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On the Optimum Organization of Calculations in a Decision System*

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The paper deals with the choice of the optimum sequence and time intervals for the solution of some statistical problems calculated on computers with limited speed.

1. INTRODUCTION

A decision set may be generally regarded as a computing machine which is specialized so as to realize a statistical test and to make statistical decision. A radar receiver detecting echo signals and measuring echo parameters in the background of noise, a servomechanism controlling the position of a gun in dependence on actual coordinates of a target estimated with a statistical error, a perceptron which recognizes geometrical figures, speech or graphical signs, a diagnostical set, etc., are typical examples of decision sets. The operating algorithm of a decision set is chosen in the optimum way in the sense of an arbitrary optimality criterion. However, because of some technical or exploitational reasons, the algorithm usually desired should be optimum in a narrowed class of algorithms subjected to some additional conditions of technical realization. The difficulties of technical realization of a statistical decision algorithm increase as the statistical properties of signals submitted to mathematical operations become more complicated. That is why suitably programmed large electronic computers are preferably used to solve such problems. The cost of obtaining a solution may be considerable, nevertheless it is justified in some particular cases. It depends on "usefulness" of the decisions, defined in some way, and this is the case beyond any doubt in many military or industrial applications of decision sets.

Some specific technical troubles are associated with the application of an electronic computer as a statistical decision set and, consequently some theoretical problems arise in this case. The first one is caused by the limited calculation speed of the com-.

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puter and by the necessity of maximum utilization of its potential possibilities in this field.

In order to illustrate the weight of this problem let us consider the case of long distances radar detection of targets. Assuming constant echo amplitudes and a gaussian noise with a white power spectrum, an optimum method of signal reception (noncoherent reception) consists in arithmetical averaging of squares of envelopes of signals received in suitable time intervals corresponding to different points of the space. This averaging operation should therefore be performed for a considerably large number of sample sequences, according to the total number of elementary volume-cells into which the observed space sector may be fractionized taking into account the angular and radial resolution of the radar. Since the summation of signals is a relatively simple operation, it may be realized by means of analogue techniques, a storage tube for example. Any element of the mosaic of the storage tube may be regarded as an analogue computer operating independently of other similar computers, all of which realize the summation of signals. If statistical properties of the signals are more complicated, the optimum algorithm of signal reception cannot be realized in such a simple manner. Application of a digital computer is necessary in this case, but a limited number of arithmetical and logical operations per second requires a reasonable limitation of the number of simultaneously operated signals. Something like a "statistical microscope" which could be focused on selected elements of the space sector under observation presents itself as a possible technical realization of a decision set using a digital computer. It needs, however, statistical independency of signals corresponding to different points of the space sector.

Besides noise and other undesirable factors, the constraints due to the limited rate of signal transformations limit the effectiveness of radar. If installation of additional computing units which would make possible full consummation of the information received cannot be taken into account, the only reasonable thing to do is to work out an optimum time-table for the computer.

Similar problems arise if digital computers are used for the control of technological processes, for the detection of impairments in complexe control systems, etc. Let us therefore analyse the problem of working out an optimum time-table for a decision set, or the problem of optimum organization of calculations, as indicated in the title.

2. OPTIMUM ORGANIZATION OF CALCULATIONS

Suppose a given finite sequence of operations or problems, say $z_1, z_2, \ldots, z_n$, is to be solved by a computing machine in a sequence which, for the time given, is unknown. The solution of every problem z_ν, $\nu = 1, 2, \ldots, n$ is followed by a cost r_ν, which depends on the time instant $(t = t_\nu)$ of finishing the problem by making a decision, and on the time interval $\tau_\nu = t_\nu - t_{\nu-1}$, $t_0 = 0 \leqq t_1 \leqq t_2 \leqq \ldots \leqq t_n$ used for the solution of this problem:

(1) $$r_\nu = \varphi(t_{\nu-1}, t_\nu; \bar{\alpha}),$$

where $\bar{\alpha} \overset{\text{def}}{=} (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)})$, $k = 1, 2, 3, \ldots$ denotes a given sequence of parameters characterizing the problem z_ν.

