the G is of the form $r = ((A_1, \beta_1 B_1), (A_2, \beta_2 B_2), \ldots, (A_k, \beta_k B_k))$ where $\beta_i \in V_{T_i}^*$ and $(B_1, B_2, \ldots$ $\ldots, B_k) \in \bigtimes_{i=1}^{k} V_{N_i} \cup \{(\Lambda, \Lambda, \ldots, \Lambda)\}$.

A kT-regular grammar G is kR-regular if $\beta_i \in V_{T_i}^{\infty}$.

A kT-regular grammar G is k-regular if $\beta_i \in V_{T_i}$.

A kT-regular grammar G is k-regular mod Λ if $\beta_i \in V_{T_i} \cup \{\Lambda\}$.

Definition 7. A kT-regular, a kR-regular, a k-regular a k-regular mod Λ grammar $G = (k, V_N, V_T, Q, S)$ is strongly kT-regular, strongly kR-regular, strongly k-regular mod Λ respectively if to every pair of k-tuples $(A_1, A_2, \ldots, A_k) \in \bigtimes_1^k V_{N_i}$ and $(\beta_1, \beta_2, \ldots$ $\ldots, \beta_k) \in \bigtimes_1^k V_{T_i}^*$ there exist at most one multirule $((A_1, \beta_1 B_1), (A_2, \beta_2 B_2), \ldots,$ $\ldots, (A_k, \beta_k B_k)) \in Q$.

Theorem 15. *The class of relations generated by kT-regular grammars is the class of k-ary transductions* (see [7]).

Proof. Let $G = (k, (V_{N_1}, \ldots, V_{N_k}), (V_{T_1}, \ldots, V_{T_k}), Q, (S_1, \ldots, S_k))$ be a kT-regular grammar. Let us construct the following automaton $\mathscr{A} = ((V_{T_1}, \ldots, V_{T_k}), I, \Phi, s_0, F)$ with k input tapes where

$$I = \{\overline{(A_1, A_2, \ldots, A_k)} \mid \overline{(A_1 \ldots, A_k)} \textit{ is for } (A_1 \ldots, A_k) \in \bigtimes_1^k V_{N_i} \textit{ an abstract symbol}\} \cup \\ \cup \{\overline{(\Lambda, \Lambda, \ldots, \Lambda)}\}$$

is the set of states,

$s_0 = \overline{(S_1, S_2, \ldots, S_k)} \in I$ is the initial state,

Φ, the transition function of A, is defined as follows

$$\Phi(\beta_1, \beta_2, \ldots, \beta_k, \overline{(A_1, A_2, \ldots, A_k)}) =$$
$$= \{\overline{(B_1, B_2, \ldots, B_k)} \mid \text{there is a } ((A, \beta_1 B_1), \ldots, (A_k, \beta_k B_k)) \in Q\} .$$

$F = \{\overline{(\Lambda, \Lambda, \ldots, \Lambda)}\}$.

It can be easily verified that $T(A) = R(G)$. Let us have an automaton A with k input tapes defined as above. Let

$$G = (k, (V_{N_1}, \ldots, V_{N_k}), (V_{T_1}, V_{T_2}, \ldots, V_{T_k}), Q, (s_{01}, \ldots, s_{0k}))$$

be a kT-regular grammar where

$$V_{N_i} = \{s_i \mid s_i \text{ is for } s \in I \text{ an abstract symbol}\} .$$

If $s_m \in \Phi(\beta_1, \ldots, \beta_k, s_l)$ then

$$((s_{l_1}, \beta_1 s_{m_1}), \ldots, (s_{l_k}, \beta_k s_{m_k})) \in Q .$$

If $s_m \in F \cap \Phi(\beta_1, \ldots, \beta_k, s_l)$ then

$$((s_{l_1}, \beta_1), \ldots, (s_{l_k}, \beta_k)) \in Q .$$

No other multirules belong to Q. It holds obviously $T(A) = R(G)$. QED.

Definition 8. A set $A \subset V_T^*$ is a transduction set if $A = L_R(G)$ for a kT-regular grammar G. $\mathfrak{N}_T$ is the class of transduction sets. $\mathfrak{N}_T^A = \{A \mid A \subset V_T^\infty, A \in \mathfrak{N}_T\}$.