Let us consider the fucntion φ in more detail. Two components of this function should be taken into account: (1) the costs r'_ν due to a random error of the decision, the statistical mean value of which is a decreasing function of the time interval τ_ν, (2) the cost r''_ν due to the delay-time t_ν of making the decision, with respect to the time instant at which the problem z_ν has been set up. A more concrete form of the functions r'_ν, r''_ν, as well as any other criterion, is the subject of an arbitrary choice. In further considerations it will be assumed that

(2) $$r'_\nu = a_\nu e^{-b_\nu \tau_\nu}, \quad a_\nu, b_\nu \geqq 0,$$

(3) $$r''_\nu = c_\nu e^{d_\nu t_\nu}, \quad c_\nu, d_\nu \geqq 0,$$

(4) $$r_\nu = r'_\nu + r''_\nu.$$

The constant coefficients a_ν, b_ν, c_ν, d_ν stand here for $\alpha^{(1)}, \ldots, \alpha^{(4)}$. They are defined as quantities depending on the initial data concerning the task z_ν. The constants a_ν, c_ν may be, for example, interpreted as parameters characterizing a given space element under observation, if the problem is in optimum reception of radar signals in noise. In this case it is possible to write $a_\nu = c_\nu = p_\nu$, where p_ν denotes an initial probability of the signal to be present in noise at a given element numbered ν. The task z_ν may consist of an identification of the target ("friend" or "foe") and of an exact estimation of its coordinates and of the components of its velocity. The value a_ν may also depend on the costs which are to be payed if the decision is based on the initial information about the problem z_ν only, without any further observation of received signals. The coefficient b_ν may be determined as the value of some increasing function of the signal to noise ratio at a given space element. The greater the S/N ratio, the greater the risk of decision made after some time of observation. Finally the coefficient d_ν may depend on the initial distance to the target and on its velocity vector, because these parameters act immediately on the increasing danger from the target, the possibility of collision with an approaching aircraft or ship, for example. Similar interpretation of the coefficients a_ν, b_ν, c_ν, d_ν is possible in the case of other applications of the decision set, as for example, in the detection of impairments of a technical system.

The existence of an absolutely optimum time interval τ_1^* for solving the problem z_1, if $\nu = n = 1$ may be stated from the formulae (2)–(4). It may be obtained by substituting (2) and (3) into (4), putting $t_0 = 0$, $t_1 = \tau_1^*$, differentiating the expression with respect to τ_1^* and making the derivative equal zero. The result is a transcendential equation of the form

(5) $$-a_1 \cdot b_1 e^{-b_1 \tau^*_1} + c_1 d_1 e^{d_1 \tau^*_1} = 0.$$

We are interested in positive solutions of this equation only which exist if $a_1 b_1 > c_1 d_1$, in the opposite case the only physically admissible solution is $\tau_1^* = 0$. This means, that the decision for z_1 must be made immediately, on the base of initial information only.

If both sides of the equation (5) are divided by the first term of its left side, one gets an equation depending on a reduced number of parameters:

$$\varkappa_1' e^{\varkappa''_1 \tau^*_1} - 1 = 0 , \tag{6}$$

where

$$\varkappa_1' = \frac{c_1 \cdot d_1}{a_1 \cdot b_1} , \tag{6a}$$

$$\varkappa_1'' = b_1 + d_1 . \tag{6b}$$

The positive solution of (6)

$$\tau_1^* = - \frac{1}{\varkappa_1''} \ln \varkappa_1' \tag{7}$$

exists if $0 < \varkappa_1' \leqq 1$. By substituting (7) into (2) and (3), one obtains after slight transformations an expression for a minimal decision cost in the form

$$R_1^{(\text{cond})}(t_0) = a_1 (\varkappa_1')^{Q_1} + c_1 (\varkappa_1')^{1-Q_1} , \quad t_0 \equiv 0 , \tag{8}$$

where

$$Q_1 = \frac{b_1}{b_1 + d_1} . \tag{8a}$$

This result holds only if $\nu = n = 1$, that is to say, if there is one problem, z_1, to be solved.

Let us consider a more general case where a sequence of groups of four numbers $a_\nu, b_\nu, c_\nu, d_\nu$, $\nu = 1, 2, \ldots, n$ is given. The problem lies in the determination of an optimum sequence of tasks $\{z_{\nu_1}, z_{\nu_2}, \ldots, z_{\nu_n}\}$ which may be presented by a permutation $\{\nu_1, \nu_2, \ldots, \nu_n\}$ of the integers $\{1, 2, \ldots, n\}$, and in the calculation of optimum time instances $t_1, t_2, \ldots, t_n$ for making the decisions after the resolution of $z_{\nu_1}, z_{\nu_2}, \ldots, z_{\nu_n}$. The optimum choice of these quantities is that one which minimizes the total cost of all decisions in the sequence:

$$R = \sum_{\nu=1}^{n} r_\nu . \tag{9}$$

This problem may be solved using Bellman's method of dynamic programming. Let us suppose for a while that the sequence $\{\nu_1, \nu_2, \ldots, \nu_n\}$ is known and that the quantities $t_1, t_2, \ldots, t_{n-1}$ are chosen in some manner. The time interval τ_n of solving the problem z_{ν_n} is to be chosen in the optimum way. It can be proved that t_n depends on the time instant t_{n-1} only when the computer starts to solve the task z_{ν_n}. Indeed, t_n is defined optimally as a positive value:

$$t_n = t_{n-1} + \tau_n^* , \quad \tau_n^* \geqq 0 , \tag{10}$$

where τ_n^* minimalizes the conditional cost of calculation z_{ν_n}:

$$r_{\nu_n}^{(\text{cond})}(t_n; t_{n-1}) \overset{\text{def}}{=} a_{\nu_n} . \exp\left[-b_{\nu_n}\tau_n\right] + c_{\nu_n} . \exp\left[d_{\nu_n}(\tau_n + t_{n-1})\right] , \tag{11}$$

given the values of $t_1, t_2, \ldots, t_{n-1}$. If

$$\tilde{c}_{\nu_n}(t_{n-1}) \overset{\text{def}}{=} c_{\nu_n} . \exp\left[d_{\nu_n} t_{n-1}\right] ,$$

the problem is reduced to minimization of the function

$$r_{\nu_n}^{(\text{cond})}(t_n; t_{n-1}) = a_{\nu_n} . \exp\left[-b_{\nu_n}\tau_n\right] + \tilde{c}_{\nu_n}(t_{n-1}) . \exp\left[d_{\nu_n}\tau_n\right] , \tag{11a}$$

the general form of which is identical with (4), and, therefore, its minimum can be obtained from the formula (6) if

$$\tilde{\varkappa}'_{\nu_n}(t_{n-1}) \overset{\text{def}}{=} \frac{\tilde{c}_{\nu_n}(t_{n-1})\, d_{\nu_n}}{a_{\nu_n} b_{\nu_n}} = \varkappa'_{\nu_n} . \exp\left[d_{\nu_n} t_{n-1}\right] \tag{12}$$

is used instead of $\varkappa'_1$. The formula (6a) holds for the parameter $\varkappa''_{\nu_n}$ as well as for $\varkappa''_1$. The minimal cost $R_{\gamma_n}^{(\text{cond})}(t_n)$ may be obtained from the formula (8).

A step back can now be done and the time instant t_{n-1} can be chosen in the optimum way so as to minimize the total cost of solving two problems, $z_{\nu_{n-1}}, z_{\nu_n}$, given the time instant t_{n-2}:

$$r_{\nu_{n-1}}^{(\text{cond})}(t_{n-1}; t_{n-2}) = R_{\nu_n}^{(\text{cond})}(t_{n-1}) + r_{\nu_{n-1}}^{(\text{cond})}(t_{n-1}, t_{n-2}) , \tag{13}$$

where $R_{\nu}^{(\text{cond})}$ denotes the minimal cost of solving z_{ν_n} given the time instant t_{n-1}, and $r_{\nu_{n-1}}^{(\text{cond})}(t_{n-1}, t_{n-2})$ is the cost of solving $z_{\nu_{n-1}}$ if the time instants t_{n-2}, t_{n-1} are known.