Corollary 2. *If $A, B \in \mathfrak{N}_T$ then $A \cup B \in \mathfrak{N}_T$, $AB \in \mathfrak{N}_T$, $A^{\sim n} \in \mathfrak{N}_T$.*

Proof can be realized in the same way as the proof of the theorem 11.

Corollary 3. *If FA denotes the class of regular sets then*

$$FA \subsetneqq \mathfrak{N}_T \subset \mathfrak{N}_R(CF)$$

Proof. Obviously $FA \subset \mathfrak{N}_T \subset \mathfrak{N}_R(CF)$. The example 1 indicates that $FA \neq \mathfrak{N}_T$.

Corollary 4. *For kT-regular grammars the problem $L_R(G) = \emptyset$ is recursively decidable.*

Proof. Apply the corollary 2.

Theorem 16. *The following problems are not recursively decidable for kR-regular grammars (and therefore for kT-regular grammars)*

(1) $L_R(G_1) \cap L_R(G_2)$ *is an empty, finite or infinite set*,

(2) $L_R(G) = V_T^*$,

(3) $L_R(G_1) \subset L_R(G_2)$,

(4) $L_R(G_1) - L_R(G_2)$ *is an empty, finite or infinite set*.

Proof is a modification of the proofs of the similar assertions for context-free grammars (see [2]). We prove (1) in order to show how to modify corresponding proofs from [2]. Let $a_1 a_2 \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be $2n$ strings over an alphabet V_T containing two symbols at least.

Let $E = \{c_1, c_2, \ldots, c_n\}$ be a set of new symbols and let $G_1 = (2, (\{C\}, \{A\}), (E, V_T), Q_1, (C, A))$ where

$$Q_1 = \{((C, c_i C), (A, a_i A)), ((C, c_i), (A, a_i)) \mid i = 1, 2, \ldots, n\}$$

Let $G_2 = (2, (\{C\}, \{A\}), (E, V_T), Q_2, (C, A))$ where

$$Q_2 = \{((C, c_i C), (A, b_i A)), ((C, c_i), (A, b_i)) \mid i = 1, 2, \ldots, n\}.$$

Obviously $L_R(G_1) = \{c_{i_1}, c_{i_2} \ldots, c_{i_m} a_{i_1} a_{i_2} \ldots a_{i_m} \mid m \leqq 1, n \geqq i_j \geqq 1\}$ and $L_R(G_2) = \{c_{i_1} c_{i_2} \ldots c_{i_m} b_{i_1} b_{i_2} \ldots b_{i_m} \mid m \geqq 1, 1 \leqq i_j \leqq n\}$. Therefore $L_R(G_1) \cap L_R(G_2) = \emptyset$ if and only if the Post correspondence problem for $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ has a solution. Moreover $L_R(G_1) \cap L_R(G_2) \neq \emptyset$ if and only if $L_R(G_1) \cap L_R(G_2)$ is an infinite set. QED.

Proposition. *The intersection of two transduction set not containing Λ is a context-sensitive set.*

Proof. Transduction sets not containing Λ are context-sensitive (see theorem 20) sets and intersection of context-sensitive sets is a context-sensitive set.

Remark 10. The problem whether $\mathfrak{R}_T$ is closed under the intersection is open.

Theorem 18. *To every kT-regular grammar G there is a strongly k-regular* mod Λ *grammar G′ such that* $R(G) = R(G')$.

Proof. See [5] where the theorem is stated in the terms of the theory of generalized automata.

Corollary 5. $\mathfrak{R}_T$ *is the class* $\{A \mid A = L_R(G)$ *for a strongly k-regular* mod Λ *grammar* $G\}$.

Theorem 19. *If* $A \in \mathfrak{R}_T$ *and B is a regular set then* $A \cap B \in \mathfrak{R}_T$.

Proof is a slight modification of the proof of the theorem 10. See also [14].

Theorem 20. $\mathfrak{N}_T^4 \subset \mathfrak{M}_1(CF)$.