Let us denote by $t_{n-1}^*(t_{n-2})$ the optimal time instant of making the decision for $z_{\nu_{n-1}}$ obtained by minimization of the last expression. Let $R_{\nu_n}^{(\text{cond})}(t_{n-2})$ be the minimum of $r_{\nu_{n-1}}^{(\text{cond})}(t_{n-1}; t_{n-2})$. Both quantities are related to the given time instant t_{n-2}. Assuming $t_{n-1}^*(t_{n-2})$ and $R_{\nu_n}^{(\text{cond})}(t_{n-2})$ to be known, our considerations can be extended taking into account the optimum time instant $t_{n-2}^*(t_{n-3})$ of making decision for $z_{\nu_{n-2}}$. A minimum value of the expression

$$r_{\nu_{n-2}}^{(\text{cond})}(t_{n-2}; t_{n-3}) = R_{\nu_{n-1}}^{(\text{cond})}(t_{n-2}) + r_{\nu_{n-2}}^{(\text{cond})}(t_{n-2}, t_{n-3}) \tag{14}$$

is to be found out.

The optimal value of t_{n-2} would be regarded as a function of t_{n-3}, hence it is right to put $R_{\nu_{n-2}}^{(\text{cond})} = R_{\nu_{n-2}}^{(\text{cond})}(t_{n-3})$.

It is evident that the method can be extended up until obtaining the values $t_1^*(t_0)$, and $R_{\nu_1}^{(\text{cond})}(t_0)$, $t_0 = 0$, where $R_{\nu_1}^{(\text{cond})}(0)$ gives the mimimum total cost R (see formula (9)) of n decisions corresponding to the tasks $z_{\nu_1}, z_{\nu_2}, \ldots, z_{\nu_n}$.

The foregoing method gives only a conditionally optimum solution for a fixed sequence $\{\nu_1, \nu_2, \ldots, \nu_n\}$. It would be applied $n!$ times for different permutations of integers $1, 2, \ldots, n$. Comparing the values $R_{\nu_1}^{(\text{cond})}(0)$ obtained in each of these cases, the optimum permutation which leads to minimum costs of calculations would be taken. The optimum solution of our problem consists, therefore, in the optimum sequence of integers $\{\nu_1, \nu_2, \ldots, \nu_n\}$ and in the sequence of time instants $t_1 \leqq t_2 \leqq \leqq \ldots \leqq t_n$.

Let us illustrate the method by the following numerical example:

Let us assume there are to be solved three statistical problems, z_1, z_2, z_3, characterized by their parameters $a_\nu, b_\nu, c_\nu, d_\nu$; $\nu = 1, 2, 3$:

$$(0{,}1\ \ 4\ \ 0{,}1\ \ 1)\,,\ (0{,}3\ \ 0{,}8\ \ 0{,}3\ \ 2)\,,\ (1\ \ 2\ \ 0{,}7\ \ 1{,}5)$$

b_ν and d_ν can be determined as some dimensionless quantities, therefore the time instants t_1, t_2, t_3 can be related to a fixed time unit. For $n = 3$, $3! = 6$, there exist 6 different permutations of integers (1, 2, 3):

$$(1, 2, 3)\,,\ (1, 3, 2)\,,\ (2, 1, 3),\ (2, 3, 1)\,,\ (3, 1, 2)\,,\ (3, 2, 1)\,.$$

According to (6a)–(8) one obtains:

ν	$\varkappa'_\nu$	$\varkappa''_\nu$	Q_ν	τ^*_ν	r_ν
1	0,25	5	0,8	0,277	0,111
2	2,5	2,8	0,286	0	0,6
3	0,525	3,5	0,571	0,187	1,217

The calculations can be facilitated in an actual case by using the diagram of Fig. 1. It can be seen from the obtained results, that the decision corresponding to the problem z_2 must be made immediately, based only on the initial information. However, it is quite reasonable because of the initial values of parameters a, b, c, d, and their physical meaning. Hence it is to be chosen between the following permutations: $\{2, 1, 3\}$ and $\{2, 3, 1\}$, both beginning with the integer $\nu_1 = 2$.

Let us take into account the first of these permutations. If $\tau_1 = 0$, $\tau_2 \geqq 0$, the optimum value of τ_3

$$\tilde{\tau}_3^* \stackrel{\text{def}}{=} -\frac{1}{\varkappa''_3} \ln \tilde{\varkappa}_1 (\tau_1 + \tau_2) = -\frac{1}{\varkappa''_3} (\ln \tilde{\varkappa}'_3 + d_3 \tau_2) = \tau_3^* - (1 - Q_3)\, \tau_2$$

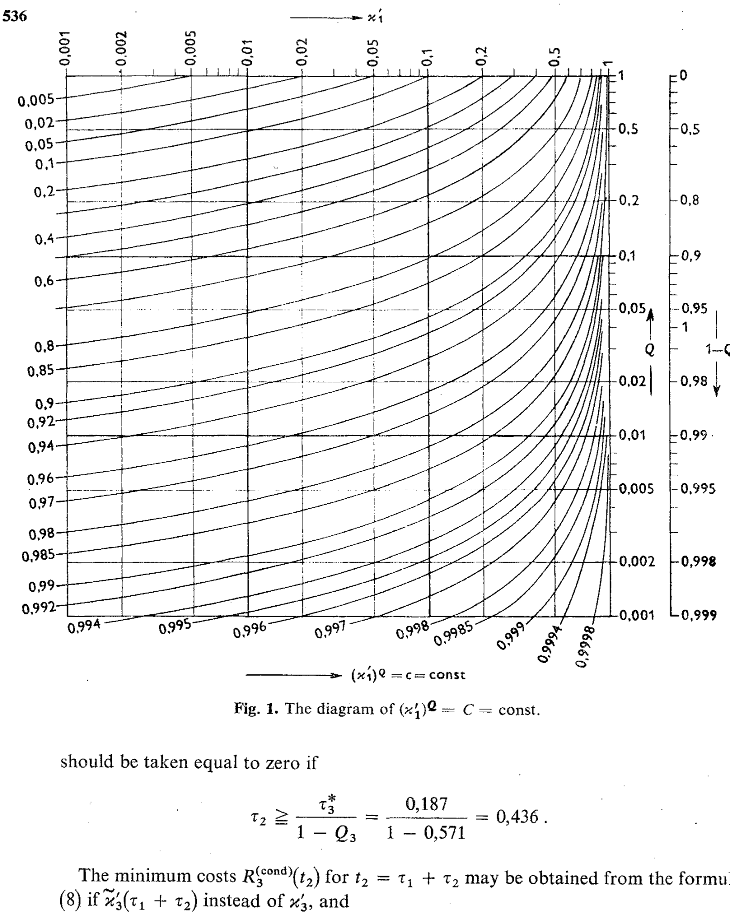

Fig. 1. The diagram of $(\varkappa_1')^Q = C = \text{const}$.

should be taken equal to zero if

$$\tau_2 \geqq \frac{\tau_3^*}{1 - Q_3} = \frac{0{,}187}{1 - 0{,}571} = 0{,}436\,.$$

The minimum costs $R_3^{(\text{cond})}(t_2)$ for $t_2 = \tau_1 + \tau_2$ may be obtained from the formula (8) if $\tilde{\varkappa}_3'(\tau_1 + \tau_2)$ instead of $\varkappa_3'$, and

$$\tilde{c}_3(\tau_1 + \tau_2) \stackrel{\text{def}}{=} c_3 e^{d_3(\tau_1 + \tau_2)}$$

instead of c_3 is used. The calculations can be facilitated using the diagrams of $(\varkappa')^Q = C = \text{const}$ in Fig. 1.