Proof. Let $A \in \mathfrak{N}_T^4$. We prove the assertion of the theorem for the case that $A = L_R(G)$ where G is a $2T$-regular grammar. The proof for general k is similar. We prove, that $A \in \mathfrak{M}_2(CF)$. Let $G = (2, (V_{N_1}, V_{N_2}), (V_{T_1}, V_{T_2}), Q, (S_1, S_2))$. We can assume that G is a strongly 2-regular mod Λ grammar (theorem 18). Let us form a multiple Λ-free grammar $G' = (V_N', V_{T_1} \cup V_{T_2}, Q', \overline{S_1S_2})$ where

$$\begin{aligned} V_N' &= V_{N_1}' \cup V_{N_2}' \cup V_{N_3}' \cup V_{N_4}\,, \\ V_{N_1}' &= \{\bar{\alpha}, \bar{\bar{\alpha}} \mid \alpha \in V_{N_1}V_{T_2}V_{N_2}\}\,, \\ V_{N_2}' &= \{\bar{\alpha} \mid \alpha \in V_{T_1}V_{N_1}V_{N_2}\}\,, \\ V_{N_3}' &= \{\bar{\alpha} \mid \alpha \in V_{N_1}V_{N_2}\}\,, \\ V_{N_4}' &= \{\bar{\alpha} \mid \alpha \in V_{T_i}V_{N_i}, i = 1, 2\}\,. \end{aligned}$$

We can assume, that $V_{N_1} \cap V_{N_2} = \emptyset$.

(I) If $((A_1, B_1), (A_2, B_2)) \in Q$, $A_i, B_i \in V_{N_i}$ then Q'

contains the following multirules:

(Ia) $((\widetilde{a_1A_1}, \widetilde{a_1B_1}), (\widetilde{a_2A_2}, \widetilde{a_2B_2}))$ for each $a_i \in V_{T_i}^{*1}$. Here and below $\widetilde{aA_i} = A_i$ for $a = \Lambda$ and $\widetilde{a_iA_i} = \overline{a_iA_i}$ for $a_i \in V_{T_i}$.

(Ib) $(\overline{A_1A_2}, \overline{B_1B_2}), (\overline{A_1a_2A_2}, \overline{B_1a_2B_2}), (\overline{A_1A_2}, \overline{B_1B_2}), (\overline{a_1A_1A_2}, \overline{a_1B_2B_1}),$ $(\overline{\overline{A_1a_2A_2}}, \overline{\overline{B_1a_2B_2}}) \in Q'$ for each $a_1 \in V_{T_1}$, $a_2 \in V_{T_2}$.

(II) If $(A_1, cB_1), (A_2, B_2)) \in Q$, $c \in V_{T_1}$, then for each $a \in V_{T_1}^{*1}$, $b \in V_{T_2}^{*1}$:

(IIa) $((\widetilde{aA_1}, \widetilde{acB_1}), (\widetilde{bA_2}, \widetilde{bB_2})) \in Q'$ for each $a, b \in V_{T_i} \cup \{\Lambda\}$ and $i = 1, 2$,

(IIb) $((\overline{A_1aA_2}, \overline{cB_1aB_2}) \in Q'$, $(\overline{aA_1A_2}, \overline{acB_1B_2}) \in Q'$ for each $a \in V_{T_2}$,

(IIc) $(\overline{A_1A_2}, \overline{cB_1B_2}) \in Q'$;

(III) If $(A_1, B_1), (A_2, cB_2)) \in Q$, $c \in V_{T_2}$, then

(IIIa) $((\widetilde{aA_1}, \widetilde{aB_1}), (\widetilde{bA_2}, \widetilde{bcB_2})) \in Q'$,

(IIIb) $(\overline{A_1aA_2}, a\,\overline{\overline{B_1cB_2}}) \in Q'$, $(a\,\overline{A_1A_2}, \overline{aA_1}\,\overline{cA_2}) \in Q'$ $(\overline{\overline{A_1aA_2}}, a\,\overline{\overline{B_1cB_2}}) \in Q'$

(IIIc) $(\overline{A_1A_2}, \overline{B_1cB_2}) \in Q'$;

(IV) If $((A_1, cB_1), (A_2, dB_2)) \in Q$, $c \in V_{T_1}$, $d \in V_{T_2}$, then

(IVa) $((\widetilde{aA_1}, a\,\overline{cB_1}), (\widetilde{bA_2}, b\,\overline{dB_2})) \in Q'$ for each $a \in V_{T_1}$, $b \in V_{T_2}$.

(IVb) $((\overline{bA_1A_2}, b\,\overline{cB_1}\,\overline{dB_2})) \in Q'$ for each $b \in V_{T_1}^{*1}$

(IVc) $(\overline{A_1\,a\,A_2}, \overline{cB_1}\,a\,\overline{dB_2}) \in Q'$

(V) If $((A_1\beta_1), (A_2, \beta_2)) \in Q$ where $\beta_1 \in V_{T_1}^{*1}$ then

(Va) $((\widetilde{aA_1}, a\beta_1), (\widetilde{bA_2}, b\beta_2)) \in Q'$ for $a\beta_1 \neq \Lambda$, $b\beta_2 \neq \Lambda$

(Vb) $((\overline{aA_1A_2}, a\beta_1\beta_2)) \in Q'$

(Vc) $(\overline{A_1aA_2}, \beta_1 a\beta_2) \in Q'$ $(\overline{\overline{A_1aA_2}}, a\beta_2) \in Q'$ for $\beta_1 = \Lambda$

(Vd) $(\overline{A_1A_2}, \beta_1\beta_2) \in Q'$ for $\beta_1\beta_2 \neq \Lambda$.

We firstly note that G' is a two-multiple Λ-free grammar. By the direct inspection of possible derivations over G we obtain that $L_R(G) \subset L_1(G)$. In fact if $(\Lambda, x) \in R(G)$ then in derivation of x over G' we can use only the 3^{th}, 2^{nd} and 5^{th} multirules in (Ib), the multirules in (IIIc) and then, possibly, the multirules from (V). Similar arguments can be used in all the remaining cases.

By induction according to the number of steps in derivations it can be shown that it must be $L_1(G) \subset L_R(G)$. QED.

Theorem 21. *The class $\mathscr{LK}$ of the symetrically localy finite transductions (see [7]) is the class of the relations generated by the kR-regular grammars.*

Proof. See [14].

Denote $LK = \{A \mid A = L_R(G)$ for a kR-regular grammar $G\}$.

Theorem 22. *If $A \in LK$ and B is a regular set then $A \cap B \in LK$.*

Proof. It is easily shown that the proof of the theorem is the same the proof of the theorem 16.

Theorem 23. *The class $\mathfrak{N}R$ of the sets A such that $\Lambda \notin A$, $A = L_R(G)$ for a strongly k-regular grammar, is the class of sets not containing Λ acceptable by k-multiple automata (see [4]). The class*

$$\mathfrak{N}DR = \{A \mid A = L_R(G) \textit{ for a k-regular grammar}\}$$

is a the class of sets accetable by k-multiple nondeterministic automata. If $A \in \mathfrak{N}R$ ($A \in \mathfrak{N}DR$) and B is a regular set then $A \cap B \in \mathfrak{N}R$ ($A \cap B \in \mathfrak{N}DR$).

Proof. In fact to every (nondeterministic) k-multiple automaton A an generalized finite (nondeterministic) automaton $\mathscr{A} = ((V_{T_1}, \ldots, V_{T_k}), I, \Phi, s_0, F)$ with k-input tapes in the sense [7] exists so that Φ is defined on $(\bigtimes_i V_{T_i}) \otimes I$ and the set $T(A)$ of strings accepted by A in the sense [4] is the set $\{x_1 \ldots x_k \mid (x_1, \ldots, x_k)$ belongs to the relation accepted by $\mathscr{A}\}$. Using this fact we can prove the all assertions of the theorem quite similary as the parallel assertions for the class $\mathfrak{N}_T^A$.