The conditional costs of solving both z_1 and z_3, according to (13), are

$$r_3^{(\text{cond})}(\tau_1 + \tau_2; \tau_1) = R_3^{(\text{cond})}(\tau_1 + \tau_2) + r_1^{(\text{cond})}(\tau_1 + \tau_2; \tau_1), \quad \tau_1 = 0,$$

or

$$r_3^{(\text{cond})}(\tau_1 + \tau_2, \tau_1) = a_3[\varkappa_3'(\tau_1 + \tau_2)]^{\varrho_3} + \\ + \tilde{c}_3(\tau_1 + \tau_2)[\tilde{\varkappa}_3'(\tau_1 + \tau_2)]^{1-\varrho_3} + a_1 e^{-b_2\tau_2} + \tilde{c}_1(\tau_1)\, e^{d_1(\tau_1+\tau_2)}.$$

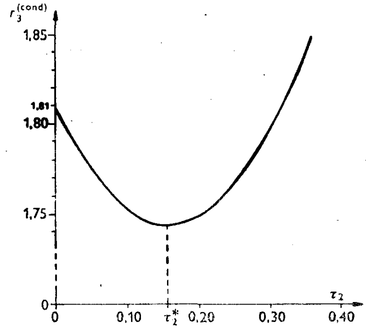

Fig. 2. The cost $r_3^{(\text{cond})}$ as a function of the time interval τ_2.

Substituting $\tau_1 = 0$ and other numerical values, one obtains

$$r_3^{(\text{cond})}(\tau_2; 0) = 1 \,.\, (0{,}525 e^{1{,}5\tau_2})^{0{,}571} + 0{,}7 \,.\, e^{1{,}5\tau_2}(0{,}525 \,.\, e^{1{,}5\tau_2})^{0{,}429} + \\ + 0{,}1 \,.\, e^{-4\tau_2} + 0{,}1 \,.\, e^{1\tau_2},$$

$$\tau_2 \geqq 0.$$

Calculating first the ordinates of this function, one can observe their increasing character. Therefore, it takes its minimum $(R_3^{(\text{cond})} = r_3^{(\text{cond})}(0; 0) = 1{,}423)$ if $\tau_2^* = 0$.

In a similar manner, taking the permutation $\{2, 3, 1\}$ of problems, one obtains the following expression for the cost of solving z_3:

$$r_3^{(\text{cond})}(\tau_2; 0) = 0{,}1 \,.\, (0{,}25 \,.\, e^{\tau_2})^{0{,}8} + 0{,}1 \,.\, e^{\tau_2}(0{,}25 \,.\, e^{\tau_2})^{0{,}2} + 1 \,.\, e^{-2\tau_2} + \\ + 0{,}7 \,.\, e^{1{,}5\tau_2},$$

$$\tau_2 \geqq 0.$$

A diagram of this function is given in Fig. 2. The minimum conditional cost $R_3^{(\text{cond})} = 1{,}745$ is obtained for $\tau_2^* = 0{,}16$.

It is evident from the obtained results that the optimum variant of performing the calculations is (z_2, z_1, z_3) for $\tau_1^* = \tau_2^* = 0$, $\tau_3^* = 0{,}187$, which gives the total cost of calculations $R = R_3^{(\text{cond})} = 0{,}6 + 0{,}2 + 1{,}217 = 2{,}017$.

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O optimální organizaci výpočtů v rozhodovacím systému

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Článek se zabývá otázkou výběru optimální posloupnosti řešení statistických úloh na počítačích s konečnou rychlostí. Uvažuje se, že s řešením každé úlohy jsou spojeny jisté náklady o dvou složkách: 1. r' — náklady způsobené nepřesností řešení, 2. r'' — náklady způsobené zpožděním mezi okamžikem t_0, kdy byla úloha zadána, a okamžikem jejího vyřešení. V případě, kdy obecný tvar nákladů je dán vzorcem (5) pro jednotlivé úlohy a vzorcem (9) pro úlohy $z_1, z_2, \ldots, z_n$, je možno s použitím metod dynamického programování R. Bellmana pro úlohy, charakterisované předem koeficienty $a_v, b_v, c_v, d_v, v = 1, 2, \ldots, n$, určit jejich optimální uspořádání $\{v_1^*, v_2^*, \ldots, v_n^*\}$ a optimální okamžiky $\tau_1^*, \tau_2^*, \ldots, \tau_n^*$, kdy jednotlivé úlohy mají být řešeny. Uvedený postup je ukázán na číselném příkladu.

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