For further properties of the classes $\mathfrak{N}$ and $\mathfrak{N}R$ see [4], [13]). Let us denote RE: the class of recursively enumerable sets, and let $\mathfrak{M}_\alpha(CF)$, $\alpha = R, 1, 2, 3$, $\mathfrak{N}_\alpha(CF)$,

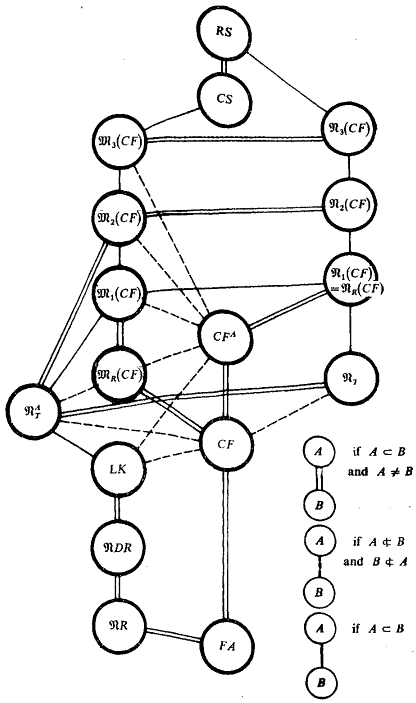

Fig. 1.

$\alpha = 1, 2, 3, R, CF, CS, \mathfrak{N}DR, \mathfrak{N}_T, LK, \mathfrak{N}R, \mathfrak{N}DR, CF^A, \mathfrak{N}_T^A$ have the above introduced meaning, let FA denote the class of regular sets. Then the above given results are shown graphically in the figure 1.

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VÝTAH

Násobné gramatiky

Jaroslav Král

Je-li $W = (w_0, \ldots, w_{n-1}, xpy)$ nějaké odvození nad gramatikou $G = (V_N, V_T, R, S)$ a obsahuje-li množina pravidel R pravidlo (p, q) je i $W' = (w_0, \ldots, w_{n-1}, xpy, xqy)$ odvození nad G. Pravidlo (p, q) může být tedy použito v n-tém kroku odvození W nezávisle na tom, zda může být v tomtéž kroku použito jiné pravidlo a nezávisle na tom, jaká pravidla byla použita v předchozích krocích odvození W. Tento předpoklad nezávislosti je možné oslabit různými způsoby. Lze např. stanovit, že je-li použito nějaké pravidlo v jistém kroku odvození W, pak „spolu s ním" musí být použita nějaká další pravidla. Pravidla jsou tedy aplikována ve skupinách. Násobné gramatiky o nichž pojednává článek představují formalizaci této koncepce.

Skupina pravidel je multipravidlo. Násobná gramatika je gramatika s multipravidly. Podle způsobu použití pravidel multipravidla jsou uvažovány tři typy odvození nad násobnou gramatikou. Je ukázáno, že se stačí omezit na násobné gramatiky násobnosti dvě (t.j. multipravidlo obsahuje nejvýše dvě pravidla). Násobná gramatika je kontextová, resp. bezkontextová, resp. bezkontextová bez prázdného slova jsou-li taková všechna pravidla multipravidel. Je dokázáno, že generativní síla násobných gramatik (násobných kontextových gramatik) není větší, než generativní síla (obyčejných) gramatik (resp. kontextových gramatik). Třídy množin generovatelných různými typy odvození nad násobnými bezkontextovými gramatikami bez prázdného slova tvoří hiearchii mezi třídou bezkontextových a třídou kontextových množin.

Dále jsou studovány problémy rozhodnutelnosti pro násobné gramatiky, problémy uzavřenosti vůči množinovým a jazykovým operacím, substituci, průniku s regulární událostí atd. Kromě toho jsou studovány relační gramatiky generující k-tice slov. Je dokázáno, že třídy množin $\{A \mid A = \{x_1 x_2 \ldots x_k \mid (x_1, \ldots, x_k)$ je k-tice generovatelná relační gramatikou $G\}\}$ má úzký vztah k třídě množin generovatelných jedním typem odvození nad násobnými gramatikami. Je ukázáno, že jisté podtřídy relačních gramatik generují právě třídu relací akceptovatelných zobecněnými automaty (viz [4], [5] a [7]).

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